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The Quantum Zeno Effect and its Application in a Three Level System

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Περίληψη

Στην παρούσα πτυχιακή εργασία θα ασχοληθούμε με το Κβαντικό Φαινόμενο του Ζήνωνα. Αρχικά θα παρουσιαστεί ένας ευρύτερος μαθηματικός φορμαλισμός, ο πίνακας πυκνότητας, που χρησιμοποιείται κυρίως για θέματα κβαντικών μετρήσεων. Θα γίνει πλήρης ανάλυση των εννοιών των καθαρών και μικτών καταστάσεων, καθώς και της χρονικής εξέλιξης και αναπαράστασης μιας μικτής ή καθαρής κατάστασης στην Σφαίρα του *Bloch*. Στην συνέχεια θα ακολουθήσει μια μικρή εισαγωγή στην έννοια των κβαντικών μετρήσεων, στο προβολικό αξίωμα και στην μέθοδο της αποσυνοχής. Χρησιμοποιώντας όλες αυτές τις γνώσεις θα είμαστε σε θέση να περιγράψουμε πλήρως φαινομενολογικά και φορμαλιστικά την εμφάνιση της "κατάπνιξης" των μεταπτώσεων ενός συστήματος, κάτω από συνεχείς μετρήσεις της κατάστασης του (ΚΦΖ). Έπειτα, θα γίνει μια θεωρητική εφαρμογή με την χρήση του προβολικού αξιώματος σε ένα σύστημα τριών επιπέδων. Τέλος, θα παρουσιαστεί το ανάλογο πείραμα, θα γίνει σύγκριση των θεωρητικών και πειραματικών αποτελεσμάτων και θα συζητηθεί η αναγκαιότητα του προβολικού αξιώματος με βάση την αριθμητική μελέτη των λύσεων των εξισώσεων *Bloch* για το σύστημα τριών επιπέδων.

Abstract

In this bachelor thesis we will concern ourselves with the Quantum Zeno Effect. Firstly, a broader mathematical framework that is mainly used in measurement theory, that of the density matrix, will be introduced. A complete analysis of the concepts of mixed and pure states, their time evolution and their representation in the Bloch Sphere, will follow. After that, we will take a brief look at some basic ideas of Quantum Measurement Theory, like the projection postulate and the decoherence method. Using all the above, we will be able to describe phenomenologically and formally the suppression of the transition of a system, under continuous measurements of its state (QZE). Then, we will theoretically apply the effect in a three level system, useing the projection postulate. Finally, a similar experiment will be presented, a comparison between the theoretical calculations and experimental data will follow and the necessity of the use of the projection postulate will be discussed, through numerical calculation based on the three level Bloch equations.

Contents

Chapter 1

Mathematical Formalism

1.1 Fundamental Postulates

To develop the necessary mathematical framework that is used in Quantum Measurement theory we used some core postulates of Quantum Mechanics that where not formally presented. Here we will write some of them and add some essential parts that will make the basic concepts that will follow understandable.

1.1.1 The Schrödinger Evolution

Postulate 1. Any isolated physical system is associated with a Hilbert space $\mathcal{H}$, known as the state space of the system. The system is completely described by its normalised state vector $|\psi\rangle$, which is a unit vector in Hilbert space.

Postulate 2. If a quantum system $\mathcal{S}$ can be in either of two states, then it can also be in any linear combination (superposition) of them.

This is frequently used when we write down a state vector as a linear combination of each other eigenvector of the system, for example in the usual two level system

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$$

Postulate 3. The probability of having a determinate measurement result – an eigenvalue a_n of the measured observable $\hat{A}$ – is given by

$$\mathcal{P}(a_n, \psi) = |c_n|^2 \tag{1.1}$$

where the complex coefficient c_n is the amplitude $c_n = \langle a_n|\psi\rangle$, and the eigenvector $|a_n\rangle$ of $|A\rangle$ corresponds to the eigenvalue a_n .

This postulate is of particular relevance because it provides the general mathematical connection between the coefficients of the expansion of the system state

$$|\psi\rangle = \sum_n c_n |a_n\rangle \tag{1.2}$$

into a given orthonormal basis $|a_n\rangle$:

$$\langle a_i|a_j\rangle = \delta_{ij}$$

and the probabilities of the corresponding outcomes of a measurement process. It is also evident of the normalisation condition

$$\sum_n |c_n|^2 = 1 \tag{1.3}$$

Postulate 4. The continuous time evolution of *any* system is governed by the *Schrödinger Equation*

$$\frac{d}{dt}|\psi(t)\rangle = \frac{1}{i\hbar}H|\psi(t)\rangle \tag{1.4}$$

A state vector at any given time t_1 is related to a state vector of another specific time t_2 by a *unitary transformation*

$$|\psi(t_2)\rangle = U(t_1, t_2)|\psi(t_1)\rangle \tag{1.5}$$

and in order to preserve the probability norm the operator U must be unitary

$$\langle\psi(t_2)|\psi(t_2)\rangle = \langle\psi(t_1)|U^\dagger U|\psi(t_1)\rangle \longrightarrow U^\dagger U = 1 \tag{1.6}$$

During the time period (t_1, t_2) we say that the system is under a *free Schrödinger evolution.*

Postulate 5. The state space of a compound physical system is the tensor product of the state spaces of the physical subsystems

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ...$$

Moreover, if we have systems numbered 1 to n, and the system number i is prepared in the state $|\psi_i\rangle$ then the joint state of the total system $|\Psi\rangle$ is

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ...$$

1.2 Density Matrix

An essential tool that provides assistance in describing a quantum mechanical system and doing calculations in a simple manner, is the *density matrix* formalism, also referred as the *density operator*. In the second section of this chapter the basic ideas and properties behind this formalism will be discussed.

1.2.1 Basic Definitions

Definition 1. The operator which projects the state vector $|\psi\rangle$ onto the $|i\rangle^{th}$ eigenstate is called the *projection operator* or *projector* and can be written as

$$\hat{P}_i = |i\rangle \langle i| \tag{1.7}$$

Some properties of the projector include:

1) $\hat{P}_i$ is hermitian.

Proof.

$$\hat{P_i}^\dagger = (|i\rangle \langle i|)^\dagger = ((\langle i|)^\dagger (|i\rangle))^\dagger = |i\rangle \langle i| = \hat{P}_i \tag{1.8}$$

2) $\hat{P_i}^2 = \hat{P}_i$.

Proof.

$$\hat{P_i}^2 = \hat{P}_i \hat{P}_i = |i\rangle \langle i|i\rangle \langle i| = |i\rangle \langle i| \tag{1.9}$$

Definition 2. We define the *Trace* of an operator $\hat{A}$ as

$$\mathrm{Tr}\left(\hat{A}\right) = \sum_n \langle n| \hat{A} |n\rangle \tag{1.10}$$

where $\{|n\rangle\}$ is the set of the basis vectors.

