Lessons from CFTs in nontrivial geometries

Based on T.P & A. Stergiou 1806.02340 (PRL) and forthcoming work with A. Stergiou and C. Wen

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Introduction and Motivation

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- E.g. Poincare transformations of flat space are reparametrizations that preserve exactly the norm of vectors. The line element ds^2 is the norm of the vector dx^μ. Under $x^\mu \mapsto x'^\mu(x)$ we have

$$x^\mu \mapsto x'^\mu : ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \mapsto ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu \equiv ds^2$$

$$\Rightarrow \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = \eta_{\rho\sigma}$$

CFTs in nontrivial geometries: generalities

- For conformal transformations we demand

$$x^\mu \mapsto x'^\mu : ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \mapsto ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu \equiv \Omega^2(x) ds^2$$
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- In $d > 2$ $x'^\mu(x)$ are at most quadratic in x^μ.

CFTs in nontrivial geometries: generalities

CFTs are those QFTs that have a number (sometime finite in $d = 2$, certainly infinite in $d > 2$) of quasiprimary local operators $\mathcal{O}(x)$ that under conformal transformations behave as

$$x^\mu \mapsto x'^\mu : \mathcal{O}(x)\Big|_\eta \mapsto \mathcal{O}'(x')\Big|_\eta = \Omega^\Delta(x)\mathcal{O}(x')\Big|_\eta \equiv \Omega^\Delta(x)\mathcal{O}(x)\Big|_{\Omega^2\eta}$$

where Δ is the scaling dimension. We considered scalars for simplicity.

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$$\equiv \Omega^\Delta(x)\langle\mathcal{O}(x)...\rangle\Big|_{\Omega^2\eta} \equiv \langle\mathcal{O}(x)...\rangle\Big|_\eta$$

- The simplest example to see how this works is the 2pt function of scalar field $\phi(x)$ with dimension Δ under the scale transformation $x^\mu \to x'^\mu = \lambda x^\mu$ for which $\Omega = \lambda$. We have

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{x_{12}^{2\Delta}} \to \langle\phi'(x_1')\phi'(x_2')\rangle = \lambda^{2\Delta}\frac{1}{(x_{12}')^{2\Delta}} \equiv \langle\phi(x_1)\phi(x_2)\rangle$$

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- Consider the 2-pt function of scalars in flat space. Going to spherical coordinates $\vec{x} \to (r, \Omega_{d-1})$ with $ds^2 = dr^2 + r^2 d\Omega_{d-1}^2$ one has

$$\langle \mathcal{O}(\vec{x}_1) \mathcal{O}(\vec{x}_2) \rangle = \frac{1}{\vec{x}_{12}^{2\Delta}} \to$$

$$\to \langle \mathcal{O}(r_1, \theta) \mathcal{O}(r_2, 0) \rangle = \frac{1}{(\vec{r}_1 - \vec{r}_2)^{2\Delta}} = \frac{1}{(r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta)^{\Delta}}$$

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- Next, if there was a *conformal transformation* of flat space with metric $\eta_{\mu\nu}$ to the metric $\Omega^2(x)\eta_{\mu\nu}$, then Ward identity would give us

$$[\Omega(r_1, \theta)\Omega(r_2, 0)]^{-\Delta}\langle \mathcal{O}(r_1, \theta)\mathcal{O}(r_2, 0)\rangle\Big|_{\eta} = \langle \mathcal{O}(r_1, \theta)\mathcal{O}(r_2, 0)\rangle\Big|_{\Omega^2\eta}$$

  In other words, knowledge of the 2-pt function in flat space with metric $\eta_{\mu\nu}$ fully determines the 2-pt function on the conformally-flat metric $\Omega^2(x)\eta_{\mu\nu}$.

CFTs in nontrivial geometries: example in $d = 2$

- In $d = 2$ all analytic transformations are conformal transformations since for $ds^2 = dx^2 + dy^2 = dzd\bar{z}\,,\ \ z = x + iy\,,\ \ \bar{z} = x - iy$ and under the general analytic transformations $z \mapsto z' = f(z)$, $\bar{z} \mapsto \bar{z}' = \bar{f}(\bar{z})$ we have

$$ds^2 \mapsto ds'^2 = \partial_z f(z)\partial_{\bar{z}}\bar{f}(\bar{z})dzd\bar{z} \equiv \Omega^2(z,\bar{z})ds^2$$

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- Choosing $f(z) = L\ln(z/L)$ and $\bar{f}(\bar{z}) = L\ln(\bar{z}/L)$ we find

$$ds^2 \mapsto ds'^2 = \frac{L^2}{z\bar{z}}dzd\bar{z} = \frac{L^2}{x^2+y^2}[dx^2 + dy^2] \equiv \frac{L^2}{r^2}[dr^2 + r^2 d\theta^2]$$

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- Now, we observe the the metric shown in the last equality is actually the reparametrization of the metric on $\mathbb{R} \times S^1$ i.e. under

$$r = Le^{\frac{\rho}{L}} \ \Rightarrow\ ds^2 \equiv \frac{L^2}{r^2}[dr^2 + r^2 d\theta^2] = d\rho^2 + L^2 d\theta^2$$

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- So, the Ward identity gives

$$\begin{aligned}\langle \mathcal{O}(r_1,\theta)\mathcal{O}(r_2,0)\rangle\Big|_{\Omega^2\eta} &\equiv \langle \mathcal{O}(\rho_1,\theta)\mathcal{O}(\rho_2,0)\rangle\Big|_{\mathbb{R}\times S^1} \\ &= \frac{1}{L^{2\Delta}}\frac{1}{\left(2\cosh\frac{\rho_1-\rho_2}{L}-2\cos\theta\right)^{\Delta}}\end{aligned}$$

