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The Quantum Measurement Process and the Quantum Eraser

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Abstract

Στην παρούσα πτυχιακή εργασία θα ασχοληθούμε με την διαδικασία της μέτρησης και μια σειρά πειραμάτων που ονομάζονται Quantum Eraser . Αρχικά θα παρουσιάσουμε τα βασικά θεωρήματα της μη-σχετικιστικής Κβαντομηχανικής και στην συνέχεια θα εισάγουμε τον φορμαλισμό της μήτρας πυκνότητας, η οποία αποτελεί μία εναλλακτική αναπαράσταση της κβαντικής κατάστασης. Έπειτα, υπό τον φορμαλισμό αυτό θα μελετήσουμε την κβαντική διμπλοκή, έννοια που καθίσταται απαραίτητη για την μελέτη των κβαντικών συστημάτων ως ανοιχτά συστήματα. Στην συνέχεια θα μας απασχολήσει το πρόβλημα της μέτρησης στην Κβαντομηχανική. Τέλος θα δείξουμε μέσα από μερικές πειραματικές διατάξεις ¨διπλής σχισμής' πως η μέτρηση εμπεριέχει 2 βασικές πτυχές: Καταστροφή των φαινομένων συμβολής μέσω της ¨ανάγνωσης' της διαδρομής και ξεκάθαρη δυνατότητα να 'χειραγωγίσουμε' το παρελθόν.

Abstract

In this thesis we will concern ourselves with the quantum measurement process and with various experiments called Quantum Eraser. In order to do this, initially we will present the fundamental postulates of non-relativistic Quantum Mechanics and we will introduce the density matrix formalism, which form an alternative representation of quantum state of a system. Under this formalism we will describe the quantum entanglement, a notion which is vital for the examination of quantum systems as open systems. Then, we will deal with the problem of measurement in Quantum Mechanics. Finally we will show, through some thought experiment and experimental realizations of the "double slit experiment" that the quantum measurement has two core ideas: destruction of interference when we obtain the which-path information and definite possibility to "manipulate" the past.

Contents

Part I

Theoretical Background

Chapter 1

The fundamental postulates of Quantum Mechanics

Quantum mechanics is an extremely successful physical theory due to its accurate empirical predictions. This chapter is an introduction to fundamental postulates in Quantum Mechanics, which we will present them not formally.

Postulate 1: A quantum system is described using a Hilbert space $\mathcal{H}$. Often, this Hilbert space $\mathcal{H}$ is assumed to be separable. A pure state of a quantum system is represented by a normalized vector $|\psi\rangle$ in $\mathcal{H}$. State vectors differing only by a phase factor of absolute value 1 represent the same state. In the position representation, where the Hilbert space is the space of square integrable functions of a position vector $|x\rangle$, $\psi(x)$ is called the wave function of the system.

Postulate 2: The time evolution of an isolated quantum system represented by the state vector $|\psi(t)\rangle$ is given by:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

where $\hat{H}$ is the Hamilton operator. This is ***Schrödinger equation.*** This rule is valid in the formulation of quantum mechanics called the Schrödinger picture. There are other, equivalent formulations of the time evolution, especially the Heisenberg picture and the Dirac (interaction) pictures, where time evolution is entirely or partially shifted from the state vector to the operators.

Postulate 3: An observable of a quantum system is represented by a Hermitian operator $\hat{A}$

with real spectrum acting on a dense subspace of $\mathcal{H}$.

Postulate 4: The possible measured values of a measurement of an observable are the spectral values of the corresponding operator $\hat{A}$. In case of a discrete spectrum, these are the eigenvalues a satisfying

$$\hat{A}\,|a\rangle = a\,|a\rangle$$

Postulate 5: Let $|a\rangle$ be a complete set of (generalized) eigenvectors of the self-adjoint operator $\hat{A}$ with spectral values a. Let the quantum system be prepared in a state represented by the state vector $|\psi\rangle$. If a measurement of the observable corresponding to $\hat{A}$ is performed, the probability $p_\psi(a)$ to find the measured value a is given by

$$p_\psi(a) = |\,\langle a|\psi\rangle\,|^2$$

This is Born rule, in a formulation that assumes that all eigenvalues are nondegenerate.

Postulate 6: For successive, non-destructive projective measurements with discrete results, each measurement with measuring value o_k can be regarded as preparation of a new state whose state vector is the corresponding eigenvector $|\psi_k\rangle$, to be used for the calculation of subsequent time evolution and further measurements. This is the ***von Neumann projection postulate***.

Chapter 2

Density Matrix Formalism

2.1 Two basic ideas

2.1.1 Projection Operator

Mathematically, the operator which performs such projection is called a projector. The projector can be written as

$$\hat{P}_j = |j\rangle \langle j|$$

Where $|j\rangle$ is the complete set of eigenfunctions of an arbitrary state vector $|\psi\rangle = \sum_n c_n |n\rangle$. In particular,

$$\hat{P}_j |\psi\rangle = |j\rangle \langle j|\psi\rangle = c_j |j\rangle$$

which is the projection of the vector $|\psi\rangle$ on the basis vector $|j\rangle$. This kind of operators have some important properties, which we are going to prove them formally.

1. $\hat{P}_j$ is hermitian

Proof.

$$\hat{P}_j^\dagger = (|j\rangle \langle j|)^\dagger = \langle j|^\dagger |j\rangle^\dagger = |j\rangle \langle j| = \hat{P}_j$$

□

2. $\hat{P}_j^2 = \hat{P}_j$

Proof.

$$\hat{P}_j^2 = \big(\left|j\right\rangle\left\langle j\right|\big)\big(\left|j\right\rangle\left\langle j\right|\big) = \left|j\right\rangle\left\langle j|j\right\rangle\left\langle j\right| = \left|j\right\rangle\left\langle j\right| = \hat{P}_j$$

□

3. $\hat{P}_j\hat{P}_{j'} = 0$

Proof.

$$\hat{P}_j\hat{P}_{j'} = \big(\left|j\right\rangle\left\langle j\right|\big)\big(\left|j'\right\rangle\left\langle j'\right|\big) = \left|j\right\rangle\left\langle j|j'\right\rangle\left\langle j'\right| = 0$$

□

4. $\sum_j \left|j\right\rangle\left\langle j\right| = 1$

Proof.

$$\big(\sum_j \left|j\right\rangle\left\langle j\right|\big)\left|\psi\right\rangle = \sum_j c_j\left|j\right\rangle = \left|\psi\right\rangle \Rightarrow \sum_j \left|j\right\rangle\left\langle j\right| = 1$$

□

Finally, the projector operator can be useful in calculation of inner product of two arbitrary state vectors $\left|\psi\right\rangle = \sum_j c_j\left|j\right\rangle$ and $\left|\phi\right\rangle = \sum_j d_j\left|j\right\rangle$:

$$\left\langle\phi|\psi\right\rangle = \left\langle\phi\right|1\left|\psi\right\rangle = \left\langle\phi\right|\big(\sum_j \left|j\right\rangle\left\langle j\right|\big)\left|\psi\right\rangle = \sum_j \left\langle\phi|j\right\rangle\left\langle j|\psi\right\rangle = \sum_j \left\langle j|\phi\right\rangle^*\left\langle j|\psi\right\rangle = \sum_j d_j^* c_j$$

2.1.2 Statistical ensemble

Until now for the development of quantum mechanical formalism and his statistical interpretation we have limited in the case of pure quantum ensemble that comprises a collection of quantum systems in the same quantum state $\left|\psi\right\rangle$.

Definition: Statistical ensemble is an idealization consisting of a large number of virtual copies (sometimes infinitely many) of a system, considered all at once, each of which represents a possible state that the real system might be in. In other words, a statistical ensemble is a probability distribution for the state of the system.

As we will see, the particularity of quantum measurement demands an ensemble of identical quantum systems so that with repeated measurements we are able to test the statistical predictions of the theory. This occurs because the state of the system after the measurement is the eigenstate than corresponds in the measured eigenvalue. In the core of this process lies the calculation of mean value from system's observables.

$$\langle \hat{O} \rangle = \langle \psi | O | \psi \rangle$$

Ensembles can be not only pure. There is the possibility to composed from ensembles which one corresponds in another quantum state $|\psi_i\rangle$. This mixed ensemble described from the statistical weight of its parts.

2.2 Density Matrix

In this chapter we will develop an alternative way to describe the quantum state of a system. The Density Matrix is vital if we want to approach compound quantum systems. In a quantum system, the wave function specifies all the physical properties of the system. The wave function provides the maximal information. A system with a wave function description is said to be in a pure state. In contrast sometimes there is incomplete information about a quantum system and this may be more common. Von Neumann and Landau proposed to use the density matrix to describe the state of a quantum system in this situation. It is the basic tool for the study of quantum entanglement, channel capacity in quantum communication and so on.

2.2.1 Definition, Mean Value and Probabilities

We assume the statistical mixture quantum mechanical states: $\{p_1, |\psi_1\rangle\}, \{p_2, |\psi_2\rangle\}, \ldots, \{p_n, |\psi_n\rangle\}$ and the one of the observables of the system $\hat{O}$. So, the expectation value of this observable will be:

$$\langle \hat{O} \rangle = \sum_i p_i \langle \psi_i | \hat{O} | \psi_i \rangle \equiv \sum_i p_i \langle \hat{O} \rangle_i$$

This is the weighted mean of quantum mechanical mean values. We assume that the eigenstates of the operator $\hat{O}$ are the $|j\rangle$ and the eigenequation will be:

$$\hat{O} |j\rangle = o_j |j\rangle$$

So,

$$\langle \hat{O} \rangle = \sum_i P_j o_j = \sum_{i,j} p_i |\langle j|\psi_i\rangle|^2 o_j$$

where $P_j = \sum_i p_i |\langle j|\psi_i\rangle|^2$. Also through the term

$$|\langle j|\psi_i\rangle|^2 = \langle j|\psi_i\rangle \langle j|\psi_i\rangle^*$$

we can expand the above mean value:

$$\langle \hat{O} \rangle = \sum_{i,j} p_i \langle j|\psi_i\rangle \langle \psi_i| o_j |j\rangle = \sum_{i,j} p_i \langle j|\psi_i\rangle \langle \psi_i| \hat{O} |j\rangle = \sum_j \langle j| \big(\sum_i p_i |\psi_i\rangle \langle \psi_i| \big) \hat{O} |j\rangle$$

Definition: For a finite-dimensional function space, the most general density operator is of the form:

$$\hat{\varrho} \equiv \sum_i p_i |\psi_i\rangle \langle \psi_i| \tag{2.1}$$

where the coefficients p_i are non-negative and add up to one, and $|\psi_i\rangle \langle \psi_i|$ is an outer product written in bra-ket notation. This represents a mixed state, with probability p_i that the system is in the pure state $|\psi_i\rangle$.

To finalizing the calculation of the mean value we use the trace of the linear operator with respect to this basis :

$$\langle \hat{O} \rangle = \sum_j \langle j| \hat{\varrho} \hat{O} |j\rangle = tr\big(\hat{\varrho} \hat{O}\big) \tag{2.2}$$

In quantum mechanics, a probability amplitude is a complex number used in describing the behaviour of systems. The modulus squared of this quantity represents a probability or probability density. Probability amplitudes provide a relationship between the wave function (or, more generally, of a quantum state vector) of a system and the results of observations of that system, a link first proposed by Max Born. As we mentioned in Chapter 1 , Postulate 5 the probability $p_\psi(o_j)$ to find the measured value o_j is given by

$$p_\psi(o_j) = |\langle j|\psi\rangle|^2$$

where $|j\rangle$ be a complete set of (generalized) eigenvectors of the self-adjoint operator $\hat{O}$ with spectral values o_j and $|\psi\rangle$ and the quantum system be prepared in a state represented by the state vector $|\psi\rangle$. Through density matrix formalism the above result can be written:

$$p_\psi(o_j) = |\langle j|\psi\rangle|^2 = \langle j|\psi\rangle \langle \psi|j\rangle = \langle j| \hat{\varrho} |j\rangle \tag{2.3}$$

In terms of operator's trace (see Appendix 1) :

$$tr\big(\hat{\varrho}\ \hat{P}_j\big) = \sum_i \langle i|\ \hat{\varrho}\ \hat{P}_j\ |i\rangle = \langle j|\ \hat{\varrho}\ |j\rangle\ \langle j|j\rangle + \sum_{i\neq j} \langle i|\ \hat{\varrho}\ |j\rangle\ \langle j|i\rangle = \langle j|\ \hat{\varrho}\ |j\rangle$$

From equation (2.3) :

$$tr\big(\hat{\varrho}\ \hat{P}_j\big) = p_\psi(o_j) \tag{2.4}$$

2.2.2 Pure and Mixed States

In this chapter ,we shall carefully inspect the definition of the density matrix, and that of the mixed state. The density operator has the same structure like the projection operator, which we discuss in previous chapter. Furthermore, the nature of statistical ensemble of our systems can distinguished in two categories: pure state and mixed state

1. Pure State: This is the case of ordinary quantum mechanics which the state of the system described by a ket vector in Hilbert space. So in a statistical ensemble all the systems are identical in exactly the same state. So, the density matrix can be written:

$$\hat{\varrho} = |\psi\rangle\ \langle\psi| \tag{2.5}$$

We can recognize that the structure of density matrix in pure state is exactly the same with the projection operator but here the projection is on the state vector $|\psi\rangle$. Two essential properties are:

i. $\hat{\varrho}^2 = \hat{\varrho}$

Proof.

$$\hat{\varrho}^2 = \hat{\varrho}\ \hat{\varrho} = |\psi\rangle\ \langle\psi|\psi\rangle\ \langle\psi| = |\psi\rangle\ \langle\psi| = \hat{\varrho}$$

while the wavefunction $|\psi\rangle$ is orthornormal, so $\langle\psi|\psi\rangle = 1$ □

ii. $tr(\hat{\varrho}) = 1$

Proof.

$$tr(\hat{\varrho}) = \sum_j \langle j|\psi\rangle\ \langle\psi|j\rangle = \sum_j \langle\psi|j\rangle\ \langle j|\psi\rangle = \langle\psi|\psi\rangle = 1$$

□

From the two properties we can easily extract that :

$$tr(\hat{\varrho}^2) = 1 \tag{2.6}$$

2. Mixed State: This is the most general case, it characterizes the mixture of states with respect to its statistical properties. A mixed state cannot be described with a single ket vector. Instead, it is described by its associated density matrix, which represents the "state" of an averaged system from this ensemble: a system in this ensemble has $p_i = N_i/N$ probability in state $|\psi_i\rangle$:

$$\hat{\tilde{\varrho}} = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \tag{2.7}$$

i. $tr(\hat{\tilde{\varrho}}) = 1$

Proof.

$$\begin{aligned} tr(\hat{\tilde{\varrho}}) &= \sum_i \sum_j \langle j| \big(p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|\big) |j\rangle = \sum_{i,j} p_i \left\langle j|\psi_i\right\rangle \left\langle\psi_i|j\right\rangle \\ &= \sum_{i,j} p_i \left\langle\psi_i|j\right\rangle \left\langle j|\psi_i\right\rangle = \sum_i p_i \langle\psi_i|\psi_i\rangle = \sum_i p_i = 1 \end{aligned}$$

$$\Rightarrow tr(\hat{\tilde{\varrho}}) = 1 \tag{2.8}$$

□

ii.$tr(\hat{\tilde{\varrho}}^2) < 1$

Proof.

$$\begin{aligned} \hat{\tilde{\varrho}}^2 &= \big(\sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|\big)\big(\sum_k p_k \left|\psi_k\right\rangle \left\langle\psi_k\right|\big) \\ &= \sum_{i,k} p_i p_k \left|\psi_i\right\rangle \left\langle\psi_i|\psi_k\right\rangle \left\langle\psi_k\right| = \sum_{i,k} (p_i p_k \left\langle\psi_i|\psi_k\right\rangle) \left|\psi_i\right\rangle \left\langle\psi_k\right| \end{aligned}$$

$$\Rightarrow \; tr(\hat{\tilde{\varrho}}^2) = \sum_{i,k} (p_i p_k \left\langle\psi_i|\psi_k\right\rangle) tr(\left|\psi_i\right\rangle \left\langle\psi_k\right|) \tag{2.9}$$

Now we assume that : $|\psi_i\rangle = \sum_j c_j \left|j\right\rangle$ and $|\psi_k\rangle = \sum_j d_j \left|j\right\rangle$, so :

$$tr(\left|\psi_i\right\rangle \left\langle\psi_k\right|) = \sum_j \left\langle j|\psi_i\right\rangle \left\langle\psi_k|j\right\rangle = \sum_j \left\langle j|\psi_i\right\rangle \left\langle j|\psi_k\right\rangle^* = \sum_j c_j d_j^* = \left\langle\psi_k|\psi_i\right\rangle$$