In a matrix representation $A_{mn} = \langle m| \hat{A} |n\rangle$ of the $\hat{A}$ operator, the trace is the sum of the diagonal matrix elements. A useful property of the trace is that it is invariant under a change of basis, and that

$$\mathrm{Tr}(P_1 P_2) = \mathrm{Tr}(P_2 P_1) \tag{1.11}$$

Definition 3. For any orthonormal vector basis in a Hilbert space $\mathcal{H}$, we define the identity operator $\mathbf{1}$ as

$$\mathbf{1} = \sum_n |n\rangle \langle n| \tag{1.12}$$

Eq. (1.12) is also called the *completeness relation*.

With the above definitions, we conclude to another property of the projector:

3) The trace of $\hat{P}_i$ is 1.[1]

Proof.

$$\mathrm{Tr}\Big(\hat{P}_i\Big) = \sum_n \langle n|i\rangle \langle i|n\rangle = \sum_n \langle i|n\rangle \langle n|i\rangle = \langle i|i\rangle = 1 \tag{1.13}$$

A general case of the previously defined projector, is the *density matrix* $\hat{\rho}$

Definition 4. For a state vector $|\Psi\rangle$ that contains the maximal amount of information about a system, the density matrix of the system is

$$\hat{\rho} = |\Psi\rangle \langle\Psi| \tag{1.14}$$

It is important to note that the density matrix is not referred as an arbitrary operator, but as a unique quantity that fully describes a quantum state. Usually, instead of the ket vector $|\Psi\rangle$ the density matrix $\hat{\rho}$ is used to symbolise the state of the system, especially when dealing with compound systems. Naturally, the density matrix and the projector share the same properties.

A first application for the density matrix, is the calculation of the expectation value of an arbitrary operator $\hat{A}$ in respect to a $|\Psi\rangle$ state as follows:

$$\begin{aligned}\langle\hat{A}\rangle = \langle\Psi|\hat{A}|\Psi\rangle &= \sum_n \langle\Psi\,|n\rangle \langle n|\,\hat{A}|\Psi\rangle = \sum_n \langle n|\,\hat{A}|\Psi\rangle\langle\Psi\,|n\rangle \\ &= \sum_n \langle n|\,\hat{A}\hat{\rho}\,|n\rangle = \mathrm{Tr}\Big(\hat{\rho}\hat{A}\Big)\end{aligned} \tag{1.15}$$

[1]This is true for a normalised projector. Generally if the projector is not normalised then $\mathrm{Tr}\Big(\hat{P}_i\Big) \neq 1$, and the normalised projector is given by $\hat{P}_i{}' = \frac{\hat{P}_i}{\mathrm{Tr}\left(\hat{P}_j\right)}$

therefore, the probability that the system is *found* in the eigenstate $|\phi\rangle$ can be written as:

$$\begin{aligned}
|\langle\phi|\Psi\rangle|^2 &= \langle\phi|\Psi\rangle\langle\Psi|\phi\rangle = \sum_n \langle\phi|n\rangle\langle n|\Psi\rangle\langle\Psi|\phi\rangle \\
&= \sum_n \langle n|\Psi\rangle\langle\Psi|\phi\rangle\langle\phi|n\rangle = \sum_n \langle n|\hat{\rho}\hat{P}_\phi|n\rangle \\
&= \mathrm{Tr}\left(\hat{\rho}\hat{P}_\phi\right)
\end{aligned} \tag{1.16}$$

with Eq. (1.15) and Eq. (1.16) one can make full use of the density matrix formalism in almost any simple problem.

1.2.2 Pure and Mixed States of a System

The real value of the density matrix formalism arises when one has to make calculations about a system without a well defined state vector $|\Psi\rangle$. Until now, we used the density matrix according to Def. (4), while we had in mind that the system can be described by one state vector $|\Psi\rangle$. Systems with that property will be called *systems in a pure state*. In other words, all the previous equations and properties referred to a *pure state* system.

A quantum system that has no well defined state vector is said to be in a *mixed state*. Systems like that in general, are consisted of a statistical ensemble of *pure* states. In that case, the system is described by a *mixture* $\hat{\hat{\rho}}$ that satisfies:

$$\hat{\hat{\rho}} = \sum_i w_i |\Psi_i\rangle\langle\Psi_i| \tag{1.17}$$

where w_i are the statistical weights of the ensemble, which satisfy the normalisation relation

$$\sum_i w_i = 1$$

and $\{|\Psi_i\rangle\}$ an orthonormal basis on the Hilbert space, that describes the pure states.

Having Eq. (1.17) as the definition of the mixture, the trace of a system in a mixed state can be calculated

$$\mathrm{Tr}\left(\hat{\hat{\rho}}\right) = \sum_{i,n} w_i \langle n|\Psi_i\rangle\langle\Psi_i|n\rangle = \sum_i w_i \langle\Psi_i|\Psi_i\rangle = \sum_i w_i = 1 = \mathrm{Tr}(\hat{\rho}) \tag{1.18}$$

whereas[1]

$$
\begin{aligned}
\mathrm{Tr}\Big(\hat{\tilde{\rho}}^2\Big) &= \sum_{i,j,n} w_i w_j \langle n|\Psi_i\rangle \langle \Psi_i|\Psi_j\rangle \langle \Psi_j|n\rangle = \sum_{i,j} w_i w_j |\langle \Psi_i|\Psi_j\rangle|^2 \\
&\leq \mathrm{Tr}\Big(\hat{\tilde{\rho}}\Big) = 1
\end{aligned}
\tag{1.19}
$$

which is a crucial property that separates density matrices that describe mixed and pure states. [2]

The above equation also proves that $\tilde{\rho}^2 \neq \tilde{\rho}$

We can also try to apply the right hand side of Eq. (1.15) for a mixed state:

$$
\begin{aligned}
\mathrm{Tr}\Big(\tilde{\rho}\hat{A}\Big) &= \sum_i w_i \,\mathrm{Tr}\Big(|\Psi_i\rangle \langle \Psi_i| \hat{A}\Big) = \sum_i w_i \,\mathrm{Tr}\Big(|\Psi_i\rangle \langle \Psi_i| \hat{A}\Big) \\
&= \sum_i w_i \sum_n \langle n|\Psi_i\rangle \langle \Psi_i| \hat{A} |n\rangle = \sum_i w_i \sum_n \langle \Psi_i| \hat{A} |n\rangle \langle n|\Psi_i\rangle \\
&= \sum_i w_i \langle \hat{A}\rangle_i
\end{aligned}
\tag{1.20}
$$

which confirms that the view of a statistical mixture also applies for the expectation value of a system in a mixed state.