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$$= \frac{1}{L^{2\Delta}}\frac{1}{\left(2\cosh\frac{\rho_1-\rho_2}{L} - 2\cos\theta\right)^{\Delta}}$$

- The metric reparametrization above generalises for all $d > 2$ as

$$r = Le^{\frac{\rho}{L}} \;\Rightarrow\; ds^2 \equiv \frac{L^2}{r^2}[dr^2 + r^2 d\Omega_{d-1}^2] = d\rho^2 + L^2 d\Omega_{d-1}^2$$

However, there is no *conformal* transformation of flat d-dimensional space with $\Omega^2(r,\Omega_{d-1}) = L^2/r^2$, so *CFT correlation functions on $\mathbb{R}\times S^{d-1}$ cannot be determined by those on $\mathbb{R}^d$.*

CFTs in nontrivial geometries: example in $d = 2$

- One possibility is to consider *Weyl invariant QFTs*. These are QFTs that have operators $\mathcal{O}(x)$ with a definite behaviour under general *local Weyl rescalings* of the flat metric

$$\eta_{\mu\nu} \mapsto \Omega^2(x)\eta_{\mu\nu} \;\Rightarrow\; \mathcal{O}(x) \mapsto \mathcal{O}'(x) = \Omega^{\Delta}(x)\mathcal{O}(x)$$

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- The above behaviour is *independent of the spin of the operator* $\mathcal{O}(x)$. This leads to the corresponding Ward identities expressing Weyl invariance of correlation functions as

$$\Omega^{\Delta}(x)\langle\mathcal{O}(x)...\rangle\Big|_{\Omega^2\eta} \equiv \langle\mathcal{O}(x)...\rangle\Big|_{\eta} \tag{1}$$

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- So, for *Weyl invariant theories*

$$\begin{aligned}\langle\mathcal{O}(r_1,\hat{e}_1)\mathcal{O}(r_2,\hat{e}_2)\rangle\Big|_{\Omega^2\eta} &\equiv \langle\mathcal{O}(\rho_1,\hat{e}_1)\mathcal{O}(\rho_2,\hat{e}_2)\rangle\Big|_{\mathbb{R}\times S^{d-1}} \\ &= \frac{1}{L^{2\Delta}}\frac{1}{\left(2\cosh\frac{\rho_1-\rho_2}{L} - 2\hat{e}_1\cdot\hat{e}_2\right)^{\Delta}}\end{aligned}$$

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In two-dimensional CFTs correlators in the *thermal* geometry $\mathbb{R} \times S^1_L$ are fully determined by those on $\mathbb{R}^2$.

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However, the thermal geometries $S^1_L \times \mathbb{R}^{d-1}$, although conformally flat, are not related *neither* by a conformal transformation *nor* by a Weyl rescaling to $\mathbb{R}^d$ for $d > 2$.

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We generally need additional data to describe CFTs in thermal geometries for $d > 2$.

The conformal OPE in nontrivial geometries

The conformal OPE in nontrivial geometries: generalities

- The conformal OPE is the statement that quasiprimary operators form a complete basis for operator products in a CFT i.e. for scalars

$$\phi(x_1)\phi(x_2) = \frac{1}{x_{12}^{2\Delta}}\mathbb{1} + \sum_{\mathcal{O}_s} \frac{1}{x_{12}^{2(\Delta - \frac{\Delta_s}{2} + \frac{s}{2})}}[x_{12} \cdot \mathcal{O}_s(x_2)]$$

where $[x_{12} \cdot \mathcal{O}_s(x_2)]$ denotes the spin-s, dimension-Δ_s contribution with all its descendants.

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- Using the OPE for correlators in nontrivial geometries we could evaluate them <u>if we knew</u> the 1-pt functions $\langle \mathcal{O}_s(x) \rangle$ for the relevant quasiprimary operators.

- Nevertheless, for 1-pt functions we generically have

$$\langle \mathcal{O}(x) \rangle_{\Omega^2 \eta_{\mu\nu}} = [\Omega(x)]^{-\Delta} \langle \mathcal{O}(x) \rangle_{\eta_{\mu\nu}} = 0 \;\text{ for }\; \mathcal{O}(x) \neq \mathbb{1}$$

  and we need to be careful.

The conformal OPE in nontrivial geometries: $d = 2$ example

- Since since *all* 1-pt functions vanish on $\mathbb{R}^d$, namely $\langle \mathcal{O}(x) \rangle \Big|_\eta = 0$ and in $d = 2$ the plane is conformally related to the *thermal* geometry it would appear that *all* 1-pt functions of quasiprimary operators vanish.

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- Nevertheless, in $d = 2$ there exist operators transforming anomalously i.e. the energy momentum tensor

$$T(z) \to T'(z') = [f'(z)]^2 T(z') + \frac{c}{12}\{f(z), z\}\mathbb{1}\,, \quad \{f, z\} = \frac{f''' f' - \frac{3}{2} f''^2}{f'^2}$$

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- Using the above one obtains

$$\langle T(z) \rangle_{\mathbb{R}\times S^1_\beta} = -\frac{c}{24}\frac{1}{L^2}$$

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- We conclude that thermal correlation functions in $d = 2$ do receive contributions from nontrivial 1pt functions of *non-quasiprimary* operators i.e. from conformal anomalies. Setting $u = \rho \cos\phi$ and $L\theta = \rho \sin\phi$ the thermal 2-pt function becomes [J. Cardy (1986)]

$$\langle \phi(\rho, \phi)\phi(0, 0) \rangle = \frac{1}{\rho^{2\Delta_\phi}} \left[1 - \frac{\Delta_\phi}{12}\frac{\rho^2}{L^2} \cos 2\phi + \cdots \right]$$