And we return to (2.9) :

$$tr(\hat{\tilde{\varrho}}^2) = \sum_{i,k} p_i p_k \langle\psi_i|\psi_k\rangle \langle\psi_k|\psi_i\rangle = \sum_{i,k} p_i p_k |\langle\psi_i|\psi_k\rangle|^2$$

Through the Schwarz inequality $|\langle\psi_i|\psi_k\rangle| \leq ||\psi_i\rangle| \; ||\psi_k\rangle| = 1$ so,

$$tr(\hat{\tilde{\varrho}}^2) \leq \sum_i p_i \sum_k p_k \Rightarrow tr(\hat{\tilde{\varrho}}^2) \leq 1$$

The equality can be in effect, if only our system is in pure state and $i \equiv k$. So, for mixed state :

$$tr(\hat{\tilde{\varrho}}^2) < 1 \tag{2.10}$$

□

The differences between $\hat{\varrho}$ and $\hat{\tilde{\varrho}}$ is an essential point in understanding the nature of quantum systems. Through an example of a two-state system, we will analyse the two cases. Let us consider the example of the polarization state :

$$|\psi\rangle = c_h |h\rangle + c_v |v\rangle \tag{2.11}$$

of a photon, where $|h\rangle$ and $|v\rangle$ are the states of horizontal and vertical polarization, and $|c_h|^2 + |c_v|^2 = 1$. It is the case of pure state and the density matrix corresponding to such a state can be written as

$$\begin{aligned} \hat{\varrho} &= |\psi\rangle \langle\psi| \\ &= |c_h|^2 |h\rangle \langle h| + |c_v|^2 |v\rangle \langle v| + c_h c_v^* |h\rangle \langle v| + c_h^* c_v |v\rangle \langle h| \end{aligned} \tag{2.12}$$

In matrix form, where the eigenstates of the two-dimensional polarization Hilbert space are defined as

$$|h\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |v\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Equation (2.12) may be cast in the form

$$\hat{\varrho} = \begin{pmatrix} c_h \\ c_v \end{pmatrix} \begin{pmatrix} c_h^* & c_v^* \end{pmatrix} = \begin{bmatrix} |c_h|^2 & c_h c_v^* \\ c_h^* c_v & |c_v|^2 \end{bmatrix} \tag{2.13}$$

Equations (2.12) and (2.13) describe an ensemble of N photons which are all in the same state $|\psi\rangle$. Note that the diagonal elements of $\hat{\varrho}$ are the square moduli of the coefficients of the basis vectors $|h\rangle$, $|v\rangle$ and since we know that we must have $|c_h|^2 + |c_v|^2 = 1$ we see here the reason of trace property of density matrix. This property expresses the conservation of probability.

We may also consider the case of a classical statistical ensemble of N photons in which a fraction $N_h/N = |c_h|^2$ of the photons is in the state $|h\rangle$ of horizontal polarization and a fraction of $N_v/N = |c_v|^2$ is in the state $|v\rangle$ of vertical polarization. Such an ensemble can described only by a density operator of the type

$$\hat{\hat{\varrho}} = |c_h|^2 |h\rangle \langle h| + |c_v|^2 |v\rangle \langle v| = \begin{bmatrix} |c_h|^2 & 0 \\ 0 & |c_v|^2 \end{bmatrix} \tag{2.14}$$

where the off-diagonal terms are not present. This is a mixed state or mixture.

Now it is clear that $\hat{\varrho} \neq \hat{\hat{\varrho}}$ but if we try to calculate probabilities p_h and p_v of detection horizontal and vertical polarization, respectively,via equations (2.3) or (2.4) we will see that are in both cases equal $|c_h|^2$ and $|c_v|^2$. The fundamental difference between pure and mixed state lies in the state of the system before the observation. When our system is in a pure state like (2.5), before the measurement, there is a 0% chance that the system is in either $|h\rangle$ either $|v\rangle$, and a 100% chance the system is in the state $|\psi\rangle$. On the other hand in the case of mixed state before measurement the probabilities are $|c_h|^2$ for $|h\rangle$ and $|c_v|^2$ for $|v\rangle$. We will show through another modified example this difference.

We consider the polarization in a rotated basis, for example in the basis $|\nearrow\rangle$, $|\nwarrow\rangle$ where

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle + |v\rangle \big) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.15a}$$

$$|\nwarrow\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle - |v\rangle \big) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.15b}$$

we can obtain the new form of basis vectors $|h\rangle$ and $|v\rangle$

$$|\nearrow\rangle + |\nwarrow\rangle = \frac{2}{\sqrt{2}} |h\rangle \Rightarrow |h\rangle = \frac{1}{\sqrt{2}}\big(|\nearrow\rangle + |\nwarrow\rangle \big) \tag{2.16a}$$

$$|\nearrow\rangle - |\nwarrow\rangle = \frac{2}{\sqrt{2}} |v\rangle \Rightarrow |v\rangle = \frac{1}{\sqrt{2}}\big(|\nearrow\rangle - |\nwarrow\rangle \big) \tag{2.16b}$$

The state vector in the basis rotated by 45° will be

$$|\psi\rangle = \frac{c_h + c_v}{\sqrt{2}} |\nearrow\rangle + \frac{c_h - c_v}{\sqrt{2}} |\nwarrow\rangle \tag{2.17}$$

which corresponds to a density matrix

$$\hat{\varrho}_{\nwarrow\nearrow} = \left(\frac{c_h + c_v}{\sqrt{2}}\left|\nearrow\right\rangle + \frac{c_h - c_v}{\sqrt{2}}\left|\nwarrow\right\rangle\right)\left(\frac{c_h + c_v}{\sqrt{2}}\left|\nearrow\right\rangle + \frac{c_h - c_v}{\sqrt{2}}\left|\nwarrow\right\rangle\right)^*$$

$$= \frac{(c_h + c_v)(c_h + c_v)^*}{\sqrt{2}}\left|\nearrow\right\rangle\left\langle\nearrow\right| + \frac{(c_h - c_v)(c_h - c_v)^*}{\sqrt{2}}\left|\nwarrow\right\rangle\left\langle\nwarrow\right| \Rightarrow$$

$$\begin{aligned}\hat{\varrho}_{\nwarrow\nearrow} = {} & \frac{|c_h + c_v|^2}{2}\left|\nearrow\right\rangle\left\langle\nearrow\right| + \frac{|c_h - c_v|^2}{2}\left|\nwarrow\right\rangle\left\langle\nwarrow\right| \\ & + \frac{(c_h + c_v)(c_h - c_v)^*}{2}\left|\nearrow\right\rangle\left\langle\nwarrow\right| + \frac{(c_h + c_v)^*(c_h - c_v)}{2}\left|\nwarrow\right\rangle\left\langle\nearrow\right|\end{aligned} \tag{2.18}$$

Also we can use the (2.15a) and (2.15b) to express the new density operator in a matrix form:

$$\hat{\varrho}_{\nwarrow\nearrow} = \frac{1}{2}\begin{bmatrix} |\ c_h + c_v\ |^2 & (c_h + c_v)(c_h - c_v)^* \\ (c_h + c_v)^*(c_h - c_v) & |c_h - c_v|^2 \end{bmatrix} \tag{2.19}$$

For the mixed state the equation (2.14) through the transformations (2.16) will be

$$\begin{aligned}\hat{\hat{\varrho}}_{\nwarrow\nearrow} = {} & |c_h|^2\left[\frac{1}{\sqrt{2}}\left(\left|\nearrow\right\rangle + \left|\nwarrow\right\rangle\right)\right]\left[\frac{1}{\sqrt{2}}\left(\left\langle\nearrow\right| + \left\langle\nwarrow\right|\right)\right] + |c_v|^2\left[\frac{1}{\sqrt{2}}\left(\left|\nearrow\right\rangle - \left|\nwarrow\right\rangle\right)\right]\left[\frac{1}{\sqrt{2}}\left(\left\langle\nearrow\right| - \left\langle\nwarrow\right|\right)\right] \\ = {} & \frac{|c_h|^2 + |c_v|^2}{2}\left|\nearrow\right\rangle\left\langle\nearrow\right| + \frac{|c_h|^2 + |c_v|^2}{2}\left|\nwarrow\right\rangle\left\langle\nwarrow\right| + \frac{|c_h|^2 - |c_v|^2}{2}\left[\left|\nearrow\right\rangle\left\langle\nwarrow\right| + \left|\nwarrow\right\rangle\left\langle\nearrow\right|\right]\end{aligned} \tag{2.20}$$

And the corresponding density operator in matrix form will be

$$\hat{\hat{\varrho}}_{\nwarrow\nearrow} = \frac{1}{2}\begin{bmatrix} |c_h|^2 + |c_v|^2 & |c_h|^2 - |c_v|^2 \\ |c_h|^2 - |c_v|^2 & |c_h|^2 + |c_v|^2 \end{bmatrix} \tag{2.21}$$

Now we are ready to calculate the probabilities of detecting polarization along 45° and 135° directions respectively, by the equations (2.3) and (2.18), (2.20).

For pure state:

$$\begin{aligned} p_\psi(45^\circ) &= \left\langle\nearrow\right| \varrho_{\nwarrow\nearrow} \left|\nearrow\right\rangle = \frac{|c_h + c_v|^2}{2} \\ p_\psi(135^\circ) &= \left\langle\nwarrow\right| \varrho_{\nwarrow\nearrow} \left|\nwarrow\right\rangle = \frac{|c_h - c_v|^2}{2} \end{aligned} \tag{2.22}$$

For mixed state

$$\begin{aligned} p_{\psi}^{'}(45^{\circ}) &= \langle\nearrow|\,\hat{\hat{\varrho}}_{\nwarrow\nearrow}\,|\nearrow\rangle = \frac{|c_h|^2+|c_v|^2}{2} \\ p_{\psi}^{'}(135^{\circ}) &= \langle\nwarrow|\,\hat{\hat{\varrho}}_{\nwarrow\nearrow}\,|\nwarrow\rangle = \frac{|c_h|^2-|c_v|^2}{2} \end{aligned} \tag{2.23}$$

These values strongly differ and the reason is the interference effects of quantum mechanics, which can be reflected in the off-diagonal terms of density matrix (2.13). These terms called quantum coherences or coherent terms.

Another more abstract example,for distinguish these types of ensembles, is the following :

We assume the pure state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(\,|\psi_1\rangle + |\psi_2\rangle\,\big)$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are the eigenfunctions of an observable of system called $\hat{A}$. There's not a 50% chance the system is in the state $|\psi_1\rangle$ and a 50% it is in the state $|\psi_2\rangle$. There is a 0% chance that the system is in either of those states, and a 100% chance the system is in the state $|\Psi\rangle$. The point is that these statements are all made before any measurements. It is true that if we measure the observable $\hat{A}$ then there is a 50% chance after collapse the system will end up in the state $|\psi_1\rangle$ or $|\psi_2\rangle$. Nevertheless, we choose another observable, let's say the observable called $\hat{B}$ with eigenvectors $|\phi_1\rangle$ and $|\phi_2\rangle$. We assume that $\hat{A}$, $\hat{B}$ are incompatible observables in the sense that as operators $\big[\hat{A},\hat{B}\big] \neq 0$. The incompatibility means that ψ_1 is not just proportional to $|\phi_1\rangle$, it is a superposition of ϕ_1 and $|\phi_2\rangle$ (the two operators are not simultaneously diagonalized).

At this point, we assume the following transformation of basis

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}\big(\,|\phi_1\rangle + |\phi_2\rangle\,\big) \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}\big(\,|\phi_1\rangle - |\phi_2\rangle\,\big) \end{aligned} \tag{2.24}$$

so the expression in new basis of state vector $|\Psi\rangle$ will be

$$|\Psi\rangle = |\phi_1\rangle$$

and the probability of measuring our system in the state $|\phi_1\rangle$ is equal to 100%.

However, when our system is in a mixed state with 50% chance that the system is in the pure state $|\psi_1\rangle$, and a 50% chance the system is in the pure state $|\psi_2\rangle$, not a superposition,the result is totally different. If the state is $|\psi_1\rangle$, then there is a 50% chance that measuring $\hat{B}$ will collapse the system into the state $|\phi_1\rangle$. Meanwhile, if the state is $|\psi_2\rangle$, we get a 50% chance of finding the system in $|\phi_1\rangle$ after measuring. So the probability of measuring the system in the state $|\phi_1\rangle$ after measuring $\hat{B}$, is

$$(50\%\, being\ in\ |\psi_1\rangle)(50\%\, measuring\ |\phi_1\rangle) + (50\%\, being\ in\ |\psi_2\rangle)(50\%\, measuring\ |\phi_1\rangle) = 50\%$$

So the difference between a mixed state and a pure state lies in the ability of quantum amplitudes to interfere, which you can measure by preparing many copies of the same state and then measuring incompatible observables. A mixture of states describes a situation in which a system really is in one of these two states, and we merely do not know which state this is. On the contrary, when a system is in a superposition of states, it is definitely not in either of these states.

2.2.3 Full and Reduced density matrix

The state of a single, closed quantum system is always described by a state vector $|\psi\rangle$. This is dictated by the basic postulate of quantum mechanics (Postulate 1) : "the state of the particle is represented by a vector $|\psi(t)\rangle$ in a Hilbert space." The density matrix for such a system is $\varrho = |\psi\rangle \langle\psi|$. In this case, the density matrix description and the wave function description are strictly equivalent, and the unphysical overall phase in the wave function description is naturally eliminated in the density matrix description. It contains ALL the physical information about the system. We call the density matrix in this case a **full density matrix**.

A single, closed quantum system may have complicated structures. A quantum system of N 2-level molecules(or qubits) could be viewed as a single quantum system if we want to describe all the details of the whole N qubits. The density matrix for such an N-qubits system is of 2^N X 2^N dimension. Though the state of such a system is always pure, a mixed state description for such a system is possible when one considers only the average properties of the system over a period of time, or over some parameters. For instance if the system is under rapid change in time and only average property of the system over a relatively long period of time is concerned, the state of the system could be described by a density matrix obtained by averaging over the time

$$\hat{\hat{\varrho}} = \int_{t_1}^{t_2} \lambda(t)\hat{\varrho}\; dt \tag{2.25}$$

where $\lambda(t)$ is some probability distribution function. The resulting density matrix describes a mixed state.

The density matrix obtained by tracing out partial degrees of freedom of a compound system is called **reduced density matrix**. We will prove that the process of **partial trace** can give us the reduce density matrix as a pure state or as a mixed state and that depends on the nature of state vector. We assume that

$$|\psi\rangle_1 = \frac{1}{\sqrt{2}}\big(\,|\uparrow\rangle_1 + |\downarrow\rangle_1\,\big) \quad , \quad |\psi\rangle_2 = \frac{1}{\sqrt{2}}\big(\,|\uparrow\rangle_2 + |\downarrow\rangle_2\,\big)$$

are the state vectors that describe system 1 and system 2 respectively. We can imagine the two systems like the spin of two electron electrons. Also we assume that system 1 and system 2 consist a compound system **S** whose the state vector given by

$$\begin{aligned} |\Psi\rangle_{12} &= |\psi\rangle_1 \otimes |\psi\rangle_2 \\ &= \frac{1}{2}\big(\,|\uparrow\rangle_1 |\uparrow\rangle_2 + |\downarrow\rangle_1 |\downarrow\rangle_2 + |\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2\,\big) \end{aligned} \tag{2.26}$$

we note here that the state vector is in **factorized form**. The corresponding density matrix

$$\begin{aligned} \hat{\varrho}_{12} &= \big(\,|\psi\rangle_1 \langle\psi| \otimes |\psi\rangle_2 \langle\psi|\,\big) \\ &= \frac{1}{2}\big(\,|\uparrow\rangle_1 \langle\uparrow| + |\downarrow\rangle_1 \langle\downarrow| + |\uparrow\rangle_1 \langle\downarrow| + |\downarrow\rangle_1 \langle\uparrow|\,\big) \otimes \frac{1}{2}\big(\,|\uparrow\rangle_2 \langle\uparrow| + |\downarrow\rangle_2 \langle\downarrow| + |\uparrow\rangle_2 \langle\downarrow| + |\downarrow\rangle_2 \langle\uparrow|\,\big) \end{aligned} \tag{2.27}$$

If we want to write the density matrix for, say, system 1 irrespectively of system 2, we average $\hat{\varrho}_{12}$ over all possible states of system 2, that is, we perform the *trace* of the density matrix (2.22) with respect to system 2:

$$\begin{aligned} \hat{\varrho}_1 = Tr_2(\hat{\varrho}_{12}) &= \sum_{j=\uparrow,\downarrow} {}_2\langle j|\, \hat{\varrho}_{12}\, |j\rangle_2 \\ =&\frac{1}{2}\big(\,|\uparrow\rangle_1 \langle\uparrow| + |\downarrow\rangle_1 \langle\downarrow| + |\uparrow\rangle_1 \langle\downarrow| + |\downarrow\rangle_1 \langle\uparrow|\,\big) \\ \otimes&\; \frac{1}{2}\sum_{j=\uparrow,\downarrow} {}_2\langle j|\,\big(\,|\uparrow\rangle_2 \langle\uparrow| + |\downarrow\rangle_2 \langle\downarrow| + |\uparrow\rangle_2 \langle\downarrow| + |\downarrow\rangle_2 \langle\uparrow|\,\big)\,|j\rangle_2 \Rightarrow \end{aligned}$$

$$\hat{\varrho}_1 = \frac{1}{2}\big(\left|\uparrow\right\rangle_1 \left\langle\uparrow\right| + \left|\downarrow\right\rangle_1 \left\langle\downarrow\right| + \left|\uparrow\right\rangle_1 \left\langle\downarrow\right| + \left|\downarrow\right\rangle_1 \left\langle\uparrow\right| \big) = \left|\psi\right\rangle_1 \left\langle\psi\right| \tag{2.28}$$

and with exactly the same way,

$$\hat{\varrho}_2 = \frac{1}{2}\big(\left|\uparrow\right\rangle_2 \left\langle\uparrow\right| + \left|\downarrow\right\rangle_2 \left\langle\downarrow\right| + \left|\uparrow\right\rangle_2 \left\langle\downarrow\right| + \left|\downarrow\right\rangle_2 \left\langle\uparrow\right| \big) = \left|\psi\right\rangle_2 \left\langle\psi\right| \tag{2.29}$$

The reduce density matrices have the structure of density operators in pure state. This result establish that, we can describe each one of subsystem without any loss of information about them.