Example 1. Suppose a two level system with eigenstates $|1\rangle$ and $|2\rangle$. Let the state of the system be

$$|\Psi\rangle = c_1 |1\rangle + c_2 |2\rangle$$

The density matrix of this *pure* state is

$$\rho = \sum_{i,j=1,2} \rho_{ij} |i\rangle \langle j| = \begin{bmatrix} |c_1|^2 & c_1^* c_2 \\ c_2^* c_1 & |c_2|^2 \end{bmatrix}$$

Now, if the system was in a mixed state, of lets say N two-level systems, each one with a possible state $|\Psi\rangle_1 = |1\rangle$ or $|\Psi\rangle_2 = |2\rangle$. For the general case, the weight of

[1] taking into consideration the Cauchy–Schwartz inequality, which leads to

$$\sum_{i,j} w_i w_j |\langle \Psi_i|\Psi_i\rangle|^2 \leq \sum_{i,j} w_i w_j \langle \Psi_i|\Psi_i\rangle \langle \Psi_i|\Psi_j\rangle = \sum_i w_i \sum_j w_j = 1$$

[2] from now on, we exclude the (ˆ) from the density matrix symbol, for the sake of notion economy

each pure state $|\Psi\rangle_n$ will be $w_i = N_i/N = |c_i|^2$ (where N_i is the number of the pure state subsystems in the state $|\Psi_i\rangle$). Applying Eq.(1.17) for the mixture:

$$\tilde{\rho} = |c_1|^2 |1\rangle \langle 2| + |c_2|^2 |2\rangle \langle 2| = \begin{bmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{bmatrix}$$

As we can see the off-diagonal terms (called the *coherences* or *coherent terms*) of the density matrix become zero in this case of mixture. In quantum mechanics, the coherent terms are responsible for interference effects that are usually present in quantum systems, so their absence from a "classical" statistical ensemble, that characterises a mixture, is not surprising.

1.2.3 Time Evolution and the Von Neumann Equation

As it is known (see Postulate 4), in order to study the time evolution of a fuction, the use of a unitary matrix U (that $U^\dagger U = 1$) is needed:

$$|\Psi(t)\rangle = U |\Psi\rangle$$

given that, for a *pure state* system, the time evolution of the density matrix (Eq.(1.14)) can be written as:

$$\rho(t) = |\Psi(t)\rangle \langle \Psi(t)| = U |\Psi\rangle \langle \Psi| U^\dagger = U\rho\, U^\dagger \tag{1.21}$$

Note 1. A system that initially is in a pure state, will *never* evolve in a mixed state with a unitary transformation.

Proof. Suppose the unitary transformation $\rho' = U\rho U^\dagger$, then the trace of the new state will be

$$\mathrm{Tr}\big(\rho'^2\big) = \mathrm{Tr}\big(U\rho U^\dagger U\rho U^\dagger\big) = \mathrm{Tr}\big(U\rho^2 U^\dagger\big) = \mathrm{Tr}\big(\rho^2 U^\dagger U\big) = \mathrm{Tr}\big(\rho^2\big) = 1$$

thus, (in combination with Eq. (1.19)) the new state will also be *pure*. This is an important note for the measurement problem in quantum measurement theory that will be discussed later.

Now, we will try to write down a differential equation that describes the time evolution of the density matrix. We start by differentiating Eq. (1.21)

$$\dot{\rho}(t) = \frac{d}{dt}\left|\Psi(t)\right\rangle \left\langle\Psi(t)\right| + \left|\Psi(t)\right\rangle \frac{d}{dt}\left\langle\Psi(t)\right|$$

Next, one can apply the Schrödinger Equation

$$\frac{d}{dt}\left|\Psi(t)\right\rangle = -\frac{i}{\hbar}H\left|\Psi(t)\right\rangle$$

in order to write:

$$\dot{\rho}(t) = -\frac{i}{\hbar}H\left|\Psi(t)\right\rangle \left\langle\Psi(t)\right| + \frac{i}{\hbar}\left|\Psi(t)\right\rangle \left\langle\Psi(t)\right| H$$

or

$$\dot{\rho} = \frac{i}{\hbar}[\rho, H] \tag{1.22}$$

the last equation is called the *von Neumann equation* and it basically tells us that pure states evolve into pure states under a Schrödinger evolution. In Quantum Measurement Theory however, it is more usual to write down the time evolution in the *Dirac Picture* rather than the Schrödinger picture . For this a more complex mathematical notion is needed, called the *Master Equation*.

1.2.4 Reduced Density Matrix and Partial Trace

As mentioned earlier, a mixed state can be described as the ensemble of many pure states. However, it is possible to reduce a pure state in a mixed state through the use of the *partial trace*. The reason behind this is that sometimes when dealing with a compound system with a known wavefunction (a pure state), one can find themselves unable to admit each of the subsystems in a similar description in terms of an independent wavefunction (subsystems in mixed state).

Let's assume a compound system in a pure state ρ_{AB}, composed of two subsystems, A and B. Now we will apply a *partial trace in respect to B*, to the density matrix

$$\mathrm{Tr}_B(\rho_{AB}) = \sum_n \left\langle n\right| \rho_{AB} \left|n\right\rangle_B \tag{1.23}$$

we can demonstrate that Eq. (1.23) indeed represents a mixed state ρ_A of one of the subsystems, in a simple example

Example 2. Let the compound system be composed of two two-level systems, system A and system B, with wavefunctions

$$\left|\Psi\right\rangle_k = \frac{1}{\sqrt{2}}(\left|1\right\rangle_k + \left|2\right\rangle_k) \quad , \quad k = A, B$$

and the pure state wavefunction of the compound system be

$$\left|\Psi\right\rangle_{AB} = \frac{1}{\sqrt{2}}(\left|1\right\rangle_A \otimes \left|2\right\rangle_B + \left|2\right\rangle_A \otimes \left|1\right\rangle_B)$$

the corresponding density matrix is written as

$$\begin{aligned}
&\rho_{AB} = \left|\Psi\right\rangle_{AB} \left\langle\Psi\right| \\
&\rho_{AB} = \frac{1}{2}(\left|1\right\rangle_A \left\langle 1\right| \otimes \left|2\right\rangle_B \left\langle 2\right| + \left|1\right\rangle_A \left\langle 2\right| \otimes \left|2\right\rangle_B \left\langle 1\right| + \left|2\right\rangle_A \left\langle 1\right| \otimes \left|1\right\rangle_B \left\langle 2\right| + \left|2\right\rangle_A \left\langle 2\right| \otimes \left|1\right\rangle_B \left\langle 1\right|)
\end{aligned} \tag{1.24}$$