The conformal OPE in nontrivial geometries: $S^1_\beta \times \mathbb{R}^{d-1}$

- In the $S^1_\beta \times \mathbb{R}^{d-1}$ geometry the 1-pt functions of scalar quasiprimaries can only depend on a single parameter as

$$\langle \mathcal{O}(x) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \langle \mathcal{O}(0) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}$$

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- For $SO(d)$ irreducible tensors we have

$$\langle T_{\mu\nu\ldots}(0) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \frac{b_T}{\beta^{\Delta_T}} (\hat{e}_\mu \hat{e}_\nu \ldots - \text{traces})$$

where $x^\mu = (\tau, \mathbf{x})$ are coordinates on $S^1_\beta \times \mathbb{R}^{d-1}$ with period $\tau \sim \tau + \beta$, $r = |x|$ and $\theta \in [0, \pi]$ is a polar angle when $\mathbb{R}^{d-1}$ is written in spherical coordinates. $\hat{e}_\mu$ are unit vectors in the τ-direction.

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$$\langle T_{\mu\nu\ldots}(0) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \frac{b_T}{\beta^{\Delta_T}} (\hat{e}_\mu \hat{e}_\nu \ldots - \text{traces})$$

where $x^\mu = (\tau, \mathbf{x})$ are coordinates on $S^1_\beta \times \mathbb{R}^{d-1}$ with period $\tau \sim \tau + \beta$, $r = |x|$ and $\theta \in [0, \pi]$ is a polar angle when $\mathbb{R}^{d-1}$ is written in spherical coordinates. $\hat{e}_\mu$ are unit vectors in the τ-direction.

- Then, the thermal two-point function takes the generic form

$$\boxed{\langle \phi(x) \phi(0) \rangle_\beta \equiv g(r, \cos\theta) = \sum_{\mathcal{O}_s} a_{\mathcal{O}_s} \left(\frac{r}{\beta} \right)^{\Delta_{\mathcal{O}_s}} \frac{C_s^\nu(\cos\theta)}{r^{2\Delta_\phi}}}$$

The conformal OPE in nontrivial geometries: $S^1_\beta \times \mathbb{R}^{d-1}$

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  with $C_{\mathcal{O}_s}$ and $g_{\phi\phi\mathcal{O}_s}$ the corresponding 2-pt and 3-pt function coefficients, and $(a)_n$ the Pochhammer symbol.

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  with $C_{\mathcal{O}_s}$ and $g_{\phi\phi\mathcal{O}_s}$ the corresponding 2-pt and 3-pt function coefficients, and $(a)_n$ the Pochhammer symbol.
- The unit operator $\mathbb{1}$ is the unique operator with dimension zero, and here

$$a_{\mathbb{1}} = \frac{2^{2\Delta_\phi - d}\Gamma(\Delta_\phi)}{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2} - \Delta_\phi)}$$

  so that the momentum-space two-point function is unit-normalized.

The conformal OPE in nontrivial geometries: $d = 2$ example

- The thermal 2-pt function gives (there is an extra factor of 2 in the normalization of the $d = 2$ Gegenbauers)

$$a_T = \frac{g_T b_T}{C_T} = -\frac{\Delta_\phi}{12}$$

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- There is yet another coefficient $\tilde{c}$ related to thermal 1-pt functions

$$\langle T_{\tau\tau} \rangle_{\mathbb{R}\times S_\beta^{d-1}} = -(d-1)[f_\beta - f_\infty] = \frac{b_T}{\beta^d} = -2(d-1)\frac{\zeta(d)}{\beta^d}\tilde{c}$$

where f_β is the free energy density. For $d = 2$ we see that $\tilde{c} \sim c$.

The conformal OPE in nontrivial geometries: lesson 2

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For general $d > 2$ we need additional data, in the form of 1-pt functions, to determine the correlators.

Nevertheless, in $d = 2$ it appears as if the thermal correlator is fully determined by the plane result. This is a consequence of the fact that the only nonzero 1-pt functions are those of anomalously transforming operators, and they depend on the central charge. The latter cancels in the OPE.

The OPE inversion formula

The OPE inversion formula: the Euclidean formula

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- Complexifying Δ one defines the spectral function $a(\Delta, s)$ via

$$g(r, \cos\theta) = \sum_s \oint_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{d\Delta}{2\pi i} a(\Delta, s) \frac{C_s^\nu(\cos\theta)}{r^{2\Delta_\phi - \Delta}}$$

whose poles at $\Delta = \Delta_{\mathcal{O}_s}$ with residues $-a_{\mathcal{O}_s}$ yield the physical spectrum.

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  whose poles at $\Delta = \Delta_{\mathcal{O}_s}$ with residues $-a_{\mathcal{O}_s}$ yield the physical spectrum.
- Assuming that the physical poles lie on the right of the imaginary axis one can close the contour clockwise for $r < 1$ (we set $\beta = 1$ from now on) if $a(\Delta, s)$ does not grow exponentially at infinity.