In quantum mechanics, there may be cases in which even though the wave function of the composite system is known, the subsystems do not admit a description in terms of an independent wave function, such that the compound state be a mere product of the states of the subsystems. In such cases, where the total wave function is not factorizable, the concept of density matrix is crucial for a proper description of these subsystems. These cases wil be discussed in the next chapter. Furthermore, we will prove that the produced states of the process of partial trace, in this special case, will be mixed states and describes each one of subsystem.

2.2.4 Time evolution

According to the Postulate 2, time evolution of an isolated quantum system represented by the state vector $\left|\psi(t)\right\rangle$ is given by:

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

where H is the Hamilton operator. We can produce the corresponding equation for bra vector

$$-i\hbar\frac{d}{dt}\left\langle\psi(t)\right| = \hat{H}\left\langle\psi(t)\right|$$

so for pure state we have

$$\begin{aligned}\hat{\varrho} = \sum_i \left|\psi_i\right\rangle\left\langle\psi_i\right| \Rightarrow i\hbar\dot{\hat{\varrho}} &= \sum_i \big(i\hbar\frac{d}{dt}\left|\psi_i\right\rangle\big)\left\langle\psi_i\right| + \sum_i \left|\psi_i\right\rangle\big(i\hbar\frac{d}{dt}\left\langle\psi_i\right|\big) \\ &= \sum_i \hat{H}\left|\psi_i\right\rangle\left\langle\psi_i\right| - \sum_i \left|\psi_i\right\rangle\left\langle\psi_i\right|\hat{H} \Rightarrow\end{aligned}$$

$$\dot{\hat{\varrho}} = \hat{H}\hat{\varrho} - \hat{\varrho}\hat{H} = [\hat{H}, \hat{\varrho}] \tag{2.30}$$

This equation is called Von Neumann Equation. Once again, knowledge of the Hamiltonian guarantees knowledge over the dynamics of our states, where in this case are described by pure state density operators.

Chapter 3

Entanglement - Non Locality

3.1 The continuous EPR problem

Assume we have a one dimensional system $\mathcal{S}$ of two particles which interact during the time interval between t_1 and t_2, after which they no longer interact . The the nature of interaction is not of importance. The first one is in a position x_1 and the second one in a position $x2$. We take into account the momenta

$$\hat{p}_x^{(1)} = -i\hbar\frac{\partial}{\partial x_1} \quad and \quad \hat{p}_x^{(2)} = -i\hbar\frac{\partial}{\partial x_2} \tag{3.1}$$

on S_1 with eigenfunction $\phi_p(x_1)$ and on S_2 with eigenfunction $\psi_p(x_2)$. Let us assume that the compound system is described by the wave function

$$\Psi(x_1, x_2) = \int_{-\infty}^{+\infty} dp\ \psi_p(x_2)\phi_p(x_1) \tag{3.2}$$

Since the two particles no longer interact both wavefunctions are placed in free space and because of that they will have the form of plane waves. In the position representation are

$$\phi_p(x_1) = \frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}px_1} \quad and \quad \psi_p(x_2) = \frac{1}{\sqrt{2\pi}}e^{\frac{-i}{\hbar}(x_2-x_0)p} \tag{3.3}$$

where p is the corresponding eigenvalue of the momentum $\hat{p}_x^{(1)} = -i\hbar\frac{\partial}{\partial x_1}$ and $\hat{p}_x^{(2)} = -i\hbar\frac{\partial}{\partial x_2}$. Also x_0 is a constant. Now we locally measure the momentum on $\mathcal{S}_1$ and suppose that we find the eigenvalue $p^{'}$. Therefore, the state (3.2) reduces to

$$\psi_{p'}(x_2)\phi_{p'}(x_1) \tag{3.4}$$

Then, it is evident that $\mathcal{S}_2$ must be in state $\psi_{p'}(x_2)$ and this result can be predicted with absolutely certainty, without disturbing $\mathcal{S}_2$.

On the other hand if we had chosen to consider another observable pertaining to $\mathcal{S}_1$ say, $\hat{x_1}$ whose eigenfunctions are $\phi_x(x_1)$ and $\hat{x_2}$ with eigenfunctions $\phi_x(x_2)$, the Ψ will be

$$\Psi(x_1, x_2) = \int_{-\infty}^{+\infty} dx\ \psi_x(x_2)\phi_x(x_1) \tag{3.5}$$

Now by applying a Fourier transform on both particles eigenfunctions of momentum before measurement we transfer them to the position eigenfunctions :

$$\begin{aligned} \phi_x(x_1) &= \delta(x_1 - x) \\ \psi_x(x_2) &= \delta(x - x_2 + x_0) \end{aligned} \tag{3.6}$$

we locally measure the position on $\mathcal{S}_1$ and suppose that we find the eigenvalue $x^{'}$. Therefore, the state (3.5) reduces to

$$\psi_{x'}(x_2)\phi_{x'}(x_1) \tag{3.7}$$

Then, it is evident that $\mathcal{S}_2$ must be in state $\psi_{x'}(x_2)$ and this result can be predicted with absolutely certainty, without disturbing $\mathcal{S}_2$.

Conclusions of the two different experiments look incompatible on the basis of the fact that position and momentum observables of particle 2 do not commute. According to the EPR argument, quantum mechanics cannot be complete, i.e. the wave function (3.2) and (3.5) cannot be considered a complete discription of the compound system.

3.2 Description of Entanglement via Density Matrix

When one deals with compound system, quantum mechanics have some extremely puzzling features if observed from a classical point of view. Let us illustrate this with the help of an example. We consider the spin state of a pair of electrons. The Hilbert space $\mathcal{H}$ of the total system will be the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of the two dimensional Hilbert spaces of the two particles and $\{|\uparrow\rangle_1, |\downarrow\rangle_1\}$ and $\{|\uparrow\rangle_2, |\downarrow\rangle_2\}$ the two basis of electron 1 and electron 2, respectively. As we know, the state vector can consist of an arbitrary superposition of the basis vectors. We assume the state

$$|\Psi\rangle_{12} = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\uparrow\rangle_1 |\downarrow\rangle_2 \big) \tag{3.8}$$

How can we interpret such a state? It is clearly a superposition state of a particular type. In each of the terms in the (3.8) the two electrons are in orthogonal states. For example, if we measure the spin of the electron 1 and find it to be up, we know with certainty that the spin of the electron 2 will be down, and *vice versa*. However, it is impossible to assign a certain state either to electron 1 or to electron 2. So we cannot write the state $|\Psi\rangle_{12}$ in the **factorized form**

$$|\Psi\rangle_{12} = |\psi\rangle_1 \otimes |\psi\rangle_2$$

This means that the two electrons (through their spin degree of freedom) constitute an inseparable whole and cannot be considered as separate systems.

Definition : Two system $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be entangled with respect to a certain degree of freedom if their total state $|\Psi\rangle_{12}$, relative to that degree of freedom, cannot be written in a factorized form as a product $|\psi\rangle_1 \otimes |\psi\rangle_2$

In other words, an entangled state is a state of a compound system whose subsystems are not probabilistically independent.

We are coming back to the example of electrons spin, the pure state vector of compound system $\boldsymbol{S}$ is

$$|\Psi\rangle_{12} = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 \big)$$

which is an entangled state. The corresponding density matrix of system $\boldsymbol{S}$ is the following

$$\begin{aligned} \hat{\varrho}_{12} &= |\Psi\rangle_{12} \langle\Psi| \\ &= \frac{1}{2}\big(|\uparrow\rangle_1 \langle\uparrow| \otimes |\downarrow\rangle_2 \langle\downarrow| \big) + |\downarrow\rangle_1 \langle\downarrow| \otimes |\uparrow\rangle_2 \langle\uparrow| \\ &\quad + |\uparrow\rangle_1 \langle\downarrow| \otimes |\downarrow\rangle_2 \langle\uparrow| + |\downarrow\rangle_1 \langle\uparrow| \otimes |\uparrow\rangle_2 \langle\downarrow| \big) \end{aligned} \tag{3.9}$$

$$\hat{\varrho}_1 = Tr_2(\hat{\varrho}_{21}) = \sum_{j=\uparrow,\downarrow} {}_2\langle j| \hat{\varrho}_{21} |j\rangle_2 = \frac{1}{2}\big(|\uparrow\rangle_1 \langle\uparrow| + |\downarrow\rangle_1 \langle\downarrow| \big) \tag{3.10}$$

In similar way,

$$\hat{\varrho}_2 = Tr_1(\hat{\varrho}_{21}) = \sum_{j=\uparrow,\downarrow} {}_1\langle j| \, \hat{\varrho}_{21} \, |j\rangle_1 = \frac{1}{2}\big(|\uparrow\rangle_2 \langle\uparrow| + |\downarrow\rangle_2 \langle\downarrow| \big) \tag{3.11}$$

From equation (2.7) it is clear that density matrices (3.10) and (3.11) are mixed states and describe ensembles of electrons, half of which have spin up and the other half spin down. In this case there is a loss of information, when we want to describe the subsystems of the whole. This is a totally different outcome from the case of factorized state vector. So in conclusion the separation of the subsystems is only virtual in the case of entangled states.

Part II

Introduction to Quantum Measurement

Chapter 4

Quantum Measurement problem

In this part we shall consider one of the most difficult problems of quantum mechanics, that is, how it is possible that we have a dynamics ruled by a reversible equation but we obtain random irreversible events when measuring. Many different interpretations have been provided for solving this puzzle. We will discuss here the von Neumann's projection postulate, one of the first attempts to address the measurement problem and its implications. The role played by the apparatus and the environment in the concept of measurement is then emphasized.

4.1 Statement of the problem

"Quantum mechanics attempts to describe not where the next particle will land on the detection screen" but statistical features of the pattern formed when many identically-prepared particles are directed at the screen. And that it manages to do impressively well. It was for this reason that Einstein (and Schrödinger too; also DeBroglie) considered quantum mechanics to 'nice so far as it goes, but obviously incomplete.' How, in such an abstract place as $\mathcal{H}$ do we secure ground to stand on? To what do we tie the thread that anchors us in experienced reality? Consider the corresponding classical question: how do we gain knowledge of the coordinates (q, p) the serve to describe the momentary state of a classical system? The answer, of course, is **"by direct observation, by measurement."** The situation in quantum mechanics is precisely the same, but with a difference: in classical physics the question is seldom posed/answered because measurement is considered classically to be conceptually straightforward, whatever may be the practical difficulties in particular cases. In quantum mechanics, on the other hand, the "measurement problem" is not at all trivial: it is central to the theory, is the source of much that is most characteristic of quantum physics and of conceptual issues that still after all these years remain profoundly surprising, sometimes baffling. But in its mathematical essentials of the quantum theory of measurement is quite simple .

Suppose a system in a superposition pure state state before measuring

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle \tag{4.1}$$

where $|\psi_n\rangle$ the eigenvector of an system's observable $\hat{O}$ and c_n the corresponding complex coefficients. As a result of measurement we obtain the component $|\psi_k\rangle$ of the initial state.

$$|\psi\rangle \xrightarrow{\text{measurement}} |\psi_k\rangle \tag{4.2}$$

There in no way to obtain this result from a superposition state through a unitary evolution.

Proof. The ordinary time evolution in quantum mechanics is

$$\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t} \tag{4.3}$$

where $\hat{H}$ is the Hamiltonian of the system with

$$\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle \tag{4.4}$$

So the time evolution of state (4.1) will be given by

$$\hat{U} |\psi\rangle = e^{-\frac{i}{\hbar}\hat{H}t} \sum_n c_n |\psi_n\rangle = \sum_n c_n e^{-\frac{i}{\hbar}E_n} |\psi_n\rangle$$

but the term $c_n e^{-\frac{i}{\hbar}E_n} \neq 0 \quad , \forall\ c_n \neq 0$. So, if for at least two eigenstates ψ_n the coefficients are $c_n \neq 0$ (superposition) the unitary evolution cannot evolve to a component $|\psi_k\rangle$. □

This is the key point in measurement problem, how can we harmonize the unitary time evolution with the experimental evidence of measurement?

An alternative description of the problem can be given by the density matrix formalism . The density matrix corresponding to the initial pure state (4.1) is

$$\hat{\varrho} = |\psi\rangle \langle\psi| = |\psi\rangle = \sum_n |c_n|^2 |\psi_n\rangle \langle\psi_n| + \sum_{n\neq m} c_n c_m^* |\psi_n\rangle \langle\psi_m| \tag{4.5}$$

We remind that after measurement the state of the system, as we assume above, is the eigenstate $|\psi_k\rangle$, but if we don't actually look at the measurement result, it could be anything: the new state could be in any one of the many eigenstates with certain probabilities. So, it is clear that the latter sum which corresponds in *off-diagonal* (coherent) term has to be zero after measurement. That means that the form of the after measurement density matrix will be a statistical mixture of eigenistates of the measurement observable.

$$\hat{\varrho} \xrightarrow{\text{measurement}} \hat{\tilde{\varrho}} \tag{4.6}$$

where $\hat{\tilde{\varrho}} = \sum_n |c_n|^2 |\psi_n\rangle \langle\psi_n|$. This requirement can be satisfied if we project our initial state on the measurement outcomes and then sum on all the possible outcomes, so we have

$$\hat{\tilde{\varrho}} = \sum_n \hat{P}_n \; \hat{\varrho} \; \hat{P}_n \tag{4.7}$$

This expression is known as *Lüders mixture*

4.2 Von Neumann - Projective Measurement

Bohr and the supporters of the Copenhagen interpretation gave no clue on how to frame the measurement problem in terms of the quantum-mechanical formalism. This is due to the fact that Bohr thought that quantum theory can only be accounted for in terms of classical, macroscopic experience. We will discuss in the next section what impact has this statement concerning the nature of apparatus. However, Von Neumann was the first who try to formalize the problem of measurement assuming that there are two different types of evolution in a quantum system. The first is usual unitary time evolution, whereas the other is characterized from quantum measurement, which presents the following features: it is a **discontinuous, non-casual, instantaneous, non-unitary and irreversible** change of state. This abrupt called by Von Neumann *reduction of wave packet* and as we have seen in first chapter he postulates the above statement:

For successive, non-destructive projective measurements with discrete results, each measurement with measuring value o_k can be regarded as preparation of a new state whose state vector is the corresponding eigenvector $|\psi_k\rangle$, to be used for the calculation of subsequent time evolution and further measurements. This is the ***von Neumann projection postulate****.*

Suppose again a system in a superposition pure state which is in the state (4.1) before measuring . According to the projection postulate if that outcome k occur we have:

$$|\psi_k\rangle = \frac{\hat{P}_k |\psi\rangle}{\sqrt{\langle\psi| \hat{P}_k |\psi\rangle}} \tag{4.8}$$

where $\hat{P}_k = |k\rangle \langle k|$ is the projection operator in the state $|k\rangle$

The post measurement density matrix will be

$$\hat{\varrho_k} = \left(\frac{\hat{P}_k |\psi\rangle}{\sqrt{\langle\psi| \hat{P}_k |\psi\rangle}}\right)\left(\frac{\langle\psi| \hat{P}_k^{\dagger}}{\sqrt{\langle\psi| \hat{P}_k |\psi\rangle}}\right) = \frac{\hat{P}_k |\psi\rangle \langle\psi| \hat{P}_k^{\dagger}}{\langle\psi| \hat{P}_k |\psi\rangle} \tag{4.9}$$

If we assume the orthonormal basis $\{|n\rangle\}$, a further expansion of this equation is

$$\langle\psi| \hat{P}_k |\psi\rangle = \sum_n \langle\psi| \hat{P}_k |n\rangle \langle n|\psi\rangle = \sum_i \langle n|\psi\rangle \langle\psi| \hat{P}_k |n\rangle = tr(|\psi\rangle \langle\psi| \hat{P}_k)$$

and through the equation (4.5) the final form of the post-measurement density matrix is

$$\hat{\varrho_k} = \frac{\hat{P}_k \ \hat{\varrho} \ \hat{P}_k^{\dagger}}{tr(\hat{\varrho} \ \hat{P}_k)} = \frac{\hat{P}_k \ \hat{\varrho} \ \hat{P}_k}{tr(\hat{\varrho} \ \hat{P}_k)} \tag{4.10}$$

We can now extract that if we sum on all the possible results of the measurement, then we will have a mixture like equation (4.7). In fact, this is the best answer that we can provide for the post-measurement state, while in the core of the measurement process lies the interrogative selection. So summing on all possible results, we have a complete description of our system after the measurement. From equation (4.10) we have

$$\sum_n \hat{P}_n \ \hat{\varrho} \ \hat{P}_n = \sum_n tr(\hat{\varrho} \ \hat{P}_n)\hat{\varrho_n} = \sum_n |c_n|^2 \hat{\varrho_n} = \sum_n |c_n|^2 |\psi_n\rangle \langle\psi_n| = \hat{\tilde{\varrho}} \tag{4.11}$$

4.3 The role of the apparatus

In order to examine more exactly what happens when measuring, we need to introduce the concept of apparatus in the context of quantum measurement. Strictly speaking, the apparatus $\mathcal{A}$ can be divided into a little measuring device, also called a meter, which interacts directly with $\mathcal{S}$, an amplifier, and a pointer on a graduated scale. The role of the meter is that of establishing a correlation between the system and the apparatus, while the role of the amplifier is that of transforming a "microscopic input" (which is a consequence of the interaction between the system and the meter) into a macroscopic output, i.e. the result that can be read by an observer. In the most simple case, the possible outcomes may be represented as ticks on a reading scale . In the discrete case, we can understand the reading scale as a partition of the space of the possible outcome values. The pointer is some device that can move along this reading scale by associating a given number to a certain outcome result.