taking the partial trace of B in that expression, and keeping in mind that it only acts in the vectors of the Hilbert subspace that is assigned to B, while its base is orthonormal, we conclude that:

$$\begin{aligned}
\mathrm{Tr}_B(\rho_{AB}) &= \sum_{n=1,2} \left\langle n\right| \rho_{AB} \left|n\right\rangle_B \\
2\,\mathrm{Tr}_B(\rho_{AB}) &= \left|1\right\rangle_A \left\langle 1\right| \otimes \left\langle 2|2\right\rangle_B \left\langle 2|2\right\rangle_B + \left|1\right\rangle_A \left\langle 2\right| \otimes \left\langle 2|2\right\rangle_B \left\langle 1|2\right\rangle_B \\
&+ \left|1\right\rangle_A \left\langle 2\right| \otimes \left\langle 1|2\right\rangle_B \left\langle 2|1\right\rangle_B + \left|1\right\rangle_A \left\langle 2\right| \otimes \left\langle 2|1\right\rangle_B \left\langle 2|2\right\rangle_B \\
&+ \left|1\right\rangle_A \left\langle 2\right| \otimes \left\langle 1|1\right\rangle_B \left\langle 2|1\right\rangle_B + \left|1\right\rangle_A \left\langle 1\right| \otimes \left\langle 1|1\right\rangle_B \left\langle 1|1\right\rangle_B \\
&= (\left|2\right\rangle_A \left\langle 2\right| + \left|1\right\rangle_A \left\langle 1\right|) = 2\rho_A
\end{aligned} \tag{1.25}$$

or

$$\mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2}(\left|1\right\rangle_A \left\langle 1\right| + \left|2\right\rangle_A \left\langle 2\right|) = \rho_A \tag{1.26}$$

it is easy to see from the matrix representation of that expression, that ρ_A is a mixture

$$\rho_A = \tilde{\rho}_A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

because its coherent elements are zero (as shown in the Example of Subsection 1.2.2). The whole operation is refereed to as "*tracing out the subsystem B*" and of course can be applied for the other system as well, by a partial trace $\mathrm{Tr}_A(\rho_A B)$.

The density matrix of such a mixed state is call *the reduced matrix* of the compound system.

While in this particular thesis we will not deal with compound systems this way, the formalism of this Subsection will be used later to understand the *decoherence* during a measurement.

Note 2. In the case of the Example 2, the compound system wavefunction cannot be written in a factorised form (as in Postulate 5). In similar cases, the partial trace will result in a mixture, whereas for a wavefunction written in factorised form, the partial trace will result in a pure state density matrix.

1.3 The Bloch Sphere Representation

Another useful tool in the representation of a state is the geometrical use of a *Bloch vector*. The simple case for this kind of representation is the application in a usual two level system (which we are also going to use later). Let the pure state of the system be

$$|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle$$

Now, we define a vector $\vec{s} = (s_x, s_y, s_z)$ in the three-dimensional space with the following (real) components:

$$s_x = \rho_{12} + \rho_{21} \tag{1.27}$$

$$s_y = i\left(\rho_{12} - \rho_{21}\right) \tag{1.28}$$

$$s_z = \rho_{22} - \rho_{11} \tag{1.29}$$

where the matrix elements are given by the density matrix of the system:

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \tag{1.30}$$

Combining Eqs. (1.27, 1.28, 1.29) with Eq. (1.30), it is not hard to conclude that

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + s_z & s_x - i s_y \\ s_x + i s_y & 1 - s_z \end{bmatrix} = \frac{1}{2}(\mathbf{I} + \vec{s}\,\hat{\sigma}\ ) \tag{1.31}$$

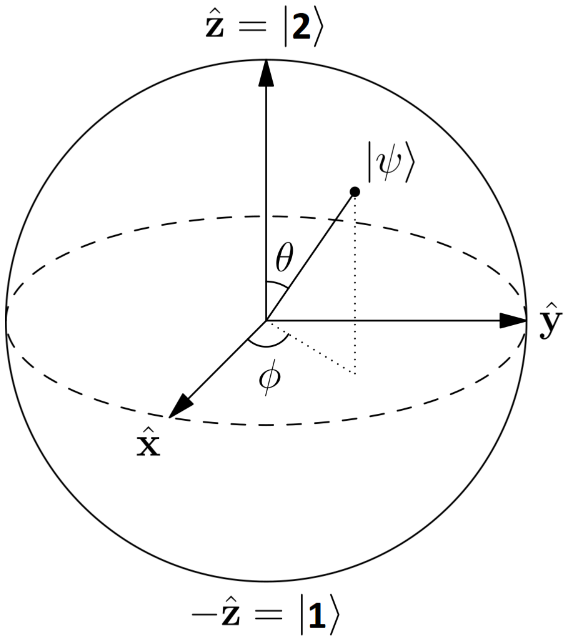

Figure 1.1: The Bloch Sphere

where $\hat{\sigma}$ is the well known *spin-1/2* operator. With all that in mind, one can prove that the state of the system can be represented as a Bloch vector inside a sphere of unit radius (Bloch Sphere), so it "moves" on the sphere's surface.

If the state is pure (as it is in our case) then the Bloch vector has $|\vec{s}|^2 = 1$.

Indeed:

$$\begin{aligned} s^2 &= s_x^2 + s_y^2 + s_z^2 \\ &= \rho_{12}^2 + \rho_{21}^2 + 2\rho_{12}\rho_{21} - \rho_{12}^2 - \rho_{21}^2 + 2\rho_{12}\rho_{21} + \rho_{11}^2 + \rho_{22}^2 - 2\rho_{12}\rho_{21} \\ &= \rho_{11}^2 + \rho_{22}^2 + 4\rho_{12}\rho_{21} - 4\rho_{11}\rho_{22} + 2\rho_{11}\rho_{22} \\ &= (\rho_{11} + \rho_{22})^2 + 4(\rho_{12}\rho_{21} - \rho_{11}\rho_{22}) \end{aligned} \tag{1.32}$$

however in a normalised state $\rho_{11} + \rho_{22} = |\langle 1|\psi\rangle|^2 + |\langle 2|\psi\rangle|^2 = 1$, so

$$s^2 = 1 + 4(\rho_{12}\rho_{21} - \rho_{11}\rho_{22}) \tag{1.33}$$

for a pure state, Eq. (1.13) must hold, so

$$\mathrm{Tr}\big(\rho^2\big) = \rho_{11}^2 + \rho_{22}^2 + \rho_{12}\rho_{21} + \rho_{21}\rho_{12} = 1$$

$$\rho_{11}^2 + \rho_{22}^2 - 1 = -2\rho_{12}\rho_{21}$$

$$2\rho_{11}\rho_{22} = 2\rho_{12}\rho_{21}$$

from Eq. (1.24) then, $s^2 = 1$.

