Defects preserving $\mathcal{N} = 2$ supersymmetry between two free CFTs

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Motivation

- To look into which conformal interfaces preserve $\mathcal{N} = 2$ worldsheet SUSY between two theories CFT_1 and CFT_2, each compactified on a D-dimensional torus.
  $\rightarrow$ Use "Folding Trick".
- Equivalently, to formulate the gluing conditions and construct the boundary states/D-branes that leave $\mathcal{N} = 2$ SUSY unbroken.
  $\rightarrow$ Use "Boundary CFT".
- Why???
- Applications to:
  - CFT (Gepner models)
  - String Theory (D-branes in Type IIB/IIA)
  - Mathematics (Kähler Geometry)
  - Statistical Mechanics

The Boundary CFT approach

How to impose boundary conditions in String Theory?

- Open string: Neumann, Dirichlet at the endpoints
- Closed string: Every point equivalent...

Idea: Need a more generic way to describe boundary conditions.

Solution: Boundary CFT $\rightarrow$ Boundary conditions encoded in boundary states, which are coherent states.

Let:

$$S(z) = \sum_{n \in \mathbb{Z}} S_n z^{-n-h}, \quad \tilde{S}(\bar{z}) = \sum_{n \in \mathbb{Z}} \tilde{S}_n \bar{z}^{-n-h}$$

be the generators of the symmetry algebra $\mathcal{A}$ defined on the upper-half complex plane and ρ the automorphisms of $\mathcal{A}$. Relate $S, \tilde{S}$ at the boundary, i.e.

$$S(z) = \rho(\tilde{S}(\bar{z})), \quad z \in \mathbb{R}.$$

Boundary states

Using this relation and mapping the upper-half plane to the (unit) circle we obtain:

$$(S_n - (-1)^h \rho(\tilde{S}_{-n})||B\rangle\rangle = 0, \text{ for all } n \in \mathbb{Z}.$$

This is called gluing condition and $||B\rangle\rangle$ are the boundary states.
Boundary states are a linear combination of Ishibashi states, i.e.

$$||B\rangle\rangle = \sum_i N_i |i\rangle\rangle,$$

where N_i non-negative integers (consistency).
In every case, the conformal symmetry should be preserved. This is translated to

$$(L_n - \tilde{L}_{-n})||B\rangle\rangle = 0.$$

!!! More symmetries $\rightarrow$ More constraints

Free bosonic field theory ($c = 1, h = 1$)

- Primary fields:

$$S \equiv \partial_z X_L(z) = -\frac{i}{2}\sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad \tilde{S} \equiv \partial_{\bar{z}} X_R(\bar{z}) = -\frac{i}{2}\sum_{n\in\mathbb{Z}} \tilde{a}_n \bar{z}^{-n-1}$$

- Symmetry algebra: $\mathcal{A} \equiv u(1), \quad \rho = \pm id$
- Generators: $S_n \equiv a_n, \;\; \tilde{S}_n \equiv \tilde{a}_n$
- Zero modes: $a_0 \equiv p_L = k, \quad \tilde{a}_0 \equiv p_R = k$
  $\rightarrow p_L = p_R = k$ (k: center-of-mass momentum)
- Gluing condition:

$$(a_n \pm \tilde{a}_{-n})||B\rangle\rangle = 0$$

  - $+ \rightarrow$ Neumann $(n = 0 \rightarrow k = 0)$
  - $- \rightarrow$ Dirichlet

Circle Compactification

Purpose: Too many dimensions (26 in bosonic string, 10 in superstring).
$\rightarrow$ Wrap some of them around small compact spaces.
Simplest case: Circle $S^1 \cong \mathbb{R}/2\pi R\mathbb{Z}$ with radius R. Identify:

$$X \sim X + 2\pi R w,$$

where w is the winding number (no analogue in particles).
Zero modes become: $p_L = \frac{k}{2R} + wR, \ \ p_R = \frac{k}{2R} - wR,$ with $k, w \in \mathbb{Z}$!!!
$\rightarrow p_L \neq p_R$
Gluing condition:

$$(a_n \pm \tilde{a}_{-n})||B\rangle\rangle = 0$$

Boundary states:

$$||0,w\rangle\rangle_N = \frac{1}{2^{\frac{1}{4}}}\sqrt{R}\sum_{w\in\mathbb{Z}} e^{iw\tilde{\phi}_0} \exp\left(-\sum_{n>0}\frac{1}{n}a_{-n}\tilde{a}_{-n}\right)|0,w\rangle_N,$$

$$||k,0\rangle\rangle_D = \frac{1}{2^{\frac{1}{4}}}\frac{1}{\sqrt{R}}\sum_{k\in\mathbb{Z}} e^{-i\frac{k}{R}\phi_0} \exp\left(\sum_{n>0}\frac{1}{n}a_{-n}\tilde{a}_{-n}\right)|k,0\rangle_D$$

Torus compactification

D-Torus: $T^D \cong \mathbb{R}^D / 2\pi\Lambda_D$. Identify:

$$X^I \sim X^I + 2\pi \sum_{i=1}^{D} w^i e_i^I, \quad w^i \in \mathbb{Z}.$$

Reduce to $D = 2$ $\rightarrow T^2 \cong S^1_{R_1} \times S^1_{R_2}$
This is a CFT with $c = 2$.
$\rightarrow$ Two real bosons, each compactified on a circle.
Zero modes:

$$p_L^\mu = \frac{k_\mu}{2R_\mu} + w_\mu R_\mu, \quad p_R^\mu = \frac{k_\mu}{2R_\mu} - w_\mu R_\mu, \quad \mu = 1, 2$$

Boundary states for rotated branes

Gluing condition:

$$\left[\begin{pmatrix} a_n^1 \\ a_n^2 \end{pmatrix} + O \begin{pmatrix} \tilde{a}_{-n}^1 \\ \tilde{a}_{-n}^2 \end{pmatrix}\right] ||B\rangle\rangle = 0, \qquad O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotate: D1-brane rotated by an angle θ. Gluing condition becomes:

$$\left[\begin{pmatrix} a_n^1 \\ a_n^2 \end{pmatrix} + \mathcal{O} \begin{pmatrix} \tilde{a}_{-n}^1 \\ \tilde{a}_{-n}^2 \end{pmatrix}\right] ||B\rangle\rangle = 0, \quad \mathcal{O} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \in O(2)$$

Rational rotation ($0 < \theta < \frac{\pi}{2}$):

$$\tan\theta = \frac{NR_2}{MR_1}, \quad N, M \in \mathbb{N} \ \ \& \ \text{coprime}$$

Boundary state:

$$||B\rangle\rangle = \sqrt{\frac{NM}{\sin 2\theta}} \sum_{k,w\in\mathbb{Z}} e^{ik\alpha - iw\beta} \exp\left(-\sum_{n>0} \frac{1}{n} a_{-n}^{\mu} \tilde{a}_{-n}^{\nu} \mathcal{O}_{\mu\nu}\right) |kN, wM; -kM, wN\rangle$$

Free fermionic field theory ($c = 1/2, h = 1/2$)

Need also fermions to talk about SUSY!

- Primary fields: $\Psi(z) = \sum_r \psi_r z^{-r-\frac{1}{2}}, \quad \tilde{\Psi}(\bar{z}) = \sum_r \tilde{\psi}_r \bar{z}^{-r-\frac{1}{2}}$
- Gluing condition:

$$\left(\psi_r - i\eta\rho(\tilde{\psi}_{-r})\right) ||B\rangle\rangle = 0,$$

  where $\eta = \pm 1$ (+: Neveu-Schwarz, -: Ramond).

  For a two-fermion theory:

$$\left[\begin{pmatrix} \psi_r^1 \\ \psi_r^2 \end{pmatrix} + i\mathcal{O}_F \begin{pmatrix} \tilde{\psi}_{-r}^1 \\ \tilde{\psi}_{-r}^2 \end{pmatrix}\right] ||B\rangle\rangle = 0, \quad \mathcal{O}_F \in O(2)$$

- Boundary state:

$$||B\rangle\rangle_{\text{NS}} = \exp\left(-i \sum_{r\in\mathbb{N}-\frac{1}{2}} \psi_{-r}^{\mu}\tilde{\psi}_{-r}^{\nu}(\mathcal{O}_F)_{\mu\nu}\right) |0\rangle_{\text{NS}}$$

! Omit the discussion for the Ramond sector.