The OPE inversion formula: the Euclidean formula

- We can then use the orthogonality of Gegenbauer polynomials to project on a spin-s state and then integrate with a suitable power in the region of convergence $r \in [0,1]$ to obtain $a(\Delta, s)$ as

$$a(\Delta, s) = \frac{1}{N_{s,\nu}} \int_0^1 \frac{dr}{r^{\Delta - 2\Delta_\phi + 1}} \int_{-1}^1 dx \, (1 - x^2)^{\nu - \frac{1}{2}} \, C_s^\nu(x) g(r, x)$$

where

$$x = \cos\theta \,, \quad N_{s,\nu} = \frac{2^{1-2\nu} \pi \Gamma(s + 2\nu)}{(s+\nu)\Gamma(s+1)\Gamma^2(\nu)}$$

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- This is termed **Euclidean inversion formula.**

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- Writing $x = \cos\theta = (w + 1/w)/2$ with $w = e^{i\theta}$ one can transform the Euclidean formula into a contour integral over the unit circle in the complex-w plane.

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- Writing $x = \cos\theta = (w + 1/w)/2$ with $w = e^{i\theta}$ one can transform the Euclidean formula into a contour integral over the unit circle in the complex-w plane.
- To exploit further the analytic structure of the 2-pt function $g(r, \cos\theta)$ one would like to allow w to explore the full complex plane. This can be done by a suitable complexification of the Euclidean variables r, θ, defining $z = rw$ and $\bar{z} = r/w$ which are now independent real variables.

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- To exploit further the analytic structure of the 2-pt function $g(r, \cos\theta)$ one would like to allow w to explore the full complex plane. This can be done by a suitable complexification of the Euclidean variables r, θ, defining $z = rw$ and $\bar{z} = r/w$ which are now independent real variables.
- As a function of w, i.e. in the w-plane, $g(r, w)$ has the cuts $(-\infty, -1/r)$, $(-r, 0)$, $(0, r)$ and $(1/r, \infty)$. One also has to assume that it does not grow faster than w^{s_0} (resp. $1/w^{s_0}$) for large (resp. small) w for some constant $s_0 > 0$.

The OPE inversion formula: the Lorenzian formula

- Moreover, one needs to use the analytic extension of the Gegenbauer polynomials to the whole complex plane as **[M. Costa et. al. 1209.4355 (JHEP)]**

$$C_s^{\nu}(w) = \frac{\Gamma(s+2\nu)}{\Gamma(\nu)\Gamma(s+\nu+1)}(F_s(1/w)e^{i\nu\pi} + F_s(w)e^{-i\nu\pi})$$

where

$$F_s(w) = w^{s+2\nu}{}_2F_1(s+2\nu, \nu; s+\nu+1; w^2)$$

The OPE inversion formula: the Lorenzian formula

- Then, the integral giving $a(\Delta, s)$ will receive contributions from the discontinuities across the **cuts** of $g(r, w)$ as well as from the **arcs** at infinity. The final result is

$$a(\Delta, s) = a_{\text{Disc}}(\Delta, s) + \theta(s_0 - s)\, a_{\text{arcs}}(\Delta, s)$$

where

$$a_{\text{Disc}}(\Delta, s) = K_s \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^\infty \frac{dz}{z} \Big[(z\bar{z})^{\Delta_\phi - \frac{\Delta}{2} - \nu} \times (z - \bar{z})^{2\nu} F_s\left(\sqrt{\frac{\bar{z}}{z}}\right) \text{Disc}[g(z, \bar{z})] \Big]$$

with

$$K_s = (1 + (-1)^s) \frac{\Gamma(s+1)\Gamma(\nu)}{4\pi\Gamma(s+\nu)}$$

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- The discontinuity relevant for the evaluation of the above integral is the one across the cut $(1/r, \infty)$, as all others are related to it.

Using the Lorentzian inversion formula

Gap equations from the inversion formula: bosonic case

Inversion formulae (i.e. spectral decompositions) are most useful when there is an independent evaluation of the correlation functions.

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- For bosons (scalars) the simplest ansatz is to consider the momentum-space thermal 2-pt function

$$G^{(d)}(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m_{\rm th}^2}\,,\, \omega_n = 2\pi n\,, n = 0, \pm 1, \pm 2, \ldots,$$

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- This is motivated by known work on thermal field theory which shows that fields develop generically a thermal mass $m_{\rm th}$ at finite temperature.
- We are actually asking whether the simple ansatz above can define a thermal CFT. We make no reference to a Lagrangian, although it is known that the 2-pt function can be obtained, for example, in the large-N limit of the $O(N)$ model [T. P. et. al. hep-th/9803149 (PLB)].

Gap equations from the inversion formula: bosonic case

- In arbitrary-d the above 2-pt function can be Fourier-transformed to

$$G^{(d)}(\tau,\mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n=-\infty}^{\infty} \left(\frac{m_{\text{th}}}{|X_n|}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(m_{\text{th}}|X_n|)\,,\ X_n = (\tau - n, \mathbf{x})$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind.

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  where $K_\alpha(x)$ is the modified Bessel function of the second kind.
- Defining $z = \tau + i|\mathbf{x}|$ we find $|X_n| = \sqrt{(n-z)(n-\bar{z})}$.
- We focus on **odd** $d = 2k+1$, $k = 1, 2, \ldots$, and in that case we may write

$$G^{(2k+1)}(\tau,\mathbf{x}) = \frac{1}{2^{k+1}\pi^k} \sum_{n=-\inf ty}^{\infty} \frac{m_{\text{th}}^{k-1}}{|X_n|^k} e^{-m_{\text{th}}|X_n|} \sum_{p=0}^{k-1} \frac{L_{k,p}}{(m_{\text{th}}|X_n|)^p}$$

  with

$$L_{k,p} = \frac{(k-1+p)!}{2^p p!(k-1-p)!}$$

Gap equations from the inversion formula: bosonic case

- The latter coefficients also appear in the Bessel polynomials

$$y_n(x) = \sum_{p=0}^{n} L_{n+1,p}\, x^p = \sqrt{\frac{2}{\pi x}}\, e^{1/x} K_{n+\frac{1}{2}}(1/x)$$

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$$f^{(k)}(x) = \frac{a^{k-1}}{(\sqrt{x})^k}\, e^{-a\sqrt{x}} \sum_{p=0}^{k-1} \frac{L_{k,p}}{(a\sqrt{x})^p}$$

across the cut due to the square-root branch point at $x = 0$.