Note 3. For a mixed state, we have $\mathrm{Tr}(\tilde{\rho}) < 1$ and therefore

$$\rho_{11}\rho_{22} < \rho_{12}\rho_{21}$$

from which, it follows that

$$s^2 = 1 + 4(\rho_{12}\rho_{21} - \rho_{11}\rho_{22}) < 1$$

that means that a *mixture* corresponds to a Bloch vector that "moves" on the *inside* of the Bloch Sphere.

Now, if one wants to formally study the "movement" of a Bloch vector, then they need a differential equation of motion. For the case of an isolated two level system that undergoes a spontaneous emission and is described by a Hamiltonian of the form

$$H = \frac{1}{2}\hbar\omega\sigma_z \tag{1.34}$$

the equations of motion, which are derived from the von Neumann equation (see Subsection 1.2.3) (one for each vector component), are called the *optical Bloch equations for a two level system* and are written as:

$$\dot{s_x} = -\gamma s_x \tag{1.35}$$

$$\dot{s_y} = -\gamma s_y \tag{1.36}$$

$$\dot{s_z} = -2\gamma(1 + s_z) \tag{1.37}$$

where γ is a coupling constant of the particular system.

Chapter 2

Introduction to Quantum Measurement Theory

2.1 The Measurement in Quantum Mechanics

Despite the sharpness and consistency of the adressed postulates of Quantum Mechanics, an important part that defines any theory and experiment is not sufficiently covered. That is concept of the *measurement*.

The most popular interpretation of Quantum Mechanics, the *Copenhagen Interpretation*, states that physical systems generally do not have definite properties (or "are not real") prior to being measured. However the act of measurement affects the system, causing the set of probabilities to reduce to only one of the possible values immediately after the measurement. This feature is known as *wave function collapse*. Yet even with that information, one cannot specifically say what exactly happens *during* the measurement itself.

2.1.1 The Measurement Problem

If a system is initially unmeasured, which means that we as observers have not acted in any way in order to determine the eigenvalue of a specific observable $\hat{A}$, then as Postulate 2 states, the state of the system will be a superposition of every possible eigenstate of the observable

$$|\psi_{before}\rangle = a_1 |a_1\rangle + a_2 |a_2\rangle + ...$$

however, we know with high certainty that after every measurement, we always get as a result only *one* eigenvalue. This means that if we act with the observable operator after the measurement we will get

$$\hat{A}\,|\psi_{after}\rangle = a_n\,|\psi_{after}\rangle$$

so every other eigenstate $|a_m\rangle$ $(m \neq n)$ somehow vanishes during the transformation

$$|\psi_{before}\rangle \longrightarrow |\psi_{after}\rangle$$

However, as Postulate 4 states, the evolution of every system is done by the act of a unitary transformation. By definition unitary transformations conserve probabilities, so a unitary transformation alone *cannot* vanish an eigenvector from the "before" superposition. This statement is called the *measurement problem* in quantum mechanics:

"A unitary transformation cannot explain the results of a measurement."

2.1.2 The Projection Postulate

Von Neumann was the first to try to formalize the problem of measurement [8]. He assumed that there are basically two types of evolution in quantum mechanics. The first is the usual unitary evolution, while the other is represented by measurement, which presents the following features: it is a discontinuous, non-causal, instantaneous, non-unitary, and irreversible change of state. Von Neumann called this abrupt change reduction of the wave packet (a passage from a superposition to one of its components) the *collapse of the wave function.* Then, von Neumann postulated that, if the observable $\hat{A}$ is measured on a system S in an arbitrary state $|\psi\rangle$, then the latter is projected after the measurement onto one of the basis vectors $|a_n\rangle$ of the representation in which $\hat{A}$ is diagonal, i.e. in an eigenstate of $\hat{A}$ for which the probability is $|\langle a_n|\psi\rangle|^2$. This has been known as the *projection postulate.*

We shall use the density matrix formalism developed earlier in this thesis, to make some mathematical conclusions that we are going to use later, about the projection postulate.

Let us write the "before" superposition state as

$$|\psi\rangle = \sum_j c_j\,|\psi_j\rangle$$

then, the "before" density matrix is written as

$$\rho_{before} = |\psi\rangle\langle\psi| = \sum_{j,k} c_j c_k^* |\psi_j\rangle\langle\psi_k| = \sum_j |c_j|^2 |\psi_j\rangle\langle\psi_j| + \sum_{j\neq k} c_j c_k^* |\psi_j\rangle\langle\psi_k| \quad (2.1)$$

we can see that the last sum represents the *coherent* matrix elements that were mentioned in the Subsection 1.2.2, while the second-to-last sum represents the diagonal matrix elements, i.e. the probabilities for each eigenvalue *separately.* This means that the "after" density matrix must include *only* the diagonal elements because they are associated with only *one* eigenvalue respectively. For that, the "after" state must somehow have *zero* coherent terms. In other words, as we saw in Subsection 1.2.2, the "after" state must be a kind[3] of *mixture.*

With all the above, the mixture is written as

$$\rho_{after} = \tilde{\rho} = \sum_j |c_j|^2 |\psi_j\rangle\langle\psi_j| \quad (2.2)$$

Despite not having any unitary transformation that transforms Eq. (2.1) to Eq. (2.2), a non-unitary transformation in the form of

$$\tilde{\rho} = \sum_j \hat{P}_j \, \rho \hat{P}_j \quad (2.3)$$

(where $\hat{P}_j = |\psi_j\rangle\langle\psi_j|$ is a projector) provides us with the requested result.

Proof.

$$\begin{aligned}
\sum_i \hat{P}_i \, \rho \hat{P}_i &= \sum_{i,j} |c_j|^2 |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_i\rangle\langle\psi_i| + \sum_{i,j\neq k} c_j c_k |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_k|\psi_i\rangle\langle\psi_i| \\
&= \sum_{i,j} |c_j|^2 |\psi_i\rangle \delta_{ij}\delta_{ji} \langle\psi_i| + \sum_{i,j\neq k} c_j c_k |\psi_i\rangle \delta_{ij}\delta_{ki} \langle\psi_i| \\
&= \sum_{i,j} |c_j|^2 |\psi_i\rangle\langle\psi_i| = \tilde{\rho}
\end{aligned}$$

Eq. (2.3) is the mathematical representation of the *projection postulate.*

[3]there are many kinds of mixtures, in this case only the ones that have zero coherences are taken into consideration, like the one from Example 1.