$\mathcal{N} = 1$-supersymmetric free field theory ($c = 3/2$)

Combine the previous results:
$\rightarrow$ Two bosons and two fermions on the upper-half complex plane.

- Symmetry algebra: $\mathcal{N} = 1$ superconformal algebra
- Generators: T ($h = 2$), G ($h = \frac{3}{2}$)
- Gluing conditions:

$$\begin{aligned} (L_n - \tilde{L}_{-n})||B\rangle\rangle &= 0 \\ (G_r - i\eta\tilde{G}_{-r})||B\rangle\rangle &= 0 \end{aligned}$$

- Boundary state:

$$||B\rangle\rangle = ||B\rangle\rangle_{\text{bos}} \otimes ||B\rangle\rangle_{\text{ferm}}$$

$\mathcal{N} = 2$-supersymmetric free field theory ($c = 3$)

! Works only for even spacetime dimensions.
Question: Which $\mathcal{N} = 1$ boundary states preserve $\mathcal{N} = 2$ SUSY as well?
Preliminaries: Combine two real bosons into a complex boson (complexification). Same for the fermions ($i = 1, 2$):

$$\begin{aligned} X^{+i} &= \frac{1}{\sqrt{2}}(X^{2i-1} + iX^{2i}), \quad X^{-i} = \frac{1}{\sqrt{2}}(X^{2i-1} - iX^{2i}) \\ \Psi^{+i} &= \frac{1}{\sqrt{2}}(\Psi^{2i-1} + i\Psi^{2i}), \quad \Psi^{-i} = \frac{1}{\sqrt{2}}(\Psi^{2i-1} - i\Psi^{2i}) \end{aligned}$$

Primary fields (left sector):

$$\begin{aligned} \partial X^{+i} &= -\frac{i}{2}\sum_{n>0} a_n^{+i} z^{-n-1}, \;\; \partial X^{-i} = -\frac{i}{2}\sum_{n>0} a_n^{-i} z^{-n-1} \\ \Psi^{+i} &= \sum_{r>0} \psi_r^{+i} z^{-r-\frac{1}{2}}, \;\; \Psi^{-i} = \sum_{r>0} \psi_r^{-i} z^{-r-\frac{1}{2}} \end{aligned}$$

$\mathcal{N} = 2$ gluing conditions

- Symmetry algebra: $\mathcal{N} = 2$ superconformal algebra
- Generators: T, G^+, G^-, J
- Gluing conditions (B-type):

$$T = \tilde{T}, \;\; G^+ = \tilde{G}^+, \;\; G^- = \tilde{G}^-, \;\; J = \tilde{J}$$

- Equivalently:

$$\begin{aligned}(L_n - \tilde{L}_{-n})||B\rangle\rangle &= 0 \\ (G^+_r - i\eta\tilde{G}^+_{-r})||B\rangle\rangle &= 0 \\ (G^-_r - i\eta\tilde{G}^-_{-r})||B\rangle\rangle &= 0 \\ (J_n + \tilde{J}_{-n})||B\rangle\rangle &= 0\end{aligned}$$

- A-type $\Leftrightarrow$ B-type (mirror symmetry)

B-type gluing conditions

B-type gluing conditions:

$$\left[\begin{pmatrix} a_n^{+1} \\ a_n^{+2} \end{pmatrix} + \Omega \begin{pmatrix} \tilde{a}_{-n}^{+1} \\ \tilde{a}_{-n}^{+2} \end{pmatrix}\right] ||B\rangle\rangle = 0$$
$$\left[\begin{pmatrix} a_n^{-1} \\ a_n^{-2} \end{pmatrix} + \Omega^{\dagger} \begin{pmatrix} \tilde{a}_{-n}^{-1} \\ \tilde{a}_{-n}^{-2} \end{pmatrix}\right] ||B\rangle\rangle = 0$$
$$\left[\begin{pmatrix} \psi_r^{+1} \\ \psi_r^{+2} \end{pmatrix} + \Omega_{\mathcal{F}} \begin{pmatrix} \tilde{\psi}_{-r}^{+1} \\ \tilde{\psi}_{-r}^{+2} \end{pmatrix}\right] ||B\rangle\rangle = 0$$
$$\left[\begin{pmatrix} \psi_r^{-1} \\ \psi_r^{-2} \end{pmatrix} + \Omega_{\mathcal{F}}^{\dagger} \begin{pmatrix} \tilde{\psi}_{-r}^{-1} \\ \tilde{\psi}_{-r}^{-2} \end{pmatrix}\right] ||B\rangle\rangle = 0$$