Bohr argued that the account of all evidence in quantum mechanics must be expressed in classical terms, since all terms, by which we perceive and experience, have an unambiguous meaning only in the frame of macroscopic ordinary experience, the only one that can unambiguously be called "experience." When we translate this approach to the terminology of the measurement theory, Bohr's requirement can be expressed as postulating the necessity of a classical apparatus. This statement appears untenable if taken literally. It can be shown that the pointer cannot be classical. Assume that the pointer observable $\hat{\mathcal{O}}_{\mathcal{A}}$ is classical. Then, in the unitary measurement coupling $\hat{U}_{\tau}^{\mathcal{SA}} = e^{i\hat{H}_{\mathcal{SA}}\tau}$, $\hat{H}_{\mathcal{SA}}$ commutes with $\hat{O}_{\mathcal{A}}$. In fact, any quantum observable has to commute with any classical variable (c-number). Therefore, $\hat{O}_{\mathcal{A}}$ also commutes with $\hat{U}_{\tau}^{\mathcal{SA}}$ and it must follow that the probability distribution of $\hat{O}_{\mathcal{A}}$ is completely independent from the measured observable. In other words, the apparatus remains uncoupled from the object system. In fact, we have

$$
\begin{aligned}
\left\langle \hat{O}_{\mathcal{A}} \right\rangle_{\tau} &= \langle \Psi_{\mathcal{SA}}(\tau)| \hat{O}_{\mathcal{A}} |\Psi_{\mathcal{SA}}(\tau)\rangle \\
&= \langle \Psi_{\mathcal{SA}}(0)| (\hat{U}_{\tau}^{\mathcal{SA}})^{\dagger} \hat{O}_{\mathcal{A}} \hat{U}_{\tau}^{\mathcal{SA}} |\Psi_{\mathcal{SA}}(0)\rangle \\
&= \langle \Psi_{\mathcal{SA}}(0)| (\hat{U}_{\tau}^{\mathcal{SA}})^{\dagger} \hat{U}_{\tau}^{\mathcal{SA}} \hat{O}_{\mathcal{A}} |\Psi_{\mathcal{SA}}(0)\rangle \\
&= \langle \Psi_{\mathcal{SA}}(0)| \hat{O}_{\mathcal{A}} |\Psi_{\mathcal{SA}}(0)\rangle
\end{aligned}
$$

This result is incompatible with measurement unless $\hat{O}_{\mathcal{S}}$ is trivial (constant).

On the other hand, let us now assume that the apparatus $\mathcal{A}$ has a quantum-mechanical definition in terms of an arbitrary eigenbasis $\{ |a_n\rangle \}$ of the pointer observable $\hat{O}_{\mathcal{A}}$ in Hilbert space $\mathcal{H}_{\mathcal{A}}$. The the generic state of the apparatus is

$$|\mathcal{A}\rangle = \sum_{n} c_{a_n} |a_n\rangle \tag{4.12}$$

The apparatus and the object system must be coupled in such a way that there is a one-to-one correspondence between values o_k of the observable $\hat{O}_{\mathcal{S}}$ and the values a_k that $\mathcal{A}$ registers. One of the most important points to be emphasized here is that the interaction between $\mathcal{S}$ and $\mathcal{A}$ should (and can) be described as a quantummechanical interaction. In general, there will be a unitary operator describing the evolution during the interaction time τ , i.e.

$$\hat{U}_{\tau}^{\mathcal{SA}} = e^{-\frac{i}{\hbar}\int_0^{\tau} dt \hat{H}_{\mathcal{SA}(t)}} \tag{4.13}$$

where $\hat{H}_{\mathcal{SA}}$ is the system-apparatus interaction Hamiltonian. A simple form of this Hamiltonian can be expressed as

$$\hat{H}_{\mathcal{SA}} = \epsilon_{\mathcal{SA}}(t)\hat{O}_{\mathcal{S}} \otimes \hat{O}_{\mathcal{A}} \tag{4.14}$$

where $\epsilon_{\mathcal{SA}}$ is coupling function. We start initial state $t_0 = 0$ when the apparatus and the system are uncoupled

$$|\Psi_{\mathcal{SA}}(0)\rangle = |\psi(0)\rangle \otimes |\mathcal{A}(0)\rangle \tag{4.15}$$

the apparatus will be in the initial state $|a_0\rangle$ and the initial state of the system is superposition state (4.1), so we have

$$\hat{O}_{\mathcal{S}}\hat{O}_{\mathcal{A}} |\Psi_{\mathcal{SA}}(0)\rangle = \sum_n c_n \hat{O}_{\mathcal{S}} |\psi_n\rangle \hat{O}_{\mathcal{A}} |a_0\rangle \tag{4.16}$$

because the states $|\psi_n\rangle$ and $|a_0\rangle$ belong to different Hilbert spaces and are initially unrelated. The interaction between $\mathcal{S}$ and $\mathcal{A}$, we would like to have

$$\begin{aligned} |\Psi_{\mathcal{SA}}(\tau)\rangle &\equiv \hat{U}_{\tau}^{\mathcal{SA}} |\psi(0)\rangle |a_0\rangle \\ &= \sum_n c_n e^{i\phi_n} |\psi_n\rangle |a_n\rangle \end{aligned} \tag{4.17}$$

where $e^{i\phi_n}$ is a phase factor. In fact the state (4.16) displays **a perfect correlation between system and apparatus states.**

We will show through an example how such a correlation may be achieved: we consider here only the meter $\mathcal{M}$ instead of the whole apparatus $\mathcal{A}$ and that both system $\mathcal{S}$ and the meter $\mathcal{M}$ are represented by two-level systems. The interaction is only between $\mathcal{S}$ and $\mathcal{M}$. We choose Pauli matrices, and in particular we consider $\hat{\sigma}_z^{\mathcal{S}}$ for the observable of $\mathcal{S}$ and $\hat{\sigma}_x^{\mathcal{M}}$ for the observable of $\mathcal{M}$.

$$\hat{\sigma}_z^{\mathcal{S}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad and \quad \hat{\sigma}_x^{\mathcal{M}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.18}$$

Also the system $\mathcal{S}$ is initially prepared in a superposition of the two eigenstates of $\hat{\sigma}_z^{\mathcal{S}}$, which are the spin-up and the spin-down state respectively,

$$|\uparrow\rangle_{\mathcal{S}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad and \quad |\downarrow\rangle_{\mathcal{S}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.19}$$

so

$$|\psi\rangle = c_\uparrow |\uparrow\rangle_{\mathcal{S}} + c_\downarrow |\downarrow\rangle_{\mathcal{S}} \tag{4.20}$$

The meter is initially in z spin-down state $|\downarrow\rangle_{\mathcal{M}}$, so that if, after the interaction, the system is in $|\downarrow\rangle_{\mathcal{S}}$, the state of $\mathcal{M}$ remains unchanged. Otherwise, it will become $|\uparrow\rangle_{\mathcal{M}}$. The interaction Hamiltonian (4.13) can be written as

$$\hat{H}_{\mathcal{SM}} = \epsilon_{\mathcal{SM}}(t)(1+\hat{\sigma}_z^{\mathcal{S}})\hat{\sigma}_x^{\mathcal{M}} \tag{4.21}$$

In order to calculate the action of the unitary operator $\hat{U}_\tau^{\mathcal{SM}}$ on the initial state $|\Psi(0)\rangle_{\mathcal{SM}}$, we have to diagonalize $\hat{H}_{\mathcal{SM}}$. The operator $\hat{\sigma}_z^{\mathcal{S}}$ is already diagonal with respect to the basis states (4.18) while,

$$\hat{\sigma}_z^{\mathcal{S}} |\uparrow\rangle_{\mathcal{S}} = |\uparrow\rangle_{\mathcal{S}} \quad and \quad \hat{\sigma}_z^{\mathcal{S}} |\downarrow\rangle_{\mathcal{S}} = -|\downarrow\rangle_{\mathcal{S}} \tag{4.22}$$

On the other hand the eigenvalues of $\hat{\sigma}_x^{\mathcal{M}}$ are ± 1 and the corresponding eigenstates are

$$|\uparrow\rangle_x^{\mathcal{M}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_{\mathcal{M}} + |\downarrow\rangle_{\mathcal{M}}) \quad and \quad |\downarrow\rangle_x^{\mathcal{M}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_{\mathcal{M}} - |\downarrow\rangle_{\mathcal{M}}) \tag{4.23}$$

So in order to write the initial state of meter $|\downarrow\rangle_{\mathcal{M}}$ in diagonalized terms we have

$$|\downarrow\rangle_{\mathcal{M}} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_x^{\mathcal{M}} - |\downarrow\rangle_x^{\mathcal{M}}) \tag{4.24}$$

So, now we are totally ready to calculate the above action of the unitary operator

$$\begin{aligned}|\Psi(\tau)\rangle_{\mathcal{SM}} =& \hat{U}_\tau^{\mathcal{SM}} |\Psi(0)\rangle_{\mathcal{SM}} = e^{-\frac{i}{\hbar}\tau\epsilon_{\mathcal{SM}}(1+\hat{\sigma}_z^{\mathcal{S}})\hat{\sigma}_x^{\mathcal{M}}}\left[\left(c_\uparrow |\uparrow\rangle_{\mathcal{S}} + c_\downarrow |\downarrow\rangle_{\mathcal{S}}\right) |\downarrow\rangle_{\mathcal{M}}\right] \\ =& \frac{1}{\sqrt{2}}\Big(c_\uparrow e^{-\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}} |\uparrow\rangle_{\mathcal{S}} |\uparrow\rangle_x^{\mathcal{M}} + c_\downarrow |\downarrow\rangle_{\mathcal{S}} |\uparrow\rangle_x^{\mathcal{M}} \\ &- c_\downarrow |\downarrow\rangle_{\mathcal{S}} |\downarrow\rangle_x^{\mathcal{M}} - c_\uparrow e^{+\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}} |\uparrow\rangle_{\mathcal{S}} |\downarrow\rangle_x^{\mathcal{M}}\end{aligned}$$

Using the equations (4.22) we can go back to eigenstates $|\uparrow\rangle_{\mathcal{M}}, |\downarrow\rangle_{\mathcal{M}}$

$$|\Psi(\tau)\rangle_{\mathcal{SM}} = \frac{1}{\sqrt{2}}\Bigg[c_\uparrow e^{-\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}}\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}} + |\uparrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}}\Big) - c_\uparrow e^{+\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}}\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}} - |\uparrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}}\Big)$$

$$+\frac{1}{\sqrt{2}}c_\downarrow\Bigg(|\downarrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}} + |\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}}\Bigg) + \frac{1}{\sqrt{2}}c_\downarrow\Bigg(|\downarrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}} - |\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}}\Bigg)\Bigg]$$

$$= \frac{1}{\sqrt{2}}\Bigg[\frac{1}{\sqrt{2}}c_\uparrow|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}}\left[e^{-\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}} - e^{+\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}}\right] + \frac{1}{\sqrt{2}}c_\uparrow|\uparrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}}\left[e^{-\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}} + e^{+\frac{i}{\hbar}2\tau\epsilon_{\mathcal{SM}}}\right]$$

$$+\sqrt{2}c_\downarrow|\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}}\Bigg] \Rightarrow$$

$$\begin{aligned}|\Psi(\tau)\rangle_{\mathcal{SM}} = \frac{1}{\sqrt{2}}\Bigg[&- c_\uparrow|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}}\frac{2i}{\sqrt{2}}sin\Big(\frac{2\tau\epsilon_{\mathcal{SM}}}{\hbar}\Big) + \sqrt{2}c_\downarrow|\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}} \\ &+ c_\uparrow|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}}\frac{2}{\sqrt{2}}cos\Big(\frac{2\tau\epsilon_{\mathcal{SM}}}{\hbar}\Big)\Bigg]\end{aligned} \tag{4.25}$$

Choosing now

$$\frac{2\tau\epsilon_{\mathcal{SM}}}{\hbar} = \frac{\pi}{2} \quad or \quad \tau = \frac{\pi\hbar}{4\epsilon_{\mathcal{SM}}} \tag{4.26}$$

i.e by fine-tuning the interaction time, we finally obtain

$$|\Psi(\tau)\rangle_{\mathcal{SM}} = +c_\downarrow|\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}} - ic_\uparrow|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}} \tag{4.27}$$

Which is the required coupling between the system and the meter. So if the initial state of $\mathcal{S}$ is prepared as a superposition relatively to the measured observable $\hat{O}_{\mathcal{S}}$, then the resulting total state of $\mathcal{S}+\mathcal{A}$ is **entangled**.

This formalism can be easily translated in density-matrix formalism

$$\begin{aligned}\hat{\varrho}_{\mathcal{SM}} = |\Psi(\tau)\rangle_{\mathcal{SM}}\langle\Psi(\tau)| = &\Bigg(+c_\downarrow|\downarrow\rangle_{\mathcal{S}}|\downarrow\rangle_{\mathcal{M}} - ic_\uparrow|\uparrow\rangle_{\mathcal{S}}|\uparrow\rangle_{\mathcal{M}}\Bigg)\Bigg(+c_\downarrow^*\langle\downarrow|_{\mathcal{S}}\langle\downarrow|_{\mathcal{M}} + ic_\uparrow^*\langle\uparrow|_{\mathcal{S}}\langle\uparrow|_{\mathcal{M}}\Bigg) \\ = &|c_\downarrow|^2|\downarrow\rangle_{\mathcal{S}}\langle\downarrow|\otimes|\downarrow\rangle_{\mathcal{M}}\langle\downarrow| + |c_\uparrow|^2|\uparrow\rangle_{\mathcal{S}}\langle\uparrow|\otimes|\uparrow\rangle_{\mathcal{M}}\langle\uparrow| \\ &- ic_\uparrow c_\downarrow^*|\uparrow\rangle_{\mathcal{S}}\langle\downarrow|\otimes|\uparrow\rangle_{\mathcal{M}}\langle\downarrow| + ic_\uparrow^* c_\downarrow|\downarrow\rangle_{\mathcal{S}}\langle\uparrow|\otimes|\downarrow\rangle_{\mathcal{M}}\langle\uparrow|\end{aligned} \tag{4.28}$$

This is definitely a pure-state density matrix with diagonal and off-diagonal terms, but as we saw in the Section 4.1 the final state after measurement is expressed as the mixture

$$\hat{\hat{\varrho}}_{\mathcal{SM}} = |c_\downarrow|^2 \left|\downarrow\right\rangle_{\mathcal{S}} \left\langle\downarrow\right| \otimes \left|\downarrow\right\rangle_{\mathcal{M}} \left\langle\downarrow\right| + |c_\uparrow|^2 \left|\uparrow\right\rangle_{\mathcal{S}} \left\langle\uparrow\right| \otimes \left|\uparrow\right\rangle_{\mathcal{M}} \left\langle\uparrow\right| \tag{4.29}$$

So the "intermediate" state (4.26) and (4.27) can be considered as a first step of measurement process, called ***premeasurement***, where the correlation between system and apparatus is provided. We should keep distinct the preparation, where some selection of a given state occurs, and the premeasurement, which is not selective as far as we expect that the interrogative selection is provided by the successive step of measurement. As we will see in the next chapter these off-diagonal terms cannot be vanished but there is a formal way to extract them from the system $\mathcal{S} + \mathcal{A}$.