2.2 Decoherence

In the following Section we will very briefly discuss one of the possible ways to deal with the measurement problem, that of the *decoherence* method. The main idea is that, the system in question is always considered to be coupled with the environment (systems with that characteristic are called *open systems*)). Then, the *loss of coherence* that is required for the transformation of a pure into a mixed state (which is also a requirement for the measurement) is achieved by "transferring" the information of the coherences from the system to the environment. That "transfer" is done by using the tool of the *partial trace* (see Subsection 1.2.4).

2.2.1 Decoherence in an Open Quantum System

Let us assume once again a two level system S with a state vector

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$$

the environment in this case is considered to be a heat bath R. The environment has its own state vector $|R\rangle$ and is considered to be a system as well, but of vast proportions. The heat bath is coupled with the system, so the initial total state of the two, will be

$$|\Psi\rangle = |\psi\rangle \otimes |R\rangle \quad (2.4)$$

where $|R_i\rangle$ are some random states of the heat bath (not strictly orthogonal). In general, after the interaction between the two subsystems, the total state cannot be written as a product state. For the sake of simplicity, let's assume that the total state after the interaction will be

$$|\Psi\rangle = C_1 |\psi_1\rangle |R_1\rangle + C_2 |\psi_2\rangle |R_2\rangle \quad (2.5)$$

we then calculate the density matrix for this (pure) state

$$\rho_{SR} = |C_1|^2 P_1 P_{R_1} + |C_2|^2 P_2 P_{R_2} + C_1 C_2^* |1\rangle \langle 2|R_1\rangle \langle R_2| + C_2 C_1^* |2\rangle \langle 1|R_2\rangle \langle R_1| \quad (2.6)$$

where $P_i = |\psi_i\rangle \langle\psi_i|$ and $P_{R_i} = |R_i\rangle \langle R_i|$. Now as we mentioned, we *trace out* the environment, by applying the respective partial trace to ρ_{SR} :

$$\rho_S = \mathrm{Tr}_R(\rho_{SR}) = \sum_j^{\sim} \langle r_j| \rho_{SR} |r_j\rangle \; (2.7)$$

where $|r_j\rangle$ is the orthonormal basis of the heat bath. Now, if we expand the non-orthogonal heat bath states we have

$$|R_i\rangle = \sum_n a_n |r_n\rangle \tag{2.8}$$

The above, in combination with Eq. (2.6) and Eq. (2.7) produces the reduce matrix:

$$\tilde{\rho}_S = \begin{bmatrix} |C_0|^2 & C_0 C_1^* \langle R_1|R_0\rangle \\ C_1 C_0^* \langle R_0|R_1\rangle & |C_1|^2 \end{bmatrix} \tag{2.9}$$

From the form of the reduced density matrix we firstly observe that the diagonal elements are intact, which required for the probability preserving. Next we see that the coherent elements will in fact diminish, if the original states of the heat bath will *somehow* become orthogonal. The more the random heat bath states become orthogonal, the smaller the coherent terms will get, until they disappear.

While in this thesis we will not deal with open systems[4], this example teaches us that decoherence may not necessarily happen instantly. The requirement is for the coherences to simply fall to zero quickly or have a tendency of falling to zero in a specific time scale (decoherence time) which is inversely proportional to the size of the heat bath/environment.

Another important aspect of this method, is that just the *presence* of the environment is enough to cause decoherence phenomena. That fact alone can be problematic if someone want to avoid *any* amount of information loss from the initial system (for example, in quantum computing).

Note 4. All the above discussion applies to the most probable type of measurement, that which does not annihilate the initial system. This type is the one that is going to be applied to the following chapters. However we should also note that another type of measurement, that which annihilates the system (i.e. measurements in photon states) can be more problematic in terms of mathematical description, so we did not cover that part (which is not going to be of any further use for the next chapters in either way).

[4]as stated before, every quantum system is open, but in the examples of the next chapters, we will consider them to be approximately isolated

Chapter 3

The Quantum Zeno Effect

The name of this phenomenon is derived from the famous Greek philosopher *Zeno of Elea* and his equally famous "paradox", the *Zeno Paradox*[5]. The paradox states that, if an archer shoots an arrow at a target in a certain distance, by logical thinking the arrow should never reach the said target. The argument is the following: if the arrow travels a distance d in a limited time t then it must travel half the distance $d/2$ first, again in a limited time t'. However one can separate the initial distance into infinite half-distances, each one adding a limited time to the total time of travel. Then the arrow must need an *infinite* amount of time to reach its target, which is in fact a paradoxical view, since by experience the arrow always reaches its target.

Of course there are many flaws in the application of Zeno's logic to the actual physical world. A physical object as we know, will not have a traditional trajectory when described in the smallest of space scales, so it is not wrong to say that an infinite space-time division like Zeno proposed is not feasible.

That age old question proposed by Zeno, which is also mirrored in some modern sayings, like "a watched pot never boils", has a connection with a peculiar effect of the Quantum world, called the Quantum Zeno Effect. According to the QZE, an unstable state that will never make a transition, *if* it is constantly being monitored. The parallels with Zeno's Paradox are obvious, the state vector of the unstable system acts like Zeno's arrow, and never moves, but instead of dividing infinitely space, according to the effect one must divide infinitely the *"measurement period"*. Such a task seems very unclear at first: What type of measurement is needed? Can formal Measurement theory reach to the same conclusion? Will there be a need of supplementary postulates or not? Can the effect be demonstrated in the real

[5]There are three different variants of this paradox, *The Race Paradox*, *The Dichotomy Paradox* and the The Arrow Paradox. The one presented is actually the Dichotomy Paradox, which is combined with the Arrow Paradox, for a better presentation of Zeno's logic behind all the three paradoxes.

world or is it another "paradox"? These types of questions will be adressed in the following chapters.