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- Assuming that the cut goes from $x = 0$ to $x = \infty$ we find that

$$\mathrm{Disc}(f^{(k)}(x)) = \frac{2}{x^{k-1}}\Big(\frac{1}{\sqrt{-x}}\, U_k(x) \cos(a\sqrt{-x}) + V_k(x) \sin(a\sqrt{-x})\Big)$$

$$U_k(x) = \tfrac{1}{2}\big(\theta_{k-1}(\sqrt{x}) + \theta_{k-1}(-\sqrt{x})\big)$$

$$V_k(x) = \frac{1}{2\sqrt{x}}\big(\theta_{k-1}(\sqrt{x}) - \theta_{k-1}(-\sqrt{x})\big)$$

with $\theta_n(x) = x^n y_n(1/x)$ the so-called reverse Bessel polynomials.

Gap equations from the inversion formula: bosonic case

- We can now calculate the spectral function $a(\Delta, s)$. For the discontinuity part we find

$$a^{(k)}_{\mathrm{Disc},0}(\Delta, s) = (1+(-1)^s)\frac{1}{2^{2s+k}s!}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k+s-\frac{1}{2})} \times \sum_{n=0}^{k-1+s} \frac{2^{n+1}}{n!}\frac{(2(k-1+s)-n)!}{(k-1+s-n)!} m_{\mathrm{th}}^n \, \mathrm{Li}_{2k-1+s-n}(e^{-m_{\mathrm{th}}})$$

where $\mathrm{Li}_\alpha(z) = \sum_{n=1}^{\infty} z^n/n^\alpha$ is the polylogarithm.

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- The result above follows just from the leading term in a $\bar{z}$-expansion of the inversion formula. It gives the contributions of higher-spin conserved currents with $\Delta = d - 2 + s$.

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- Subleading terms in the $\bar{z}$-expansion, denoted as $a^{(k)}_{\text{Disc},1}, a^{(k)}_{\text{Disc},2}, \ldots$, would give the contributions of higher-twist operators.

Gap equations from the inversion formula: bosonic case

- The arc part $a^{(d)}_{\text{arcs}}(\Delta, s)$ is nonzero only for $s = 0$. We find

$$a^{(d)}_{\text{arcs}}(\Delta, 0) = \frac{1}{2^{\Delta - \frac{d-5}{2}}\sqrt{\pi}}\, m_{\text{th}}^{\Delta}\,\Gamma\Big(-\frac{\Delta}{2}\Big)\Gamma\Big(-\frac{\Delta-d+2}{2}\Big)$$

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- Notice that for $m_{\text{th}} = 0$ only the $\Delta = 0$ term survives giving the contribution of the identity operator. This, along with the corresponding $m_{\text{th}} = 0$ contributions from $a^{(k)}_{\text{Disc}}(\Delta, s)$, yield the spectrum of generalized free CFTs.

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- When $m_{\text{th}} \neq 0$ and for $\Delta > 0$ the above yields contributions of an infinite tower of scalar operators with $\Delta = 2m$, $m = 1, 2, \ldots$, as well as contributions with $\Delta = d - 2 + 2l$, $l = 0, 1, 2, \ldots$.

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- The former correspond to operators of the form σ^m, $m = 1, 2, \ldots$, where σ is the shadow of ϕ^2.

Gap equations from the inversion formula: bosonic case

- For the latter operators we will first focus on the $l = 0$ case, corresponding to the ϕ^2 operator, which enters the spectrum *both* from the discontinuity part *as well as* from the arc part of the spectral function.

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- This is the so-called **gap equation** and it is here presented for any $d = 2k+1, k = 1, 2, \ldots$.

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- The subleading terms in the give higher poles at $\Delta = d - 2 + 2l, l = 1, 2, \ldots,$. These correspond to scalar operators of the form $\phi \partial^{2l} \phi$.

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- Such operators also arise from subleading terms in the $\bar{z}$ expansion of of the discontinuity parts of the spectral function., namely from $a^{(k)}_{\mathrm{Disc},1}, a^{(k)}_{\mathrm{Disc},2}, \ldots$. These operators **should also disappear from the spectrum** when the gap equation is satisfied.

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- Although we have verified this in a couple of cases, we do not have a general proof as yet.

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- This exactly reproduces the correct normalization of the identity operator (in our conventions).

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- It is also possible to study finite-temperature fermionic 2-pt functions using the inversion formula. The simplest case to consider is the singlet projection of the two-point functions of Dirac fermions $\psi_i(x)$, $\bar{\psi}_i(x)$ in odd dimensions,

$$\langle \psi_i(x)\bar{\psi}_i(0)\rangle_\beta \equiv \tilde{g}(r,\cos\theta) = \sum_{\tilde{\mathcal{O}}_s \neq \mathbb{1}} \tilde{a}_{\tilde{\mathcal{O}}_s} \left(\frac{r}{\beta}\right)^{\Delta_{\tilde{\mathcal{O}}_s}} \frac{C_s^\nu(\cos\theta)}{r^{2\Delta_\psi}}$$

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- The corresponding unit-normalized momentum-space 2-pt function is

$$\tilde{G}^{(d)}(\omega_n,\mathbf{p}) = \frac{\tilde{m}_{\text{th}}}{\omega_n^2 + \mathbf{p}^2 + \tilde{m}_{\text{th}}^2}$$

where the fermionic Matsubara frequencies are $\omega_n = 2\pi(n+1/2)$, $n = 0, \pm 1, \pm 2, \ldots$.