Chapter 5

Decoherence

Until now we assume the quantum systems closed,which means that there is no way for interaction between system and environment. An alternative view is to considering the quantum system open and then arises the idea of coupling a quantum system with a large reservoir. In this process we will show that the system loses coherence: the off-diagonal terms of the density matrix (4.28) tend quickly to zero. The dephasing models gave the hint in considering the environment as a possible source of solution of the measurement problem. In fact, environment can be considered as the largest reservoir at our disposal and can be defined as a large system which is permanently coupled to any microsystem and is not controllable by the observer. In other words, we are assuming that part of the initial information contained in the system is lost in the environment, so that we are not able to extract it. Since a system that is interacting with its environment cannot have a state on its own, the only possible description in the standard quantum formalism is by means of its density matrix. The local density matrix, which contains the complete information for the (probabilities of) the results of all measurements that can be performed at this system.

5.1 Zurek's model

Instead of considering the act of measurement as a mere interaction between the apparatus $\mathcal{A}$ and the object system $\mathcal{S}$, Zurek explicitly introduced the environment $\mathcal{E}$ as a third player that is always present when measuring, assuming therefore that quantum systems are essentially open to the environment. This was the first time that environment and open quantum systems had been applied to the measurement process. Zurek's fundamental idea is that, when considering the measurement process, we first write the state vector for $\mathcal{S}+\mathcal{A}+\mathcal{E}$, $|\Psi\rangle_{\mathcal{SAE}}$ and let $\mathcal{S},\mathcal{A}$ and $\mathcal{E}$ interact. Then, we perform a partial trace and obtain the reduced density matrix of $\mathcal{S}+\mathcal{A}$ only. In this way, while the total system $\mathcal{S}+\mathcal{A}+\mathcal{E}$ still remains entangled and is subjected to a unitary evolution according to the Schrödinger equation, the reduced system $\mathcal{S}+\mathcal{A}$ is a mixture relatively to the "point of view" of the apparatus. The measurement process can be schematically divided into two distinct steps followed by the partial trace which completes the process. The initial state (at time t = 0) is for the sake of simplicity a factorized state of $\mathcal{S},\mathcal{E},\mathcal{A}$. Then, at time $t=t_1$, due to the interaction between $\mathcal{S}$ and $\mathcal{A}$, these become entangled (we have a premeasurement). In the time interval $t_1 \leqslant t \leqslant t_2$, also the environment entangles with $\mathcal{S},\mathcal{A}$. At time t_2 the interaction is switched off. Formally,

$$|\Psi_{\mathcal{SAE}}(0)\rangle = |\psi\rangle\,|a_0\rangle\,|\mathcal{E}(0)\rangle \tag{5.1a}$$

$$\mapsto |\Psi_{\mathcal{SAE}}(t=t_1)\rangle = \sum_n \Big[c_n(|\psi_n\rangle\,|a_n\rangle)\Big]\,|\mathcal{E}(t_1)\rangle \tag{5.1b}$$

$$\mapsto |\Psi_{\mathcal{SAE}}(t\geqslant t_2)\rangle = \sum_n c_n(|\psi_n\rangle\,|a_n\rangle)\,|e_n\rangle \tag{5.1c}$$

where $\{|e_n\rangle\}$ is basis environment.

Now we can write trace out the environment by writing the corresponding reduced density matrix

$$\begin{aligned}\hat{\hat{\varrho}}_{\mathcal{SA}} &= Tr_{\mathcal{E}}(\hat{\varrho}_{\mathcal{SAE}}) \\ &= Tr_{\mathcal{E}}\,|\Psi_{\mathcal{SAE}}(\tau)\rangle\,\langle\Psi_{\mathcal{SAE}}(\tau)| \Rightarrow\end{aligned}$$

$$\hat{\hat{\varrho}}_{\mathcal{SA}} = \sum_n |c_n|^2\,|a_n\rangle\,\langle a_n| \otimes |\psi_n\rangle\,\langle\psi_n| \tag{5.2}$$

Equation (5.2) has been obtained under the simplified hypothesis (5.1c). The physical meaning of this tracing out is then the following: the information contained in the off-diagonal elements of the coupled system $\mathcal{S}+\mathcal{A}$ is not destroyed but downloaded in the environment. However, from the point of view of the apparatus, it is inaccessible and is for this reason completely lost (it would be accessible only under the hypothesis that we can exactly reconstruct the state of the whole system $(\mathcal{S}+\mathcal{A}+\mathcal{E})$. Since the quantum coherences are lost (see the end of Sec. 4.3), this interpretation of measurement is known as *decoherence*.

Part III

Quantum Eraser, Delayed - Choice gedanken experiments and their realizations

Chapter 6

Gedanken Experiments

Coplementarity distinguishes the world of quantum phenomena from the realm of classical physics. The lion's share of the credit for teaching us to accept complementarity as a fact and for insisting that we have to learn to live with it belongs to Niels Bohr. In 1927, when he was reviewing the subject at Como 1 in a speech delivered in honour of Count Alessandro Volta (1745-1827), quantum theory as we know it today was still new, and all examples used to illustrate complementarity referred to the position (particle-like) and momentum (wave-like) attributes of a quantum mechanical object, be it a photon or a massive particle. This is the historical reason why complementarity is often superficially identified with the 'wave-particle duality of matter'. Richard Feynman, discussing the two-slit experiment in his admirable introduction to quantum mechanics , notes that this wave-particle dual behaviour contains the basic mystery of quantum mechanics. In fact, he goes so far as to say: **"In reality it contains the only mystery."** Complementarity, however, is a more general concept. We say that two observables are 'complementary' if precise knowledge of one of them implies that all possible outcomes of measuring the other one are equally probable. We will illustrate this idea with the help of an experimental setup called *Quantum eraser*.

6.1 Quantum optical tests of complementarity by Scully , Englert , Walther

Scully, Englert, and Walther proposed for first time in 1991 an conceptual experiment which shows a variant of the complementarity between wave-like and particle-like behaviors. Since

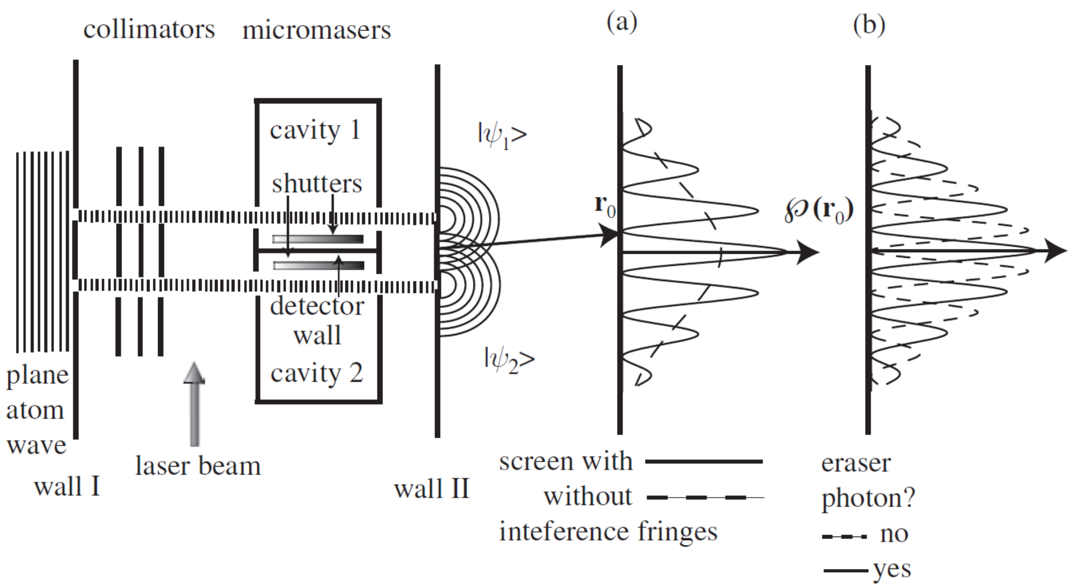

Figure 6.1: Scully, Englert, and Walther's proposed experiment.

then, the so called quantum eraser has been experimentally realized, even though with different setups. In the next section we will analyze one of these setups.

In the original proposal, we have an atomic beam which goes through two slits of wall I (see Figure 6.1 behind this wall there is a further series of slits which are used as collimators to define two atomic beams that reach the narrow slits of wall II where the interference originates. Between wall I and wall II and after the collimators the atomic beams are orthogonally intersected by an intense source of light, for instance by a LASER (Light Amplification by Stimulated Emission of Radiation) beam, which brings the internal state $|i\rangle$ of the two-level atoms from an unexcited (ground) state $|g\rangle$ into an excited state $|e\rangle$. Thereafter, each of the two beams passes through a microcavity. Finally, they fall on a screen. The atomic source is adjusted in such a way that there is at most one atom at a time in the apparatus.

In the interference region, the wave function describing the center-of-mass motion of the atoms is the superposition of the two terms referring to slit 1 and slit 2 ($\mathbf{r}$ indicates the center-of-mass coordinate), so that the total (center-of-mass plus internal) wave function is given by

$$\Psi(\mathbf{r}) = \frac{1}{\sqrt{2}}\left[\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})\right]|i\rangle \tag{6.1}$$

and the probability density of particles falling on the screen at the point $\mathbf{r} = \mathbf{r_0}$ will be given by

$$\wp(\mathbf{r_0}) = \frac{1}{2}\left[|\psi_1(\mathbf{r_0}|^2 + |\psi_2(\mathbf{r_0}|^2 + \psi_1^*(\mathbf{r_0})\psi_2(\mathbf{r_0}) + \psi_1(\mathbf{r_0})\psi_2^*(\mathbf{r_0})\right] \tag{6.2}$$

We note that the usual interference behaviour is represented by the cross-terms

$$\psi_1^*(\mathbf{r_0})\psi_2(\mathbf{r_0})\ , \psi_1(\mathbf{r_0})\psi_2^*(\mathbf{r_0})$$

The cavity frequency is tuned in resonance with the energy difference between the excited and ground states of the atoms. The velocity of the atoms may be selected in such a way that, after being prepared in an excited state by the laser beam, on passing through either one of the cavities each atom will emit a microwave photon (which stays in the cavity) and leave which-path information. After the atom has passed through the cavity it is again in force-free space and its momentum keeps the initial value.

So when atoms pass through the cavities and transit from $|e\rangle$ to $|g\rangle$, the state of the global system (atomic beam plus cavity) is given by

$$|\Psi(\mathbf{r})\rangle = \frac{1}{\sqrt{2}}\left[\psi_1(\mathbf{r})\,|1_1 0_2\rangle + \psi_2(\mathbf{r})\,|0_1 1_2\rangle\,\right]|g\rangle \tag{6.3}$$

where $|1_1 0_2\rangle$ denotes the state in there is one photon in cavity 1 and none in cavity 2 and $|0_1 1_2\rangle$ there is one photon in cavity 2 and none in cavity 1.The system and the detector have become entangled by their interaction. But now there is an important difference in the probability density on the screen

$$\begin{aligned}\wp'(\mathbf{r_0}) =& \frac{1}{2}\left[|\psi_1(\mathbf{r_0}|^2 + |\psi_2(\mathbf{r_0}|^2 + \psi_1^*(\mathbf{r_0})\psi_2(\mathbf{r_0})\,\langle 1_1 0_2|0_1 1_2\rangle\right.\\ &\left.+\,\psi_1(\mathbf{r_0})\psi_2^*(\mathbf{r_0})\,\langle 0_1 1_2|1_1 0_2\rangle\,\right]\langle g|g\rangle\end{aligned}$$

Since the two cavity vectors $|1_1 0_2\rangle$ $|0_1 1_2\rangle$ are orthogonal to each other, the interference terms vanish and diffraction fringes are washed out, so the above equation reduces to

$$\wp'(\mathbf{r_0}) = \frac{1}{2}\left[|\psi_1(\mathbf{r_0}|^2 + |\psi_2(\mathbf{r_0})|^2\right] \tag{6.4}$$

Let us now separate the detectors in the cavity by a shutter–detector combination, so that, when the shutters are closed, the photons are forced to remain either in the upper or in the lower cavity. However, if the shutters are opened, light will be allowed to interact with the photodetector wall and in this way the radiation will be absorbed and the memory of the passage erased (such an operation is called ***quantum erasure***). After the erasure, will we again obtain the interference fringes which were eliminated before? The answer is yes, so that interference effects can be restored by manipulating the which path detectors long after the atoms have passed and before reaching the final (detection) wall.

This result can be formally expressed as follows. Let us include the photodetector walls into the description. These are initially in the ground state $|g\rangle_D$, so that equation (6.3) modiefieds to

$$|\Psi(\mathbf{r})\rangle = \frac{1}{\sqrt{2}}\left[\psi_1(\mathbf{r})\,|1_1 0_2\rangle + \psi_2(\mathbf{r})\,|0_1 1_2\rangle\,\right]|g\rangle_A\,|g\rangle_D \tag{6.5}$$

If we introduce symmetric and antisymmetric atomic states

$$\psi_\pm(\mathbf{r}) = \frac{1}{\sqrt{2}}\left[\psi_1(\mathbf{r} \pm \psi_2(\mathbf{r})\right] \tag{6.6}$$

together with symmetric and antisymmetric states of the radiation fields contained in the cavities

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left[\,|1_1 0_2\rangle \pm |0_1 1_2\rangle\,\right] \tag{6.7}$$

we can rewrite the equation (6.5)

$$|\Psi(\mathbf{r})\rangle = \frac{1}{\sqrt{2}}\left[\psi_+(\mathbf{r})\,|+\rangle + \psi_-(\mathbf{r})\,|-\rangle\,\right]|g\rangle_A\,|g\rangle_D \tag{6.8}$$

The action of the quantum eraser on the system is to change the above equation into

$$\left|\Psi'(\mathbf{r})\right\rangle = \frac{1}{\sqrt{2}}\left[\psi_+(\mathbf{r})\,|0_1 0_2\rangle\,|e\rangle_D + \psi_-(\mathbf{r})\,|-\rangle\,|g\rangle_D\,\right]|g\rangle_A \tag{6.9}$$

The reason is that the interaction Hamiltonian between radiation and photodetectors only depends on symmetric combinations of radiation variables so that the antisymmetric state remains unchanged.

The corresponding density matrix of our final state of the whole system is

$$\begin{aligned}\hat{\varrho} =&\frac{1}{2}\left[\psi_+(\mathbf{r})\psi_+^*(\mathbf{r})\,|0_1 0_2\rangle\,\langle 0_1 0_2|\;\;|e\rangle_D\,\langle e| + \psi_-(\mathbf{r})\psi_-^*(\mathbf{r})\,|-\rangle\,\langle -|\;\;|g\rangle_D\,\langle g|\right.\\ &\left.+\,\psi_+(\mathbf{r})\psi_-^*(\mathbf{r})\,|0_1 0_2\rangle\,\langle -|\;\;|e\rangle_D\,\langle g| + \psi_-(\mathbf{r})\psi_+^*(\mathbf{r})\,|-\rangle\,\langle 0_1 0_2|\;\;|g\rangle_D\,\langle g|\,\right]|g\rangle_a\,\langle g|\end{aligned} \tag{6.10}$$

In this point is important to note that the states $|0_1 0_2\rangle$, $|1_1 0_2\rangle$, $|0_1 1_2\rangle$ are orthogonal. So now we are ready to calculate the probability densities of all possible results. Initially we suppose that we don't know the state of the wall detector, so we sum on all the detector degrees of freedom. Also in order to calculate the probability we have to perform the trace -and- on the atomic and field degrees of freedom . So from equation (2.3) we have

$$\begin{aligned}\wp(\mathbf{r}_0) =&Tr_{A,F,D}\left(\hat{\varrho}\right)\\ =&\frac{1}{2}\left[\psi_+(\mathbf{r})\psi_+^*(\mathbf{r}) + \psi_-(\mathbf{r})\psi_-^*\right] = \frac{1}{2}\left[\psi_1(\mathbf{r})\psi_1^*(\mathbf{r}) + \psi_2(\mathbf{r})\psi_2^*\right]\end{aligned} \tag{6.11}$$

Clearly, equation (6.11) does not show any interference terms. Without knowing anything about the photodetector the photons are possible to remain either in the upper or in the lower cavity, so this gives us the probability to obtain the which-path information and that occurs for the absence of interference. However, if we compute the probability density for finding both the photodetector excited and the atom at $\mathbf{r} = \mathbf{r_0}$ on the screen, from equation (2.4) we have

$$\begin{aligned} \wp_{e_D} =& Tr\big(\varrho \left|e\right\rangle_D \left\langle e\right| \big) \\ =& \ \psi_+(\mathbf{r})\ \psi_+^*(\mathbf{r}) = \frac{1}{2}\big[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 + \psi_1(\mathbf{r})\psi_2^*(\mathbf{r}) + \psi_2(\mathbf{r})\psi_1^*(\mathbf{r})\big] \\ =& \frac{1}{2}\big[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2\big] + Re\big(\psi_1(\mathbf{r})\psi_2^*(\mathbf{r})\big) \end{aligned} \tag{6.12}$$

In similar way, the probability of finding both the photodetector in the ground state and the atom at $\mathbf{r} = \mathbf{r_0}$ on the screen is

$$\begin{aligned} \wp_{g_D} =& Tr\big(\varrho \left|g\right\rangle_D \left\langle g\right| \big) \\ =& \ \psi_-(\mathbf{r})\ \psi_-^*(\mathbf{r}) = \frac{1}{2}\big[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 - \psi_1(\mathbf{r})\psi_2^*(\mathbf{r}) - \psi_2(\mathbf{r})\psi_1^*(\mathbf{r})\big] \\ =& \frac{1}{2}\big[|\psi_1(\mathbf{r})|^2 - |\psi_2(\mathbf{r})|^2\big] - Re\big(\psi_1(\mathbf{r})\psi_2^*(\mathbf{r})\big) \end{aligned} \tag{6.13}$$

which exhibits fringes and antifringes of interference pattern like equation (6.2).

