Conformal Field Theories, Conformal Bootstrap and Applications

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Synopsis

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2. What are their characteristics and how can they be exploited?

The Basics I

Start with quantum mechanics...

$[x_i, p_j] = i\hbar\delta_{ij}$, position and momentum $\rightarrow$ operators.
Works for **fixed** number of particles ($i, j = 1, .., N$), starts to "fail" when we include special relativity.

Can't demand fixed number of particles, virtual particles are created all the time $\rightarrow$ generalization: **quantum field theory**!

Field $\phi(x, t)$ function in spacetime $\rightarrow \infty$ degrees of freedom! Usual prescription:

- Start from Lagrangian,
- Find conjugate variables (field and a derivative),
- Promote to operators, impose commutation relations,
- Define annihilation/creation operators,
- Compute "observables", correlation functions, amplitudes..

The Basics II

The development of quantum electrodynamics. 1937 (colourised).

Major problem

Theories are usually pathological in high-energies. See left!

Renormalization: impose cutoff in some very large energy scale and "integrate out" some content of the theory $\rightarrow$ various couplings start depending on energy scale.

$\Rightarrow$ Quantum field theory after renormalization: $QFT_{UV} \rightarrow QFT_{IR}$

The Basics III

Important concepts in this framework:

1. Critical points: points where couplings don't depend on energy scale $\rightarrow$ scale invariance $\rightarrow$ usually **conformal invariance**.
2. Universality (e.g liquid - gas phase transition $\leftrightarrow$ ferromagnetic phase transition).
3. Universality classes $\rightarrow$ classified by "critical exponents", constants.

Look at "the small picture" (conformal field theories, critical points) $\rightarrow$ "the big picture" (parameter space). Any QFT can be thought of as "perturbation" of a CFT!

The Tools I

Consider scalar action,

$$S = \int d^D x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3} \frac{\lambda_n}{n!}\phi^n \right) \tag{1}$$

Spectrum of operators (scaling dimension) $\rightarrow$ **first** characteristic of these theories.

We want to stay away from Lagrangians from now on!

At the critical points,

$$\Delta = \Delta_{eng} + \gamma(\lambda_n^\star)$$

where Δ_{eng} is the dimension of an operator that we can read off the Lagrangian, γ is the anomalous correction. In general it is non-integer $\rightarrow$ continuous spectrum $\rightarrow$ ... no well-defined "particles".

The Tools II

In conformal field theories, **correlation functions are extremely constrained!**

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{c}{(x_1 - x_2)^{2\Delta}} \tag{2a}$$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{f_{123}}{(x_1 - x_2)^{\Delta_1+\Delta_2-\Delta_3}(x_2 - x_3)^{\Delta_2+\Delta_3-\Delta_1}(x_1 - x_3)^{\Delta_1+\Delta_3-\Delta_2}} \tag{2b}$$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{g(u,v)}{(x_1 - x_2)^{2\Delta}(x_3 - x_4)^{2\Delta}} \tag{2c}$$

The coefficients f_{123} on [2b] are the **second** characteristics of these theories.
Note:

- Can't move on to higher correlators, no new info. Still, these are extremely valuable.

The Tools III

More tools...

1. **Unitarity**: $\langle \mathcal{O}^\dagger(t_1)..\mathcal{O}(t_1) \rangle \geq 0$ in Lorentzian signature.
   Usually interested in unitary theories $\rightarrow$ **strong** constraints on the operator spectrum.
2. **"Operator Product Expansion"** : $\mathcal{O}_i \mathcal{O}_j \sim \sum_{ijk} \mathcal{O}_k$
   Right-hand side depends on Δ's $\rightarrow$ everything can be built from (Δ, f_{ijk}), "CFT data".
3. **"Conformal Block Decomposition"**:
   Apply the Operator Product Expansion on [2c].

$$\rightarrow g(u,v) \sim \sum_{\mathcal{O}} g_{\Delta_{\mathcal{O}},l}(u,v) \tag{3}$$

See that everything comes down to the conformal blocks, contribution to the 4-point function from a single "conformal multiplet".

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No ordering on the left-hand side:

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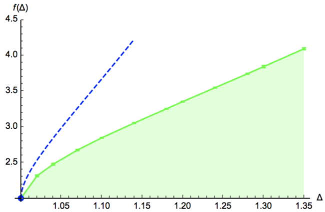
$\rightarrow$ Implications on the conformal blocks $g_{\Delta_{\mathcal{O}},l}(u,v)$!
Invariance under $(x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4)$:

$$\begin{cases} \sum_{\mathcal{O}} p_{\Delta_{\mathcal{O}},l} F_{\Delta,\Delta_{\mathcal{O}},l} = 1 \\ F_{\Delta,\Delta_{\mathcal{O}},l} = \dfrac{v^{\Delta} g_{\Delta_{\mathcal{O}},l}(u,v) - u^{\Delta} g_{\Delta_{\mathcal{O}},l}(v,u)}{u^{\Delta} - v^{\Delta}}, \quad p_{\Delta_{\mathcal{O}},l} \equiv f^2_{\phi\phi\mathcal{O}} > 0 \end{cases} \tag{5}$$

The Conformal Bootstrap II

This **highly non-trivial** sum rule is called the **bootstrap equation**. **Geometric interpretation** [Rattazzi *et al.*, JHEP **12** (2008) 031] $\rightarrow$ extract information in D=4.

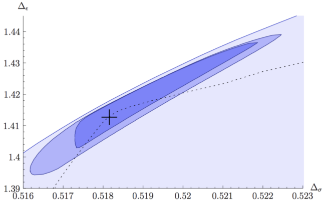
Investigate **when the bootstrap equation is satisfied**..



Start with 2 Scalar operators of dimension Δ, apply **Operator Product Expansion**.
Left, [Results from MSc dissertation]: Determine numerical **upper bound** on $f(\Delta)$ (blue and green lines) on the **allowed minimum dimensions** (green shaded area) of the **first** scalar operator present on the right-hand side of the OPE.

Conformal Bootstrap III

Important applications to many critical phenomena: e.g 3D Ising model $\rightarrow$ world-record precision for "critical exponents" [Kos *et al.*, JHEP **16** (2016) 036].



Input the operator spectrum correctly ($\mathcal{Z}_2$ discrete global symmetry, 2 relevant scalars $\sigma, \epsilon..$) $\rightarrow$ **"pushes" the method to bring us closer to Ising.** Left, [Kos *et al.*, JHEP **11** (2014) 109]:

- Cross: Known dimensions with errors.
- Blue: Bootstrap predictions with different input.

Outlook

So.. I hope that I demonstrated effectively why I chose to work on this topic, why YOU should consider it:

- Conceptually simple method,
- Non-perturbative $\rightarrow$ no ϵ- expansion. Relies only on generic features of CFTs.
- Much more rigorous nowadays than other methods, such as Monte Carlo.

Thank you for your attention!