Gap equations from the inversion formula

- The fermionic propagator vanishes for $\tilde{m}_{\text{th}} = 0$ so we will only consider $\tilde{m}_{\text{th}} \neq 0$ in the fermionic case from now on. The calculations follow closely the bosonic case e.g. it is known that fermionic Matsubara sums reduce to a linear combination of bosonic ones.

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- This gives operators of dimension $\Delta = 2m+1$ and $\Delta = d-1+2m$, $m = 0, 1, 2, \ldots$.

Gap equations from the inversion formula: fermionic case

- The former are contributions that do not arise from the discontinuity part, having the form $\tilde{\sigma}^m$ with $\tilde{\sigma}$ the shadow field of $\bar{\psi}\psi$. Note that, as expected, there is no contribution from the unit operator.

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$$\boxed{\sum_{n=0}^{k-1} \frac{2^{n+1}}{n!} \frac{(2(k-1)-n)!}{(k-1-n)!} \tilde{m}_{\text{th}}^{n+1} \, \text{Li}_{2k-1-n}(-e^{-\tilde{m}_{\text{th}}}) = -\frac{1}{2\sqrt{\pi}} \tilde{m}_{\text{th}}^{2k} \Gamma(-k+\tfrac{1}{2})}$$

Gap equations from the inversion formula: lesson 3

The Lorentzian inversion formula together with an ansatz for the form of the thermal 2-t function can be used to *bootstrap* bosonic and fermionic CFTs in arbitrary odd-d dimensions.

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The Lorentzian inversion formula together with an ansatz for the form of the thermal 2-t function can be used to *bootstrap* bosonic and fermionic CFTs in arbitrary odd-d dimensions.

The nontrivial dynamics corresponds to a rearrangement of the operator spectrum. The gap equation arises as the condition that certain classes of operators drop out from the spectrum of the nontrivial CFT.

The resulting picture for the operator spectrum corresponds to the well-known large-N CFTs that arise from a generalised Hubbard-Stratonovich transformation (see later).

Further lessons from the gap equations

Further lessons from the gap equation: solutions

- The bosonic gap equation in $d = 3$ reads

$$-m_{\text{th}} = 2\log(1 - e^{-m_{\text{th}}})$$

with the well-known solution (related to the "golden mean")

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- In $d = 5$ the bosonic gap equation becomes

$$-\tfrac{1}{6}m_{\text{th}}^3 = \text{Li}_3(e^{-m_{\text{th}}}) + m_{\text{th}}\,\text{Li}_2(e^{-m_{\text{th}}})$$

This has a complex conjugate pair of solutions given numerically by

$$m_{\text{th}}^{(d=5)} \approx 1.17431 \pm 1.19808\,i$$

Further lessons from the gap equation: solutions

- In fact, we find that for $\underline{d = 3, 7, 11, \ldots}$ the bosonic gap equation has a **unique real solution for $m_{\mathbf{th}}$** and complex solutions that come in conjugate pairs - except for $d = 3$ where there are no complex solutions. i.e. in $d = 7$ we find a real and a pair of complex conjugate solutions.

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- For $\underline{d = 5, 9, 13, \ldots}$ **we do not find any real solutions**, and the gap equation only has pairs of complex conjugate solutions. I.e. $d = 5$ we only find the solutions above, while in $d = 9$ we find four complex conjugate pairs of solutions. Notice also that $m_{\mathbf{th}} = 0$ **is never a solution of the bosonic gap equations.**

Further lessons from the gap equation: solutions

- The fermionic gap equations in $d = 3, 5$ are given respectively by

$$-\tilde{m}_{\text{th}}^2 = 2\tilde{m}_{\text{th}} \log(1 + e^{-\tilde{m}_{\text{th}}}) \, ,$$
$$-\tfrac{1}{6}\tilde{m}_{\text{th}}^4 = \tilde{m}_{\text{th}} \, \text{Li}_3(-e^{-\tilde{m}_{\text{th}}}) + \tilde{m}_{\text{th}}^2 \, \text{Li}_2(-e^{-\tilde{m}_{\text{th}}})$$

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- For $d = 3$ and $\tilde{m}_{\text{th}} \neq 0$ The fermionic gap equation has only a pair of complex conjugate imaginary solutions $\tilde{m}_{\text{th}}^{(d=3)} = \pm 2\pi i/3$. For $d = 5$ it has a pair of opposite real solutions, as well as a pair of complex conjugate imaginary ones which can be found numerically.

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- This pattern continues to higher dimensions, namely for $\underline{d = 7, 11, 15, \ldots}$ **there is no real solution to the corresponding fermionic gap equation**, while for $\underline{d = 9, 13, 17, \ldots}$ **there is always a pair of opposite real solutions and an increasing number of complex conjugate ones.**

Further lessons from the gap equation: the effective potential

The above pattern for the solutions of bosonic and fermionic gap equations for all *odd-d* fits nicely with a renormalization-group understanding of universality classes of scalars and fermions in general dimensions.

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- In the bosonic case the standard lore is that the large-N universality class for scalars in $d = 2k + 1$, $k = 1, 2, \ldots$, is accessible via the ε expansion starting from $d = 2k + 2$.
- The Hubbard–Stratonovich transformation introduces a field σ via the classically marginal interaction $\sigma\phi^2$. σ has dimension $\Delta_\sigma = 2$ in all d, and the scalars ϕ can be integrated out resulting in an effective potential of the general form

$$V_{\text{eff}}(\sigma) \sim \mathrm{Tr}_d \log(-\partial^2 + \sigma) + g_* \sigma^{\frac{d}{2}} + \cdots$$

with g_* some critical dimensionless coupling.