So when we loose the which-path information, we reproduce interference, something that tells us that these two notions are complementary. It is important not to forget that the photon is a quantum object: when the shutters are opened, the two cavities become a single larger one. Now, the photon's wave is a combination of the two partial waves, such that either the two waves reinforce each other (constructive interference) and the photosensory detects the photon, or they mutually extinguish each other, with the consequence that the photosensory detects no photon (destructive interference). In other words, there is only a 50% probability of detecting the photon. The above examination supports the statement that the complementarity principle is genuinely a fundamental principle of quantum mechanics and not a mere consequence of the uncertainty relation.

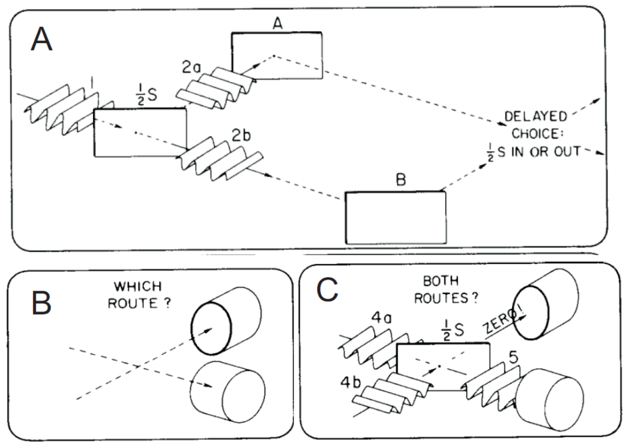

Figure 6.2: Wheeler's delayed-choice gedankenexperiment with a single photon wave packet in a Mach-Zehnder interferometer. **A** The output half-silvered mirror ($\frac{1}{2}s$) of the interferometer can be inserted or removed at will. When it is removed, as in **B**, the detectors allow one to determine which path has been followed by the photon. When it is inserted, as in **C**, detection probabilities of the two detectors depend on the length difference between the two arms. When the difference is zero, only one detector will fire.

6.2 Wheeler delayed - choice wave particle duality gedanken experiment

Wave-particle duality was one of the themes of the debates between Einstein and Bohr. Until today, it is still under discussion among physicists, as well as philosophers, and many interpretations have been proposed. The Copenhagen interpretation stands out from the others due to its consistency and austerity. One of the essential ideas of the Copenhagen interpretation is that the result of an experiment to test complementarity should depend on the experimental configuration, according to Bohr, rephrased by Wheeler: "No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon".

To translate this into a wave-particle duality experiment, imagine a setup as shown in Figure 6.2 , where we have a Mach-Zehnder interferometer and a single-photon wave packet as input. Depending on the choice made by the observer, different properties of the single-photon wave packet can be demonstrated. If the observer chooses to demonstrate the particle nature of the single-photon wave packet, he should remove the output half-silvered mirror ($\frac{1}{2}s$), which acts as a beam splitter, as shown in Figure 6.2 . Both detectors will fire with equal

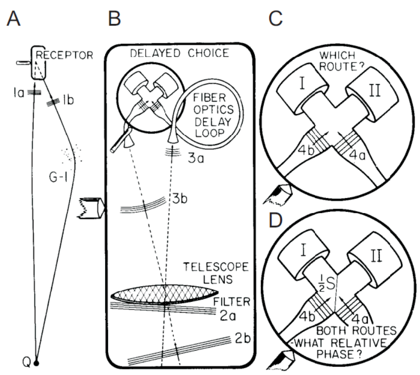

Figure 6.3: Delayed-choice gedankenexperiment at the cosmological scale. **A** Due to the gravitational lens action of galaxy G-1, light generated from a quasar has two possible paths to reach the receptor. This mimics the setup in Figure 2.1. **B**. The receptor setup. Filters are used to increase the coherence length of the light and thus allow us to perform the interference experiment. A fiber optics delay loop is used to adjust the phase of the interferometer. **C, D**. The choices to remove or insert the output half-silvered mirror, allow one to measure which route the light traveled or what the relative phase of the two routes was (thus it traveled both routes). Given the distance between the quasar and the receptor (billions of light years), the choice can be made long after the light's entry into the interferometer, an extreme example of the delayed-choice gedanken experiment.

probabilities but not at the same time. As Wheeler pointed out, "[...] one counter goes off, or the other. Thus the photon has traveled only one route" . On the other hand, if the observer chooses to demonstrate the wave nature of the single-photon wave packet, he inserts the output half-silvered mirror ($\frac{1}{2}s$) as shown in Figure 6.2 C. In this experimental configuration, only the detector on path 5 will fire and the other detector will never fire, if the path lengths of the interferometer are equal. As Wheeler pointed out, "This is evidence of interference . . . that each arriving light quantum has arrived by both routes" .This shows that whether the single-photon wave packet travels one route or both routes depends on whether the output half-silvered mirror is removed or inserted. This is counter-intuitive as it seems that we could change the propagation history of the single-photon wave packet just by changing the setting of the experimental apparatus. Furthermore, in order to rule out some naive interpretations of these phenomena, Wheeler proposed a "delayed-choice" version of this experiment, in which the choice of which property will be observed is made after the photon has passed the first beam splitter. "Thus one decides the photon 'shall have come by one route or by both routes' after it has already done its travel". In Wheeler's delayed-choice gedankenexperiment at the cosmological scale" proposal , he stated "When night comes and the telescope is at last usable we leave the half-silvered mirror out or put it in, according to our choice [...] we discover 'by which route' it came with one arrangement; or by the other, what the relative phase is of the waves associated with the passage of the photon from source to receptor by both routes' ,perhaps 50,000 light years apart as they pass the lensing galaxy G-1. But the photon has already passed that galaxy billions of years before we made our decision". Given the distance between the quasar and the receptor (billions of light years), the choice can be made long after the light's entry into the interferometer, an extreme case of the delayed-choice gedankenexperiment.

Moreover, inspired by Wheeler's proposals, one could expect that even if the choice is space-like separated from the photon's entry into the interferometer, the same result will be obtained. This shows the fact that the complementarity principle is independent of the space-time arrangement of the above mentioned events.

Chapter 7

Experimental Realizations

7.1 Quantum erasure with delayed choice

This experimental realization accomplished by Kim, Y.-H., R. Yu, S. P. Kulik, Y. Shih, and M. O. Scully in 2000. A laser beam (pump) aims photons at a double slit. After a photon passes the slits it impinges on a Barium borate (BBO) crystal placed behind the double slit. The optical crystal destroys the incoming photon and creates an entangled pair of photons via spontaneous parametric down conversion at the spot where it hit. Thus, if one of the photons of the entangled pair can later be identified by which slit it went through, we will also know whether its entangled counterpart went through the one or the other side of the crystal. By contrast, we will have no which-path information if we cannot later identify where either of the photons came from. Even though the entangled photons created at the crystal are now correlated, the experiment can manipulate them differently. We call one photon of the pair the signal photon (sent toward detector D_0) and the other one the idler photon (sent toward the prism). The naming is a matter of convention. The lens in front of detector D_0 is inserted to achieve the far-field limit at the detector and at the same time keep the distance small between slits and detector. The prism helps to increase the displacement between paths. Nothing about these parts gives which-path information and detector D_0 can not be used to distinguish between a photon coming from one slit or the other. At this point we would expect interference fringes to appear at D_0 if we were to ignore that signal photon and idler photon are entangled. The parts of the wavefunction originating at either slit should interfere and produce the well-known pattern of a double slit experiment. On the other hand,

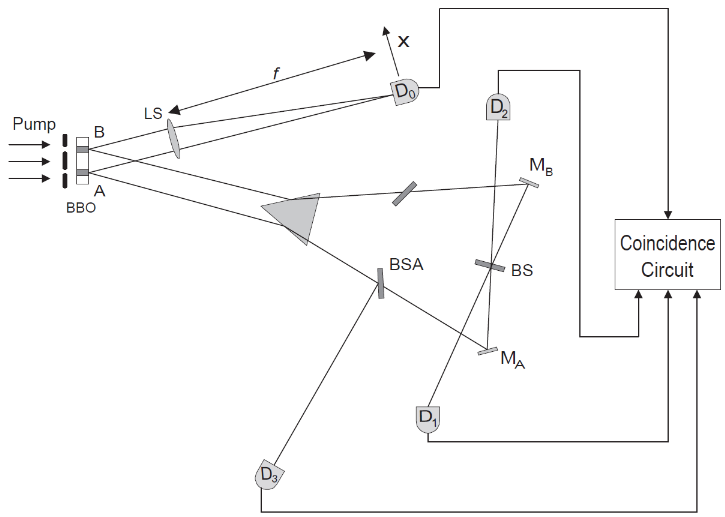

Figure 7.1: Schematic of the experimental setup. The pump laser beam of SPDC is divided by a double-slit and incident onto a BBO crystal at two regions A and B. A pair of signal-idler photons is generated either from A or B region. **The detection time of the signal photon is 8ns earlier than that of the idler.**

quantum mechanics would predict a typical clump pattern if which-path information were available. After the prism has bent the idler photon's path, the particle heads off to one of the 50-50 beamsplitters BS. The photon is reflected into the detector D_3 a random 50% of the time when it is travelling on the lower path. If the detectors D_3 clicks, a photon is detected with which-path information. That is, we know at which slit both photons of the entangled pair were generated. In that case, the formalism of quantum mechanics predicts no interference at D_0. In all of the other cases the photon passes through the beamsplitter and continues toward one of the mirrors M. Importantly, it does not matter if the choice whether the photon is reflected into the which-path detector D_3 is made by beamsplitters. The original experiment uses beamsplitters and therefore it is randomly decided which kind of measurement is performed. But we could equally replace the beamsplitters by moveable mirrors. In that way the experimenter is free to decide whether which - path information is available by either keeping the mirrors in place or removing them such that the photon can reach the eraser. After being reflected at one of the mirrors, the photon encounters another beamsplitter BS, which is the quantum eraser. This beamsplitter brings the photon in a superposition of being reflected and transmitted. To that end, for an idler photon coming from the lower mirror the beamsplitter either transmits the photon into detector D_2

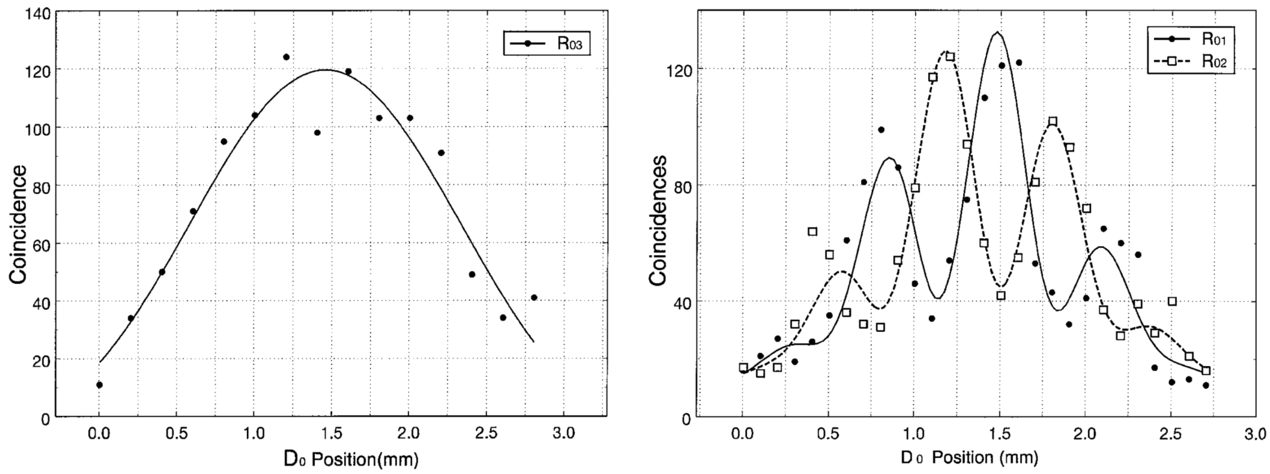

Figure 7.2: Coincidence counts between D_0 and D_3, as a function of the lateral position x_0 of D_0. Absence of interference was demonstrated and coincidence counts between D_0 and D_1 as well as between D_0 and D_2 are plotted as a function of x_0. Interference fringes were obtained.

or reflects it into detector D_1. Likewise, for an idler photon coming from the upper mirror the beamsplitter either transmits it into detector D_1 or reflects it into detector D_2. If one of the detectors D_1 or D_2 clicks, it is impossible to tell which slit the photon came from. To summarise the above, detectors D_1 and D_2 placed at the output of BS erase the which-path information, whereas a click of detector D3 provides which-path information about both the idler and the signal photon. Notably, **when the photon initially hits D_0, there is no which-path information available, only later when the entangled idler photon is detected at D_3 .**

7.2 Double-slit quantum eraser through the measurement of photon's polarization

The experiment reported here is inspired by the proposal of Scully, Englert and Walther and demonstrated by S. P. Walborn, M. O. Terra Cunha, S. Pádua, and C. H. Monken in 2000.

A linearly polarized beam of photons is incident on a double slit. If the double slit is of appropriate dimensions, the probability distribution for one-photon detection at a distant screen is given by a Young interference pattern. Suppose that in front of each slit we place a

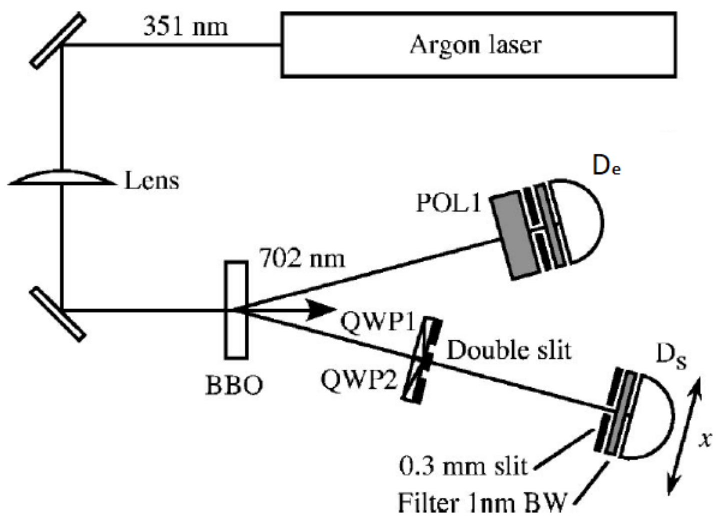

Figure 7.3: Experimental setup for the Bell-state quantum eraser. QWP1 and QWP2 are quarter-wave plates aligned in front of the double slit with fast axes perpendicular. POL1 is a linear polarizer.

quarter-wave plate, with the fast axis at an angle of 45° or −45° with respect to the photon polarization direction. Upon traversing either one of the wave plates, the photon becomes circularly polarized, and acquires a well-defined angular momentum. Supposing that the wave plate is free to rotate. Since the wave plates do not significantly modify the propagation of the beam, we have, in principle, a which-path marker with necessary characteristics for a quantum eraser.However, this scheme is far from being practical. As well as the difficulty of setting the wave plates free to rotate, the separation between the energy levels of a rotor with the mass and dimensions of a wave plate is of the order 10^{-40} eV. In addition, decoherence effects may make it impossible to use macroscopic quantum rotors to mark the path of a photon.