3.1 Phenomenological Description

To theoretically demonstrate the Quantum Zeno effect [9], a simple one-particle system is chosen. The time evolution of the system in the Schrödinger picture is given by the unitary transformation

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}\,|\psi(0)\rangle = U(t)\,|\psi(0)\rangle \tag{3.1}$$

which follows that if the system is in the initial state $|\psi(0)\rangle$ at $t = 0$ then the probability that the system's state will remain the same (evolve in the same state as the initial) after a specific time t', will be

$$P(t') = \langle\psi(0)|\psi(t')\rangle\,\langle\psi(t')|\psi(0)\rangle = |\,\langle\psi(0)|\,e^{-\frac{i}{\hbar}Ht'}\,|\psi(0)\rangle\,|^2 \tag{3.2}$$

If the first measurement is conducted shortly after $t = 0$, then t' can be arbitrary small, so that the exponential function can be approximated by a second-order Taylor series

$$\begin{aligned} P_1(t') &= |\,\langle\psi(0)|\left[1 - \frac{iHt'}{\hbar} - (\frac{Ht'}{\hbar})^2\right]|\psi(0)\rangle\,|^2 \\ &= |1 - \frac{it'}{\hbar}\,\langle\psi(0)|\,H\,|\psi(0)\rangle - \frac{1}{2}(\frac{t'}{\hbar})^2\,\langle\psi(0)|\,H^2\,|\psi(0)\rangle\,|^2 \end{aligned} \tag{3.3}$$

expanding the above, and taking up to second-order terms of time, the probability becomes

$$\begin{aligned} P_1(t') &= 1 - \frac{t'^2}{\hbar^2}\big[\,\langle\psi(0)|\,H^2\,|\psi(0)\rangle - \langle\psi(0)|\,H\,|\psi(0)\rangle^2\,\big] \\ &= 1 - \Big(\frac{t'}{\tau_z}\Big)^2 \end{aligned} \tag{3.4}$$

where the quantity $\frac{1}{\tau_z}$ carries the statistical meaning of the initial state's energy "spread" ΔE and τ_z is called *Zeno time.* Here we can also assume the interaction Hamiltonian, where $\langle\psi(0)|\,H_I\,|\psi(0)\rangle = 0$ which means that the Zeno time depends only on the interaction Hamiltonian: $\tau_z^{-2} = \langle H_I^2\rangle$. In theory, if $t' \to 0$ then the probability of transition tends quickly to zero (quadratic factor of time) and the system ”freezes” in the initial state, after the first measurement.

Now if a set number of N measurements take place in the system during a time period t, each one following successively after a time interval $\frac{t}{N}$, the probability that there is still no transition after the Nth measurement is

$$P_N(t) = P_1(t')P_2(t')P_3(t') \ ... = |\langle\psi(0)|\psi(t')\rangle|^{2N} \tag{3.5}$$

because each measurement projects the system onto the state of a particular outcome (Von Neumann measurements) and after every measurement the time evolution of Eq. (3.1) starts anew. The term measurement in this context should be understood as interactions with an external system that disturb the unitary evolution of the quantum system in a way that is effectively like a projection operator. By assuming a small time interval between each measurement the probability becomes

$$P_N(t) = \left[1 - \left(\frac{t}{N\tau_z}\right)^2\right]^N \approx 1 - \frac{t^2}{N\,\tau_z^2} \tag{3.6}$$

It is apparent that frequent measurements imply a better no-transition probability, to the limit of unit probability.

$$\lim_{N\to\infty} P_N(t) = 1 \tag{3.7}$$

The effect of no-transition under the particular measurement setting is called the *Quantum Zeno Effect*. The measurement setting however requires the ability of the observer to make an infinite series of extremely precise and frequent measurements, something that is practically unattainable. This theoretical situation can be called in that matter *ideal*, to the extend that a sufficient demonstration of the effect is considered to be the slowing of the time evolution, as opposed to the complete «freezing».

A question arises whether or not the result can be applied if the measurements happen within the "exponential law" window. It is established that the probability of transition (for example, the spontaneous radioactive decay) is described by the exponential law

$$N(t) = N(0)\,e^{-\lambda t} \tag{3.8}$$

$$P(t) = e^{-\lambda t} \tag{3.9}$$

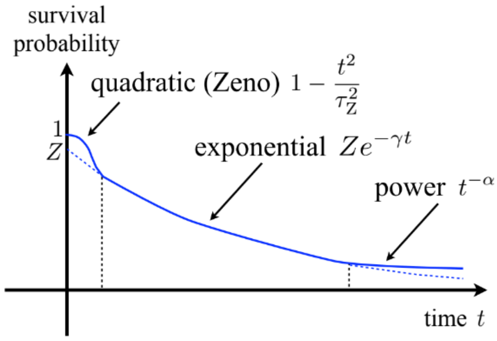

Figure 3.1: Survival probability (no-transition) of an unstable quantum system. [10]

where the constant λ depends on the properties of the nucleus. However, the above law can be regarded as classical, because for small times the probability shows a linear time dependency

$$P(t) \approx 1 - \lambda t \tag{3.10}$$

which contradicts the previous quantum mechanical result of the quadratic time dependency[2] (Eq. 3.6). A similar contradiction can be also shown for very long times (Fig. 1).

Also, in the case of the exponential decreasing probability, further measurements does not seem to effect the decaying system as QZE predicts, because

$$P_N(t) = [e^{-\frac{t}{N}}]^N = e^{-t} \tag{3.11}$$

In conclusion, the Quantum Zeno Effect arises only when the measurements in the system happen within the quadratic period, as defined by Eq. (3.6).

Note 5. The quadratic window is different for each system, and only depends on τ_z. However frequent measurements seem to occur in many situations, especially when one studies an *unstable* system, let's say a decaying particle or atom inside a bubble chamber.

[2] Another way to showcase that this quadratic time dependence is a rigorous result of any quantum theory, is by using the *Fleming's Rule*

$$P(t) \geq \cos^2 (\Delta E \, \frac{t}{\hbar})$$

which is derived by only using the Cauchy–Schwarz inequality and the Heisenberg equation of motion.

As a charged particle flies through a metastable liquid, it ionises its molecules, creating a small bubble in the process. The bubble spreading adds an uncertainty in the particle's position, around the order of the molecule's diameter. The path of the particle creates a track of bubbles that can each be considered as individual measurements of its position. For the QZE to apply in that case the bubble track must be dense enough, so the time between the appearance of each bubble must be comparable to τ_z, which is proportional to the inverse energy difference of the decayed-undecayed states. For the case of the neutron decay, that difference is of the order of some MeVs. Considering the uncertainty relation $\Delta t \, \Delta E \sim \hbar$ then the timescale between each ionisation must be around $10^{-20}s$. The neutron on that time travels only about the distance of 1000 nucleus diameters, so that means an extreme bubble track density must be observed, for the QZE to take place. That, as expected, is not experimentally achievable in a common bubble chamber, so no «state freeze» is observed and the particle is decaying normaly under Fermi's Golden Rule.

The arguement is the same for a decaying nucleus that is interacting with the environment. One can say that such interactions can trigger *decoherence* effects (as described in Subsection 2.2.1) which can be counted as a form of a measurement. Since the nucleus is classicaly decaying as expected, it is safe to say that these "measurements" do not happen that often, for the Zeno effect to apply.