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- Performing the $\mathrm{Tr}_d \log$ calculation in $d - \varepsilon$ one finds that for $\underline{d = 4, 8, 12, \ldots}$ there is a resulting contribution of the form $\sigma^{\frac{d}{2}} \log \sigma^2$, which is positive and dominates for large σ. Thus, besides various possible local minima, the effective potential has a **global minimum**.

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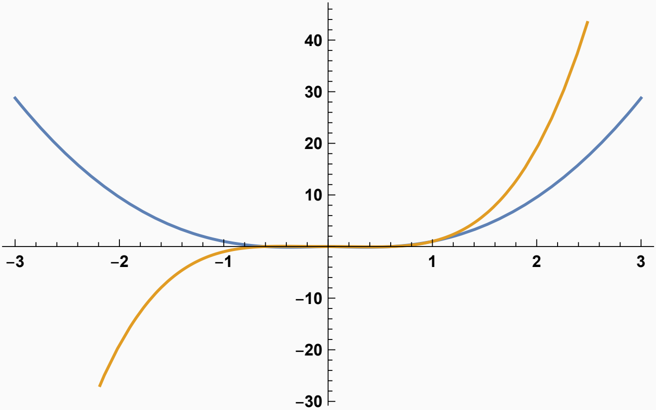
- On the other hand, for $\underline{d = 6, 10, 14, \ldots}$ the term $\sigma^{\frac{d}{2}}$ leads to an **unbounded potential,** and hence to the absence of a global minimum, regardless of the sign of the $\mathrm{Tr}_d \log$ contribution. This matches exactly the pattern we see for m_{th}: **a real m_{th} implies a global minimum, while a complex m_{th} signals unstable local extrema with nonzero decay width.**

Further lessons from the gap equation: the effective potential



Further lessons from the gap equation: the effective potential

- In the fermionic case our results are consistent with large-N universality classes in $d = 2k + 1$, $k = 1, 2, \ldots$ that are accessible via the ε expansion starting from a generalization of the Gross–Neveu–Yukawa model to $d = 2k + 2$ [P. Zinn-Justin NPB B367 (1991)]

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- The corresponding Hubbard–Stratonovich transformation introduces $\tilde{\sigma}$ via the classically marginal interaction $\tilde{\sigma}\bar{\psi}\psi$. Here $\tilde{\sigma}$ has dimension $\Delta_{\tilde{\sigma}} = 1$ in all d, and one gets an effective potential of the form **the** $\mathrm{Tr}_d \log$ **term enters with the opposite sign**

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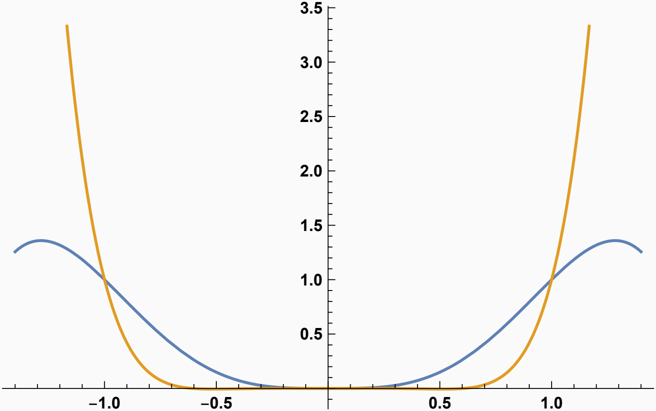
$$V_{\mathrm{eff}}(\tilde{\sigma}) \sim -\mathrm{Tr}_d \log(\partial\!\!\!/ + \tilde{\sigma}) + \tilde{g}_* \tilde{\sigma}^d + \cdots$$

- $\tilde{\sigma}^d$ gives always a bounded from below contribution (recall d is even). However, the $\mathrm{Tr}_d \log$ term changes sing as $d - \varepsilon$: for $\underline{d = 4, 8, 12, \ldots}$ it gives a negative contribution that dominates at infinity leading to an **unstable vacuum structure**, while for $\underline{d = 6, 10, 14, \ldots}$ it gives a positive contribution that guarantees the **presence of a global minimum**. In either case there can be a number of unstable extrema. This matches the obtained pattern for the $\tilde{m}_{\mathrm{th}}$.

Further lessons from the gap equation: the effective potential



Outlook & bonus material

OPE inversion formulas applied to CFTs in nontrivial geometries reveal crucial dynamical properties of critical systems at the level of the operator spectrum.
The consistency of the lift to the nontrivial geometry requires that CFTs develop thermal masses that solve a gap equation. Remarkably, these thermal masses also encode information about the vacuum structure of CFTs even at zero temperature.

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It will be interesting to apply our method to other known CFTs in all dimensions e.g. thermal $\mathcal{N} = 4$ SYM, ABJM-like models etc.

It will also be interesting to apply our method to holographic CFTs. The holographic thermal 2-pt functions can be calculated using black hole physics. What is the underlying conformal dynamics? What is the difference with the usual large-N CFTs? Can we get universal results for thermal masses or expectation values of conserved currrents from holography?

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The methods above should be adequate to understand the spectrum and dynamics of boundary CFTs. Can we extract useful physical data for such theories i.e. using bootstrap methods perhaps?