Now we will describe the above experiment formally: Let the beam of photons incident on the double slit be entangled with a second beam freely propagating in another direction, so as to define a Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\,|x\rangle_s\,|y\rangle_e + |y\rangle_s\,|x\rangle_e\,\right] \tag{7.1}$$

where the indices s and e indicate the two beams (system and environment photon) and H, V represent orthogonal linear polarizations . If beam s is incident on the double slit without wave plates, the state (7.1) is modified to

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\,|\psi_1\rangle + |\psi_2\rangle\,\right] \tag{7.2}$$

where

$$|\psi_1\rangle = \left[\,|x\rangle_{s1}\,|y\rangle_e + |y\rangle_{s1}\,|x\rangle_e\,\right] \tag{7.3a}$$

$$|\psi_2\rangle = \left[\,|x\rangle_{s2}\,|y\rangle_e + |y\rangle_{s2}\,|\rangle_e\,\right] \tag{7.3b}$$

the indices s1 and s2 refer to beams generated by slit 1 and slit 2 respectively. The probability distribution for one-photon detection on a screen placed in the far-field region of the overlapping beams s1, s2 will show the usual interference pattern Introducing the $\frac{\lambda}{4}$ plates one in front of each clit with the fast axes at angles $\theta_1 = 45°$ and $\theta_2 = -45°$ to the x direction, states $|\psi_1\rangle$ $|\psi_2\rangle$ are transformed to

$$\left|\psi_1'\right\rangle = \left[\,|L\rangle_{s1}\,|y\rangle_e + |R\rangle_{s1}\,|x\rangle_e\,\right] \tag{7.4a}$$

$$\left|\psi_2'\right\rangle = \left[\,|R\rangle_{s2}\,|y\rangle_e + |L\rangle_{s1}\,|x\rangle_e\,\right] \tag{7.4b}$$

where R and L represent right and left circular polarizations. Since $|\psi_1'\rangle$ and $|\psi_2'\rangle$ have orthogonal polarizations there is no possibility of interference.

In order to recover interference let us project the state of the system over the symmetric and antisymmetric states of the which-path detector. This is equivalent to transforming $|\psi_1\rangle$ and $|\psi_2\rangle$ in a way that expresses them as symmetric and antisymmetric combinations of polarizations:

$$|x\rangle = \left[\,|\nearrow\rangle + |\searrow\rangle\,\right] \tag{7.5a}$$

$$|y\rangle = \left[\,|\nearrow\rangle - |\searrow\rangle\,\right] \tag{7.5b}$$

$$|R\rangle = \frac{1-i}{2}\left[\,|\nearrow\rangle + i\,|\searrow\rangle\,\right] \tag{7.5c}$$

$$|L\rangle = \frac{1-i}{2}\left[i\,|\nearrow\rangle + |\searrow\rangle\,\right] \tag{7.5d}$$

where and represent polarizations $+45°$ and $-45°$ with respect to x. Rewriting the complete state $|\Psi\rangle$, we have

$$|\Psi\rangle = \frac{1}{2}\Big[\big(\,|\nearrow\rangle_{s1} - i\,|\nearrow\rangle_{s2}\,\big)\,|\nearrow\rangle_e + i\big(\,|\searrow\rangle_{s1} + i\,|\searrow\rangle_{s2}\,\big)\,|\searrow\rangle_e\,\Big] \tag{7.6}$$

According to the above expression, we can recover interference projecting the state of photon e over $|\nearrow\rangle_e$, $|\searrow\rangle_e$. Experimentally, this can be done by placing a polarizer in the path of beam e and orientating it at $+45°$ to select $|\nearrow\rangle_e$ or $-45°$ to select $|\searrow\rangle_e$. The interference

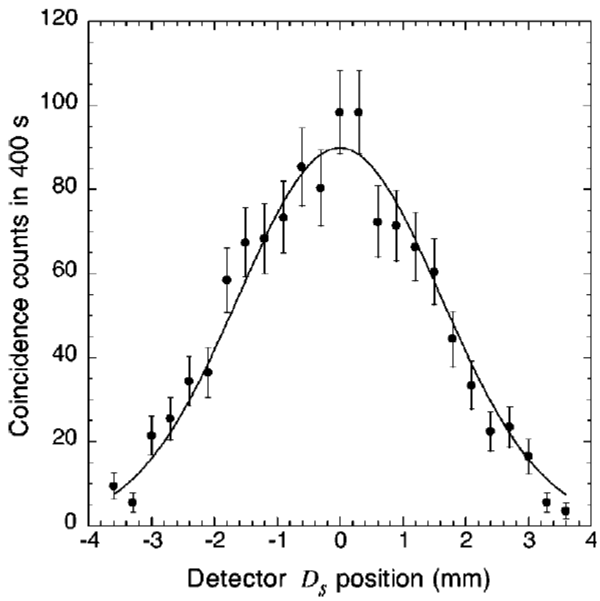

Figure 7.4: Coincidence counts when QWP1 and QWP2 are placed in front of the double slit. Interference has been destroyed.

pattern is recovered through the coincidence detection of photons s and e. Notice that the fringes obtained in the two cases are out of phase. They are commonly called fringes and antifringes.

The experimental setup is shown in Figure 7.3. An argon laser (351.1 nm at $\sim$ 200 mW) is used to pump a 1-mm-long BBO ($b - BaB_2O_4$) crystal, generating 702.2 nm entangled photons by spontaneous parametric down-conversion. The BBO crystal is cut for type-II phase matching. The pump beam is focused onto the crystal plane using a 1 m focal length lens to increase the transverse coherence length at the double slit. The width of the pump beam at the focus is approximately 0.5 mm . The orthogonally polarized entangled photons leave the BBO crystal each at an angle of 3° with the pump beam. In the path of photon e a polarizer cube (POL1) can be inserted in order to perform the quantum erasure. The double slit and quarter-wave plates are placed in path s, 42 cm from the BBO crystal. Detectors D_s and D_e are located 125 and 98 cm from the BBO crystal, respectively. QWP1 and QWP2 are quarter-wave plates with fast axes at an angle of 45°. The circular quarter-wave plates were sanded (tangentially) so as to fit together in front of the double slit. The openings of the double slit are 200 mm wide and separated by a distance of 200 mm. The detectors are

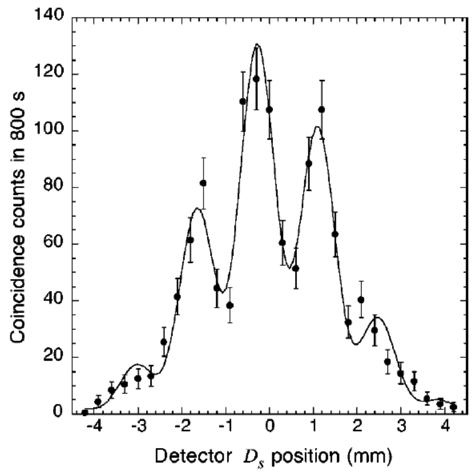

Figure 7.5: Coincidence counts when QPW1, QWP2, and POL1 are in place. POL1 was set to θ the angle of the fast axis of QWP1. Interference has been restored in the fringe pattern.

EG & G SPCM 200 photodetectors, equipped with interference filters (bandwidth 1 nm) and $300mm \times 35mm$ rectangular collection slits. A stepping motor is used to scan detector D_s.

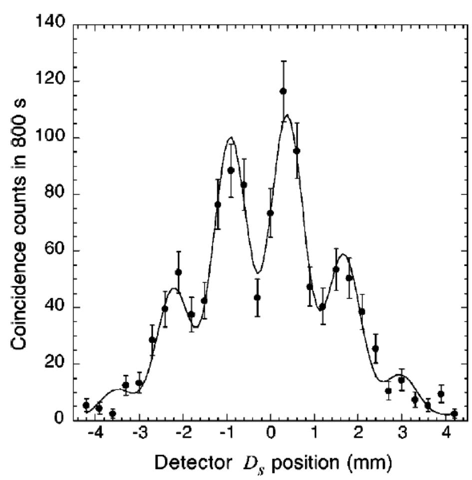

Figure 7.6: Coincidence counts when QPW1, QWP2, and POL1 are in place. POL1 was set to $\theta + \frac{\pi}{2}$, the angle of the fast axis of QWP2. Interference has been restored in the antifringe pattern.

Delayed erasure: The possibility of obtaining which-path information after the detection of photon s leads to delayed choice. In as much as our quantum eraser does not allow the experimenter to choose to observe which-path information or an interference pattern after the detection of photon s, it does allow for the detection of photon s before photon e, a situation which we refer to as delayed erasure. The question is, "Does the order of detection of the two photons affect the experimental results?"

The delayed erasure setup is similar, with two changes: i)detector D_e and POL1 were placed at a new distance of 2 m from the BBO crystal and ii) the collection iris on detector D_e has dimensions $600mm \times 35mm$.

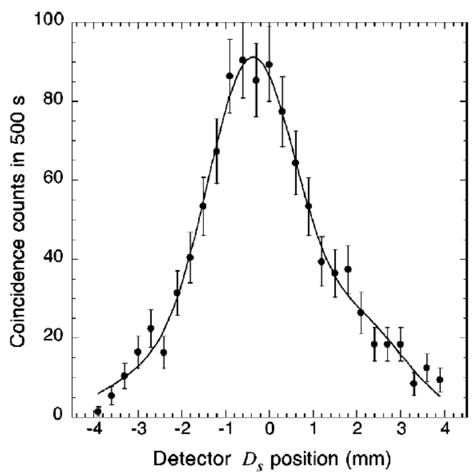

Figure 7.7: Coincidence counts in the delayed-erasure setup with QWP1 and QWP2 in place in front of the double slit. No interference is observed.

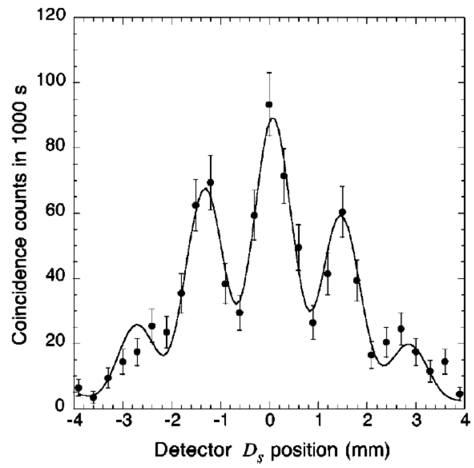

Figure 7.8: Coincidence counts in the delayed-erasure setup when QPW1, QWP2, and POL1 are in place. POL1 was set to θ, the angle of the fast axis of QWP1. Interference has been restored in the fringe pattern.

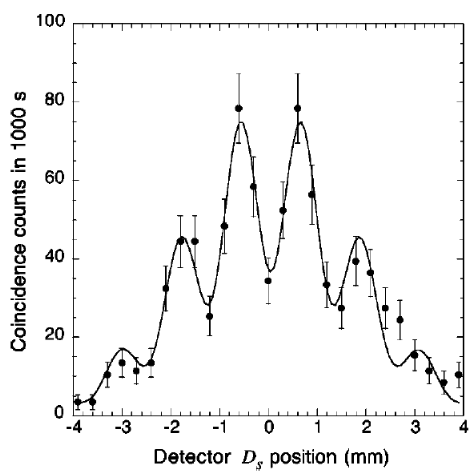

Figure 7.9: Coincidence counts in the delayed-erasure setup when QPW1, QWP2, and POL1 are in place. POL1 was set to $\theta + \frac{\pi}{2}$, the angle of the fast axis of QWP2. Interference has been restored in the antifringe pattern.

7.3 Quantum erasure with active and causally disconnected choice

In this section we present two experiments to demonstrate the non-local quantum eraser with two different length scales. These two experiments accomplished by the same research team in 2007 and 2008. This research team consist of Xiao-Song Ma, Johannes Kofler, Angie Qarry, Nuray Tetik, Thomas Scheidl, Rupert Ursin, Sven Ramelow, Thomas Herbst, Lothar Ratschbacher, Alessandro Fedrizzi, Thomas Jennewein, and Anton Zeilinger. In the first experiment (in Vienna), the system and environment photons are separated by 50 m and connected via an optical fibre between two laboratories. In the second experiment (on the Canary Islands), they are separated by 144 km and connected via a free-space link.

7.3.1 Hybrid Entanglement

Before presenting the non-local quantum eraser, we introduce a special type of entanglement: hybrid entanglement. It is the entanglement between different degrees of freedom (DOF) of a particle pair. Here we follow the proposal of hybrid entanglement between the polarization of one photon from a photon pair and the path (momentum) of its twin. A hybrid entangled state cannot be factorized into states of individual DOF. The defined joint properties are such that they link one DOF of one particle with another DOF of the other particle, where those degrees of freedom may even be defined in Hilbert spaces of different dimensionalities. For instance, in a hybrid-entangled state of photon A and B, their polarization and linear momentum are entangled as

$$\left|\Psi^{+}_{hybrid}\right\rangle = \frac{1}{\sqrt{2}}\left(|b\rangle_A |V\rangle_B + |a\rangle_A |H\rangle_B \right) \tag{7.7}$$

where $|H\rangle_B, |V\rangle_B$ denote the horizontal and vertical linearly polarized quantum states respectively, and $|a\rangle_A, |b\rangle_A$ denote two orthogonal momentum quantum states.

7.3.2 Vienna and Canary Islands experiments

The concept of this quantum eraser is illustrated in Figure 7.10. After the production of a hybrid entangled photon pairs , with entanglement between two different degrees of freedom, namely the path of one photon, denoted as the system photon, and the polarization of the

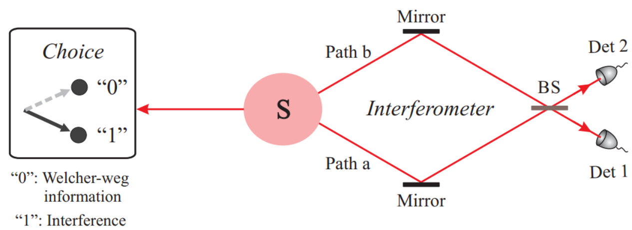

Figure 7.10: The concept of our quantum eraser under Einstein locality conditions. The hybrid entangled photon-pair source, labeled as S, emits path-polarization entangled photon pairs. The system photons are propagating through an interferometer on the right side and the environment photons are subject to polarization measurements on the left side. The choices to acquire welcher-weg information or to obtain interference of the system photons are made under Einstein locality, so that there are no causal influences between the system photons and the environment photons.

other photon, denoted as the environment photon. The system photon is sent to an interferometer, and the environment photon is sent to a polarization analyzer which performs a measurement according to a causally disconnected choice (with respect to the interferometer-related events). the environment photon's polarization carries welcher-weg information of the system photon due to the entanglement between the two photons. Depending on which polarization basis the environment photon is measured in, we are able to either acquire welcher-weg information of the system photon and observe no interference, or erase welcher-weg information and observe interference. In the latter case, it depends on the specific outcome of the environment photon which one out of two different interference patterns the system photon is showing. To test the quantum eraser concept under various spatio-temporal situations, was performed several experiments demonstrating the quantum eraser under Einstein locality on two different length scales. The first experiment performed in Vienna in 2007, the environment photon is sent away from the system photon via a 55 m long optical fibre. The second experiment performed on the Canary Islands in 2008, they are separated by 144 km and connected via a free-space link. The scheme of Vienna experiment is shown in Figure 7.11. First, a polarization-entangled state is prepared

$$\frac{|H\rangle_s |V\rangle_e + |V\rangle_s |H\rangle_e}{\sqrt{2}} \tag{7.8}$$

where $|H\rangle, |V\rangle$ denote quantum states of horizontal and vertical linear polarization, and s and e index the system and environment photon, respectively. The orthogonal polarization states of the system photon are coherently converted into two different interferometer path