Here: $\Omega, \Omega_{\mathcal{F}} \in U(2) \hookrightarrow O(4)$.

B-type boundary states (NS-sector)

- Bosonic part:

$$||B\rangle\rangle_{\text{bos}} = \sum_i N_i \exp\left\{-\sum_{n>0} \frac{1}{n}\left(a_{-n}^{+i}\tilde{a}_{-n}^{-j}\Omega_{ij} + a_{-n}^{-i}\tilde{a}_{-n}^{+j}\Omega_{ij}^{\dagger}\right)\right\} |k^+, k^-, w^+, w^-\rangle,$$

where

$$|k^+, k^-, w^+, w^-\rangle = |kN_1^+, kN_1^-, wM_1^+, wM_1^-; -kM_1^+, -kM_1^-, wN_1^+, wN_1^-\rangle$$

- Fermionic part:

$$||B\rangle\rangle_{\text{NS}} = \sum_i N_i \exp\left\{-i \sum_{r\in\mathbb{N}-\frac{1}{2}} \left(\psi_{-r}^{+i}\tilde{\psi}_{-r}^{-j}(\Omega_{\mathcal{F}})_{ij} + \psi_{-r}^{-i}\tilde{\psi}_{-r}^{+j}(\Omega_{\mathcal{F}})_{ij}^{\dagger}\right)\right\} |0\rangle_{\text{NS}}$$

- Full boundary state: $||B\rangle\rangle_{\text{full}} = ||B\rangle\rangle_{\text{bos}} \otimes ||B\rangle\rangle_{\text{NS}}$

The unfolding procedure

Conformal interfaces can be described in two equivalent ways:

1. As boundary conditions in the tensor-product theory $\mathrm{CFT}_1 \otimes \mathrm{CFT}_2^*$.
2. As operators mapping the states of CFT_2 to those of CFT_1.

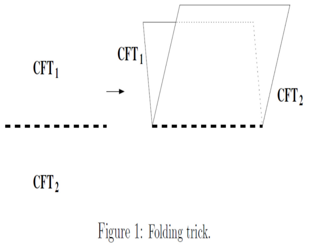

Figure 1: The folding trick

Unfolding: Hermitean conjugation and exchange of left with right movers in CFT_2.

Free-boson interfaces preserving $u(1)^2$

Conformal invariance:

$$(L_n - \tilde{L}_{-n})||B\rangle\rangle = 0$$

Equivalently: $I_{1,2} : \mathcal{H}_2 \to \mathcal{H}_1$ commutes with $\{L_n - \tilde{L}_{-n}\},\ n \in \mathbb{Z}$.
This means:

$$\begin{pmatrix} a_n^1 \\ -\tilde{a}_{-n}^1 \end{pmatrix} I_{1,2} = I_{1,2}\Lambda \begin{pmatrix} a_n^2 \\ -\tilde{a}_{-n}^2 \end{pmatrix}, \text{ for } \Lambda \in O(1,1)$$

Fold CFT_2 to CFT_2^* :

$$\begin{pmatrix} a_n^2 \\ \tilde{a}_n^2 \end{pmatrix} \mapsto \begin{pmatrix} -\tilde{a}_{-n}^2 \\ -a_{-n}^2 \end{pmatrix}, \quad k \to -k$$

After folding, the interface is written:

$$\left[\begin{pmatrix} a_n^1 \\ -\tilde{a}_{-n}^1 \end{pmatrix} + \Lambda \begin{pmatrix} \tilde{a}_{-n}^2 \\ -a_n^2 \end{pmatrix}\right] ||I_{1,2}\rangle\rangle = 0$$