Bonus material: from [E. Filothodoros et. al. 1803.05950 (NPB)]

- Recall the bosonic gap equation in $d = 5$

$$-\tfrac{1}{6}\tilde{m}_{\rm th}^3 = \tilde{m}_{\rm th}\,\mathrm{Li}_3(e^{-\tilde{m}_{\rm th}}) + \tilde{m}_{\rm th}^2\,\mathrm{Li}_2(e^{-\tilde{m}_{\rm th}})$$

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- Recall the definitions of the famous **Bloch-Wigner-Ramakrishnan** $D_m(z)$ **functions** introduced by Zagier. These are real-valued complex analytic functions. There relevant ones here are

$$D_1(z) = \Re\ln(1-z) - \frac{1}{2}\ln|z|$$

$$D_3(z) = \Re Li_3(z) - \ln|z|\Re Li_2(z) - \frac{1}{2}\ln^2|z|\Re\ln(1-z) + \frac{1}{12}\ln^3|z|$$

Bonus material: from [E. Filothodoros et. al. 1803.05950 (NPB)]

- The gap equation of $U(1)$ charged scalars in $d = 5$, coupled to Chern-Simons, and at finite temperature is [E. Filothodoros et. al. 1803.05950 (NPB)]

$$-\mathcal{N}_5\beta^3 - D_3(z_*) - \frac{1}{2}\ln^2|z_*|\left(D_1(z_*) - \frac{2\gamma}{3\pi}\right) = 0$$

while the corresponding one in $d = 3$ is

$$\mathcal{N}_3\beta + D_1(z_*) = 0$$

where $\mathcal{N}_3$, $\mathcal{N}_5$, γ are dimensionful parameters related to the coupling of the models. $z_* = e^{-\beta m_* + i\beta\alpha_*}$ with m_* the bosonic thermal mass and α_* the $U(1)$ *imaginary* chemical potential.

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- It is not hard to arrange the various parameters above in order to obtain **exacly** the bosonic gap equation arising from the inversion formula. This implies that there is a particular class of Chern-Simons coupled to matter theories that are universally described by our version of the finite temperature bootstrap. The story generalizes to all odd-d.

Bonus material: from [E. Filothodoros et. al. 1803.05950 (NPB)]

- Consider the conformal L-loop ladder integrals discussed e.g. in [J. M. Drummond 2013]

$$I^{(L)}(x_1, x_2, x_3, x_4) = \frac{x_{23}^{2(L-1)}}{\pi^{2L}} \int \frac{1}{x_{2n_1}^2} \prod_{i=1}^{L-1} \left(\frac{d^4 x_{n_i}}{x_{1n_i}^2 x_{3n_i}^2 x_{n_i n_{i+1}}^2} \right) \frac{d^4 x_{n_L}}{x_{1n_L}^2 x_{3n_L}^2 x_{4n_L}^2}$$

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- Hence, we can write the result as

$$I^{(L)}(x_1,x_2,x_3,x_4) = \frac{1}{x_{14}^2 x_{23}^2} \Phi^{(L)}(v, \frac{v}{u})$$

where the usual conformal ratios are given by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\,.$$

Bonus material: from [E. Filothodoros et. al. 1803.05950 (NPB)]

- Next, due to conformal invariance we can set $x_3 \to \infty$, $x_1 \to 0$ and $x_4 \to 1$ to obtain

  $$I^{(L)}(x_1, x_2, x_3, x_4) \to I^{(L)}(0, x_2, \infty, 1) = \Phi^{(L)}(x_2^2 = r^2, x_{24}^2 = 1+r^2-2r\cos\theta)$$

  where θ is the angle between x_2^μ and the unit vector. As expected the result depends on two real variables which we have taken to be $|x_2| = r$ and θ.

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- The remarkable result of [Ussyukina and Davydychev 1992-93] which was very nicely elucidated in [Broadhurst:1993ib] is

  $$\Phi^{(L)}(r, \cos\theta) = \frac{i}{z - \bar{z}} \sum_{k=0}^{L} \frac{(-1)^k 2^{k+1}(2L-k)!}{k!L!(L-k)!} \ln^k |z| \Im \left[Li_{2L-k}(z) \right]$$

  where $z = re^{i\theta}$, $barz = re^{-i\theta}$.

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- Suppose that we calculate the spectral function of a bosonic propagator in d-dimensions in the presence of a an imaginary chemical potential

$$G^{(d)}(\omega_n, \vec{p}; m_{th}, \alpha) = \frac{1}{(\omega_n - \alpha)^2 + \vec{p}^2 + m_{th}^2}\,, \;\; \omega_n = 2\pi n\,, \;\; n = 0, \pm 1, \pm 2, ..$$

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- It is not hard to calculate the Fourier transform of the above as

$$\begin{aligned} G^{(d)}(\tau, \vec{x}; m_{th}, \alpha) &= \sum_{n=-\infty}^{n=\infty} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} e^{-i(\omega_n - \alpha)\tau - i\vec{p}\cdot\vec{x}} \frac{1}{(\omega_n - \alpha)^2 + \vec{p}^2 + m_{th}^2} \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n=-\infty}^{\infty} e^{i\alpha n} \left(\frac{m_{th}}{|X_n|}\right)^{\nu} K_\nu(m_{th}|X_n|) \end{aligned}$$

where $\nu = \frac{d}{2} - 1$ and $X_n = (\tau - n, \vec{x})$ such that $|X_n|^2 = (n - z)(n - \bar{z})$ with $z = \tau + i\vec{x}$ and $\bar{z} = \tau - i\vec{x}$. $K_\nu(z)$ is the Bessel function of the second kind.

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- AGT-like relation?