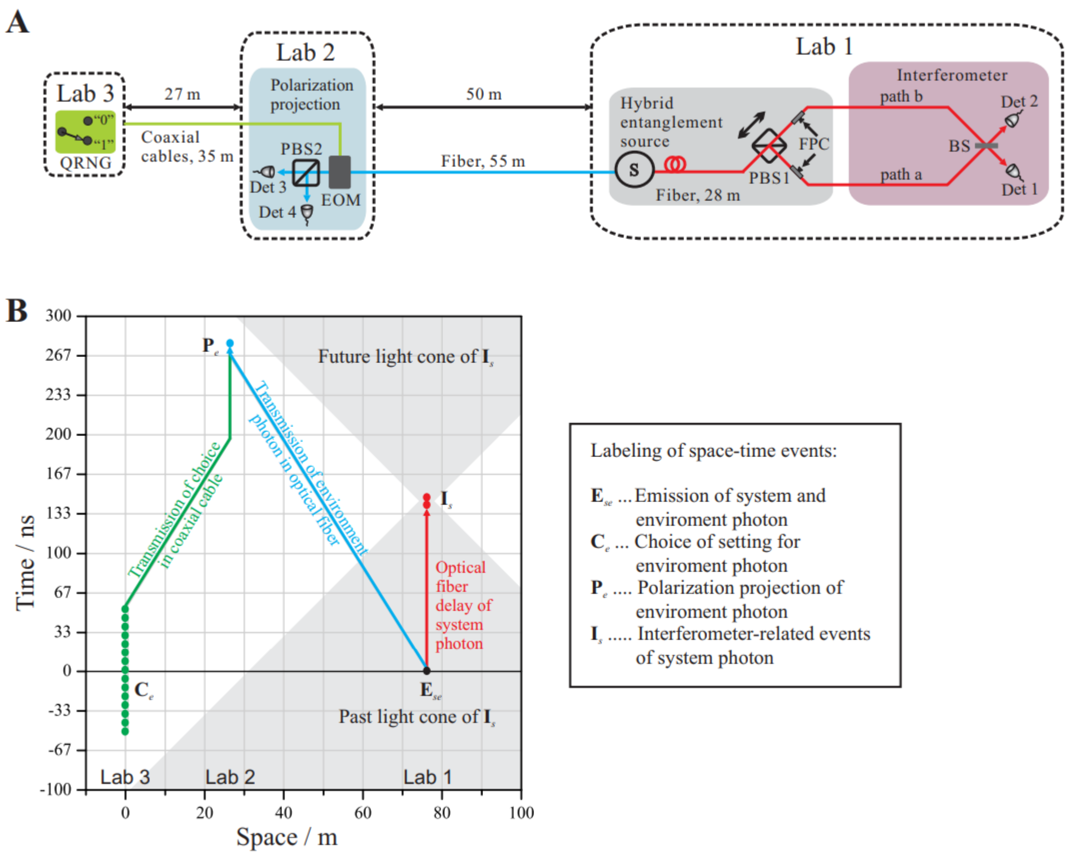

Figure 7.11: **A**. Scheme of the Vienna experiment , **B**. Space-time diagram.

states $|a\rangle_s, |b\rangle_s$ via a polarizing beam splitter and two fiber polarization controllers. This approximately generates the hybrid entangled state

$$|\Psi_{hybrid}\rangle_{se} = \frac{1}{\sqrt{2}}\big(|b\rangle_s |V\rangle_e + |a\rangle_s |H\rangle_e\big) \tag{7.9}$$

The environment photon thus carries welcher-weg information about the system photon. Therefore, we are able to perform two complementary polarization projection measurements on the environment photon and acquire or erase welcher-weg information of the system photon, respectively. (i) The environment photon is projected into the H/V basis, which reveals welcher-weg information of the system photon and no interference can be observed. (ii) the environment photon is projected into the R/L basis (with $|R\rangle = \frac{|H\rangle + i|V\rangle}{\sqrt{2}}$ and $|L\rangle = \frac{|H\rangle - i|V\rangle}{\sqrt{2}}$ of left and right circular polarization states, which erases welcher-weg information. Contrary to the first case, the detection of the environment photon in polarization R (or L) results in

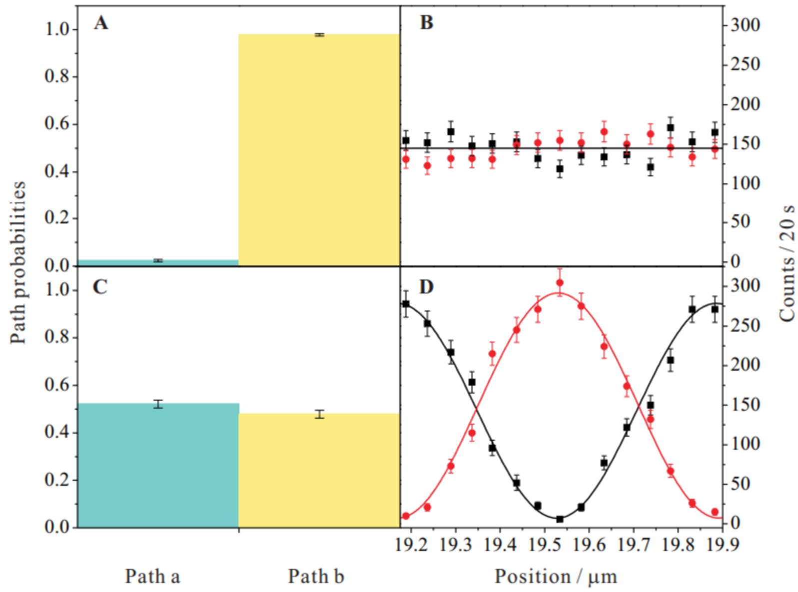

Figure 7.12: Results Vienna experiment: **A.** Choice (i) , the detection of the state $|V\rangle_e$ gives us the which-path information. This is confirmed by the fact that the system photon propagates through path a and b with probabilities 0.023 ± 0.005 (green) and 0.978 ± 0.005 (yellow), respectively. **C.** Choice (ii) , detection of the $|R\rangle_e$ erases which-path information of the system photon. The probabilities of the system photon propagating through path a and b are 0.521 ± 0.016 (green) and 0.478 ± 0.016 (yellow), respectively.

a coherent superposition with equal probabilities for the states $|a\rangle_s$ and $|b\rangle_s$ can be written

$$|\Psi_{hybrid}\rangle_{se} = \frac{1}{2}\big(\,|a\rangle_s + i\,|b\rangle_s\big)\,|L\rangle_e + \big(\,|a\rangle_s - i\,|b\rangle_s\big)\,|R\rangle_e \tag{7.10}$$

In case (ii), the polarization of the environment photon (either R or L) carries information about the relative phase between paths a and b of the system photon. This gives rise to complementary interference patterns (fringes or antifringes). The cases (i) and (ii) show that the which-path information and the fringe-antifringe information are equally fundamental. The following events are important and should be identified before the discussion of the space-time diagram (Figure 7.11): E_{se} is the emission of both the system photon and the environment photon from the source, C_e is the choice of the polarization measurement basis of the environment photon, P_e is the polarization projection of the environment photon, and I_s are all events related to the system photon inside the interferometer including its entry into, its propagation through, and its exit from the interferometer. In order to guarantee Einstein locality for a conclusive test, any causal influence between choice C_e and projection P_e of the environment photon on one hand and interferometer-related events I_s of the system photon on the other has to be ruled out. Operationally, we require space-like separation of C_e, P_e with respect to I_s. All this is achieved by setting up the respective experimental apparatus in three distant labs. The choice is performed by a quantum random number generator (QRNG). Its working principle is based on the intrinsically random detection events of photons behind a balanced beam splitter. Note that the setup also excludes any dependence between the choice and the photon pair emission , because we locate the source and QRNG in two separate labs such that space-like separation between the events C_e and E_{se} is ensured.

In Figure 7.12, are presented the experimental results for measurements of the system photon conditioned on the detection of the environment photon with Det 4. The probabilities that the system photon takes path a or b are shown when measurement (i), i.e. projection of the environment photon into the H/V basis and thus acquiring welcher-weg information, is performed. When the environment photon is subjected to measurement (i) and detected to have polarization V, the probability that the system photon propagates through path a is $P(a|V) = 0.023$, which is determined by blocking path b and summing up the coincidence counts over 120 s between both interferometer detectors and V detectors. Likewise, the probability for propagation through path b is $P(b|V) = 0.978$. In order to quantify the amount of welcher-weg information acquired, is used the so-called welcher-weg information parameter , $\mathcal{I} = |P(a|V) - P(b|V)|$. The value 0.955 of the parameter $\mathcal{I}_{(i)}$ reveals almost full welcher-weg information of the system photon. As a consequence, when the relative phase between path a and b is scanned, no interference pattern is observed. For each data point is integrated 20 s . On the other hand, when the environment photon is subjected to measurement (ii), i.e. projection of the environment photon into L/R basis, the welcher-weg information is irrevocably erased. When it is detected to have polarization R,we obtain the probabilities of the system photon propagating through path a, $P(a|R) = 0.521$, and through path b, $P(b|R) = 0.478$. In this case, $\mathcal{I}_{(ii)}$, defined as $\mathcal{I}_{(ii)} = |P(a|R) - P(b|R)|$, has the small value 0.077. Accordingly, interference shows up with the visibility of $\mathcal{V}_{(ii)} = 0.951$, where we integrate 20 s for each data point. This visibility is defined as $\mathcal{V} = \frac{C_{max} - C_{min}}{C_{max} + C_{min}}$, where C_{max} and C_{min} are the maximum and minimum counts of the system photon conditioned on the detection of the environment photon with Det 4. If the environment photon is detected to have polarization L, a $\pi - phase$ shifted interference pattern of the system photons shows up. These results together with the space-time arrangement of our experiment conclusively confirm the acausal nature of the quantum eraser concept. All results obtained indeed agree with the expectation within statistical errors. In order to quantitatively demonstrate the quantum eraser and the complementarity principle under Einstein locality, we employ a bipartite complementarity inequality namely,

$$\mathcal{I}^2 + \mathcal{V}^2 \leqslant 1 \tag{7.11}$$

which is an extension of the single-particle complementarity inequality. Here $\mathcal{I}$ and V are the parameters for two particles, as defined above. In an ideal experimental arrangement, inequality (3) is saturated. A similar setup, but with significantly larger spatial and temporal

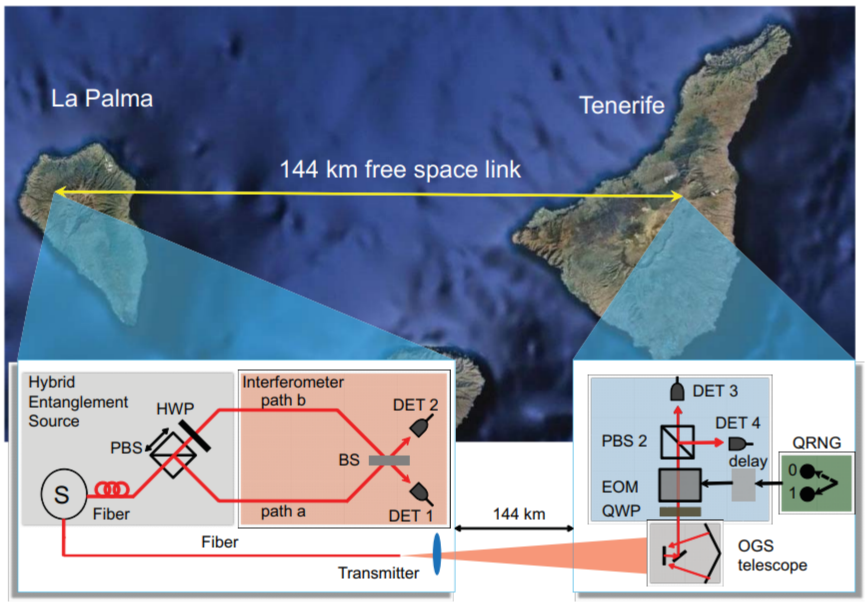

Figure 7.13: Experimental scheme of the Canary Islands experiment

separations, uses a 144 km free space link between the interferometer and the polarization projection setup . The two labs are located on two of the Canary Islands, La Palma and Tenerife . Two different space-time arrangements are realized, one of which achieves space-like separation of all relevant events. Within this scenario, different times for the choice events are chosen. One arrangement is such that the speed of a hypothetical superluminal signal from the choice event C_e to the events related to the interferometer I_s would have to be about 96 times the speed of light, ruling out an explantation by prorogation influence . The other arrangement is such that the choice event C_e happens approximately 450 μs after the events I_s in the reference frame of the source, which puts a record to the amount of delay by more than 5 orders of magnitude comparing to the previously reported quantum eraser experiment .

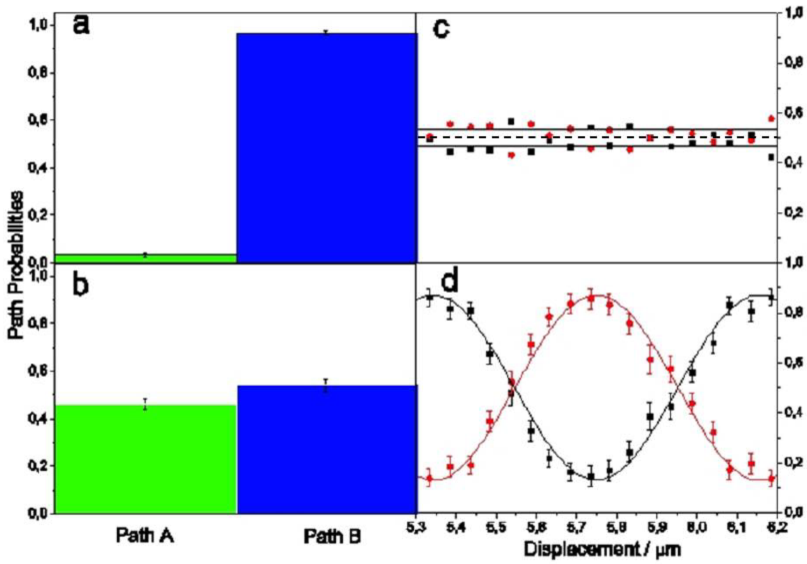

Figure 7.14: Results of of the Canary Islands experiment : Results of the Canary Island experiment. **a.** Choice (i) is performed, the probabilities of the system photon propagating through path a and b are 0.034 ± 0.08 and 0.966 ± 0.08 respectively. This reveals almost complete which-path information **b.** Choice (ii) is performed, the probabilities of the system photon propagating through path a and b are 0.461 ± 0.025 and 0.539 ± 0.025 respectively. This reveals almost no which-path information and hence interference is obtained

This work demonstrates and confirms that whether the correlations between two entangled photons reveal which-path information or an interference pattern of one (system) photon, depends on the choice of measurement on the other (environment) photon, even when all the events on the two sides that can be space-like separated, are space-like separated. The fact that it is possible to decide whether a wave or particle feature manifests itself long after—and even space-like separated from—the measurement teaches us that we should not have any naive realistic picture for interpreting quantum phenomena. Any explanation of what goes on in a specific individual observation of one photon has to take into account the whole experimental apparatus of the complete quantum state consisting of both photons, and it can only make sense after all information concerning complementary variables has been recorded. These results demonstrate that the view point that the system photon behaves either definitely as a wave or definitely as a particle would require faster-than-light communication. This would be in strong tension with the special theory of relativity.

Chapter 8

Summary

In conclusion we want to emphasize some remarkable outcomes from this thesis. We examined some fundamental features of quantum mechanics like complementarity principle, wave-particle duality and quantum measurement. Also through density matrix formalism we obtain a clear view of compound quantum system and entanglement. In particular:

- A mixture of states describes a situation in which a system really is in one of these two states, and we merely do not know which state this is. On the contrary, when a system is in a superposition of states, it is definitely not in either of these states. So the difference between a mixed state and a pure state lies in the ability of quantum amplitudes to interfere, which you can measure by preparing many copies of the same state and then measuring incompatible observables.

- Multiparticle systems can present the feature of entanglement, which describes a situation where two or more (not necessarily interacting) systems, for what concerns their physical properties, are not separable but have to be considered as a whole.

- We assumed that the the apparatus and the environment have a quantum mechanical definitions, in order to investigate the measurement problem. In the decoherence approach to the measurement problem, instead of considering only the interaction between a measurement apparatus and an object system, one also takes into account the action of the environment, which, by "absorbing" the off-diagonal elements of the density matrix describing the object system, allows the diagonalization of the matrix in the eigenbasis of the measured observable – via the formal mechanism of the partial trace. Decoherence presupposes no sharp break between the measurement process and the unitary dynamics of quantum theory. It is an interpretation that is consistent with the theory, that does not need ad hoc assumptions and is also able to make predictions that have been experimentally confirmed (decoherence time).

- In the experimental setup called quantum eraser it has been shown that the measurement act consists of two parts: the washing out of interference and the acquisition of information. When we loose the which-path information, we reproduce interference, something that tells us that these two notions are complementary.

- Delayed-choice gedanken experiments and their realizations play an important role in the foundations of quantum physics, because they serve as striking illustrations of the counter-intuitive and inherently non-classical features of quantum mechanics. They seem to imply that one may retro-act on the past. In fact, it has been shown that we never have to deal with the past but only with the present effects of past events. According to quantum mechanics, reality consists not only of events but also of non-local interdependencies and that there is a complementary dynamical relationship between these two features. Bohr said and rephrased by Wheeler: "No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon".

- Results demonstrate that the view point that the system photon behaves either definitely as a wave or definitely as a particle would require faster-than-light communication. Since this would be in strong tension with the special theory of relativity, we believe that such a view point should be given up entirely

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