Free-boson interfaces preserving $u(1)^2$ (continued)

After some linear algebra:

$$\left[\begin{pmatrix} a_n^1 \\ a_n^2 \end{pmatrix} + \mathcal{O}\begin{pmatrix} \tilde{a}_{-n}^1 \\ \tilde{a}_{-n}^2 \end{pmatrix}\right] ||I_{1,2}\rangle\rangle = 0, \text{ where } \mathcal{O} \in O(2)$$

$$\mathcal{O} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \Leftrightarrow \Lambda = \pm \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix},$$

where $\tanh\alpha = \cos 2\theta$.
The gluing matrices Λ and $\mathcal{O}$ are inverse to each other!
!!! In higher dimensions D (2D bosons) the generalization is obvious; $\Lambda \in O(D,D)$ and $\mathcal{O} \in O(2D)$.

Interface operator

Recall:

$$||B\rangle\rangle = \sum_i N_i \exp\left(-\sum_{n>0}\frac{1}{n}(a^{\mu}_{-n}\tilde{a}^{\nu}_{-n}\mathcal{O}_{\mu\nu})\right)|kN, wM; -kM, wN\rangle$$

Unfold...For $n > 0$:

$$I^{n,\text{bos}}_{1,2} = \sum_i N_i \exp\left(\sum_{n>0}\frac{1}{n}(a^1_{-n}\mathcal{O}_{11}\tilde{a}^1_{-n} - a^1_{-n}\mathcal{O}_{12}a^2_n - \tilde{a}^1_{-n}\mathcal{O}^t_{21}\tilde{a}^2_n + a^2_n\mathcal{O}^t_{22}\tilde{a}^2_n)\right)$$

For $n = 0$:

$$I^{0,\text{bos}}_{1,2} = |kN, wM\rangle\langle kM, wN|$$

Interface operator:

$$I^{\text{bos}}_{1,2} = \prod_{n\geq 0} I^{n,\text{bos}}_{1,2}$$

$\mathcal{N} = 1$-supersymmetric interfaces

Condition:

$$\left(G_r^1 - i\eta_S^1 \tilde{G}_{-r}^1\right) I_{1,2} = \eta I_{1,2} \left(G_r^2 - i\eta_S^2 \tilde{G}_{-r}^2\right)$$

Equivalently:

$$\begin{pmatrix} \psi_r^1 \\ -i\tilde{\psi}_{-r}^1 \end{pmatrix} I_{1,2} = I_{1,2} \eta \Lambda_F \begin{pmatrix} \psi_r^2 \\ -i\tilde{\psi}_{-r}^2 \end{pmatrix},$$

where

$$\Lambda_F = \eta \begin{pmatrix} 1 & 0 \\ 0 & \eta_S^1 \end{pmatrix} \Lambda \begin{pmatrix} 1 & 0 \\ 0 & \eta_S^2 \end{pmatrix} \in O(1,1).$$

Fold CFT_2 to CFT_2^*:

$$\begin{pmatrix} \psi_r^2 \\ \tilde{\psi}_r^2 \end{pmatrix} \mapsto \begin{pmatrix} -i\tilde{\psi}_{-r}^2 \\ i\psi_{-r}^2 \end{pmatrix}$$

$\mathcal{N} = 1$-supersymmetric interfaces (continued)

After folding, the interface is written:

$$\left[\begin{pmatrix} \psi_r^1 \\ \tilde{\psi}_{-r}^1 \end{pmatrix} + i\Lambda_F \begin{pmatrix} \tilde{\psi}_{-r}^2 \\ -\psi_r^2 \end{pmatrix} \right] ||I_{1,2}\rangle\rangle = 0$$

Equivalently:

$$\left[\begin{pmatrix} \psi_r^1 \\ \psi_r^2 \end{pmatrix} + i\mathcal{O}_F \begin{pmatrix} \tilde{\psi}_{-r}^1 \\ \tilde{\psi}_{-r}^2 \end{pmatrix} \right] ||I_{1,2}\rangle\rangle = 0$$

Note: $\Lambda_F \in O(1,1), \mathcal{O}_F \in O(2)$ and are inverse to each other.

Note: Both CFTs in the NS sector or in the R sector!

Interface operator

Recall:

$$||B\rangle\rangle_{\text{NS}} = \exp\left(-i\sum_{r\in\mathbb{N}-\frac{1}{2}} \psi^{\mu}_{-r}\tilde{\psi}^{\nu}_{-r}(\mathcal{O}_F)_{\mu\nu}\right)|0\rangle_{\text{NS}}$$

Unfold...For $r > 0$:

$$I^{r,\text{ferm}}_{1,2} = \exp\left(i\psi^1_{-r}(\mathcal{O}_F)_{11}\tilde{\psi}^1_{-r} - \psi^1_{-r}(\mathcal{O}_F)_{12}\psi^2_r - \tilde{\psi}^1_{-r}(\mathcal{O}_F)^t_{21}\tilde{\psi}^2_r - i\psi^2_r(\mathcal{O}_F)^t_{22}\tilde{\psi}^2_r\right)$$

For $r = 0$:

$$I^{0,\text{NS}}_{1,2} = |0\rangle^1_{\text{NS}}\ {}^2_{\text{NS}}\langle 0|$$

Interface operator:

$$I^{\text{ferm}}_{1,2} = \prod_{r>0} I^{r,\text{ferm}}_{1,2} I^{0,\text{ferm}}_{1,2}$$

Complete interface operator:

$$I^{\text{full}}_{1,2} = I^{\text{bos}}_{1,2} \otimes I^{\text{ferm}}_{1,2}$$

$\mathcal{N} = 2$-supersymmetric interfaces

Work analogously...
Unfold... Bosons:

$$\begin{pmatrix} a_n^{+1} \\ -\tilde{a}_{-n}^{+1} \end{pmatrix} I_{1,2}^{\text{bos}} = I_{1,2}^{\text{bos}} \mathcal{L} \begin{pmatrix} a_n^{+2} \\ -\tilde{a}_{-n}^{+2} \end{pmatrix}$$
$$\begin{pmatrix} a_n^{-1} \\ -\tilde{a}_{-n}^{-1} \end{pmatrix} I_{1,2}^{\text{bos}} = I_{1,2}^{\text{bos}} \mathcal{L}^\dagger \begin{pmatrix} a_n^{-2} \\ -\tilde{a}_{-n}^{-2} \end{pmatrix}$$

Fermions:

$$\begin{pmatrix} \psi_r^{+1} \\ -i\tilde{\psi}_{-r}^{+1} \end{pmatrix} I_{1,2}^{\text{ferm}} = I_{1,2}^{\text{ferm}} \mathcal{L}_{\mathcal{F}} \begin{pmatrix} \psi_r^{+2} \\ -i\tilde{\psi}_{-r}^{+2} \end{pmatrix}$$
$$\begin{pmatrix} \psi_r^{-1} \\ -i\tilde{\psi}_{-r}^{-1} \end{pmatrix} I_{1,2}^{\text{ferm}} = I_{1,2}^{\text{ferm}} \mathcal{L}_{\mathcal{F}}^\dagger \begin{pmatrix} \psi_r^{-2} \\ -i\tilde{\psi}_{-r}^{-2} \end{pmatrix}$$

Note: $\mathcal{L}, \mathcal{L}_{\mathcal{F}} \in U(1,1)$ and $\Omega, \Omega_{\mathcal{F}} \in U(2)$
In higher dimensions: $\mathcal{L}, \mathcal{L}_{\mathcal{F}} \in U(D,D)$ and $\Omega, \Omega_{\mathcal{F}} \in U(2D)$.

Symmetry defects

Conformal interface: $\text{CFT}_1 \neq \text{CFT}_2$
Defect: $\text{CFT}_1 = \text{CFT}_2$
Symmetry defects: Defects preserving symmetries between toroidal CFTs (torus automorphisms)
$\rightarrow$ topological (preserve full $u(1)^{2D}$ symmetry)
Examples:

- Trivial defect ($\theta = \frac{\pi}{4}$: Diagonal D1-brane $\rightarrow$ total transmission)
- $\mathbb{Z}_2$ symmetry
- $\mathbb{Z}_3$ symmetry
- T-duality

The End

Thank you for your attention!