3.2 The General Approach

The Zeno effect can also be shown for more general, unstable systems, mainly using the density matrix formulation. [1] Let an unstable system be prepared in a state ρ at t=0. The Schrödinger evolution by a unitary transformation still stands, so considering Eq. the probability that the system will be found to not be decayed when a measurement is conducted at the time t is

$$q(t) = Tr[\rho P_E(t)] \tag{3.12}$$

where $P_E(t) = U(t)^* P_E \, U(t)$ is the time evolution of the projection operator P_E for the initial, "undecayed" state. Correspondingly, the probability that at a instant t the system will be found after a measurement to be decayed is

$$p(t) = 1 - q(t) = Tr[\rho(I - P_E(t))] = Tr[\rho \, U(t)^* P_E^{\perp} \, U(t)] \tag{3.13}$$

Apart from the above standard probabilities that refer to a specific time instant, one may also consider another set of similar probabilities that refer to a time *interval*

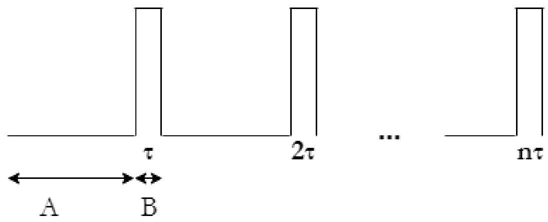

Figure 3.2: Time intervals of a system evolution during a set of measurements. In the case of the QZE we consider the limit $\tau \to 0$ and $n \to \infty$.

instead. These are:

(i) The probability $\mathcal{P}(t, \rho)$ that the system is found to be decayed sometime during the interval $\Delta t = [0, t]$.

(ii) The probability $\mathcal{Q}(t, \rho)$ that no decay is found during the interval $\Delta t = [0, t]$ (also known as survival probability).

(iii) The probability $\mathcal{R}(t, \rho)$ that the system is found undecayed throughout the interval Δt but is found to decay sometime during the subsequent period $[t, t'] = \Delta t'$.

Additionally, by definition

$$\mathcal{Q} + \mathcal{P} = 1 \tag{3.14}$$

Formally, it is assumed that each measurement takes place in an instant (ideal measurement). [Fig. 3.2] The total interval is split between these «instant» time intervals (B) that the measurement occurs and the time intervals (A) that the system is freely evolving under the Schrödinger evolution. During the measurement it is assumed that the evolution of the system is governed by the *Projection Postulate*

$$\rho' \to P_E \, \rho \, P_E \tag{3.15}$$

This assumption, which amounts to a complete absence of coherence between certain states (and replacing the smooth Schrodinger evolution by "quantum jumps"), will lead directly to the Zeno effect, as it will be shown next. The new state of the system after the first measurement undergoes a Schrödinger evolution before the next measurement

$$\rho' \to U\rho' U^* \tag{3.16}$$

then by the next measurement the state changes to

$$\rho'' \to P_E U P_E \; \rho \; P_E U^* P_E \tag{3.17}$$

and so on. As a result the final state after n-measurements, in which the system remained undecayed (the projection is always done by the operator P_E) throughout the time interval, will be

$$\rho(n,t) = T_n(t) \; \rho \; T_n^*(t) \tag{3.18}$$

where

$$T_n(t) = \left[P_E \; e^{-\frac{i}{\hbar n}Ht} P_E\right]^n \tag{3.19}$$

with that, the probability $\mathcal{Q}$ can be written as

$$\begin{aligned}\mathcal{Q}(n,\Delta t,\rho) &= Tr[\rho(n,t)P_E] = Tr\big[T_n(t)\;\rho\;T_n^*(t)P_E\big] \\ &= Tr\big[T_n(t)\;\rho\;P_E^n\;e^{\frac{i}{\hbar}Ht}P_E^nP_E\big] = Tr\big[T_n(t)\;\rho\;P_E^n\;e^{\frac{i}{\hbar}Ht}P_E^{n-1}P_E^2\big] \\ &= Tr\big[T_n(t)\;\rho\;P_E^n\;e^{\frac{i}{\hbar}Ht}P_E^{n-1}P_E\big] = Tr\big[T_n(t)\;\rho\;P_E^n\;e^{\frac{i}{\hbar}Ht}P_E^n\big] \\ &= Tr\big[T_n(t)\;\rho\;T_n^*(t)\big]\end{aligned} \tag{3.20}$$

Now the limit of this probability, at $n \to \infty$ can be calculated. Assuming the limits

$$\lim_{n\longrightarrow\infty} T_n(t) = T(t)$$

and

$$\lim_{n\longrightarrow\infty} \rho(n,t) \overset{\text{norm.}}{=} \frac{T(t)\rho T^*(t)}{Tr[T(t)\rho T^*(t)]} = \rho(t)$$

exist[2], then

$$\mathcal{Q}(\Delta t,\rho) = \lim_{n\longrightarrow\infty} \mathcal{Q}(n,\Delta t,\rho) = Tr[\rho\; T^*(t)T(t)] \tag{3.21}$$

therefore, by Eq.3.14 and setting the initial state of the system to be undecayed (which implies that $Tr[\rho P_E] = 1$) the non-survival probability is given by

[2]these assumptions can be proven to be correct by using set of theorems provided by Misra and Sudarshan [1].

$$\mathcal{P}(\Delta t, \rho) = Tr[\rho P_E] - Tr[\rho T^*(t)T(t)] \tag{3.22}$$

It can be proven mathematically that under some additional assumptions, the relation $T^*(t)T(t) = P_E$ can be applied to Eq.3.22, resulting in

$$\mathcal{P}(\Delta t, \rho) = 0$$

which is the exact outcome described by the *Quantum Zeno Effect*.

Chapter 4

Application in a Three Level System

The claims of the previous section can be supported by a simple test of the of the initial system state under the hard and continuous measurement conditions. This can be achieved easier in a system of induced transitions, rather than spontaneous decays. The standard for many modern day quantum experiments is the use of isolated ions, confined in a magnetic field and subsequently laser-cooled. This allows longer period measurements and cleaner results, without any serious perturbations. In a way, the magnetic field that is used for the "isolation trap" does not act as an environment, and does not render the system as an open system, if the measurements are made with r.f. pulses.

4.1 Theory

The simplest system for that purpose is the two-level, ground and excited level system. However, in order to be able to "measure" efficiently in which of the two states the system is at any given time, a third uncoupled state is introduced.

The original proposal by Richard J. Cook [2] consisted of a single atom three-level system, with a ground state as level-1, an excited state as level-3 and a excited metastable state (or superposition state) as level-2 (Figure 4.1). The prohibition of transition, as described by the QZE, will be studied for the $1 \to 2$ transitions.

Starting at the ground state, an on-resonance square pulse (known as the π-pulse) with a Rabi frequency $\Omega = (E_2 - E_1)/\hbar$ and duration π/Ω, is applied. As a result of the pulse, the system is driven in a coherent superposition state of level-1 and 2, with the latter being the excited state (we ignore level-3 for now). The probability

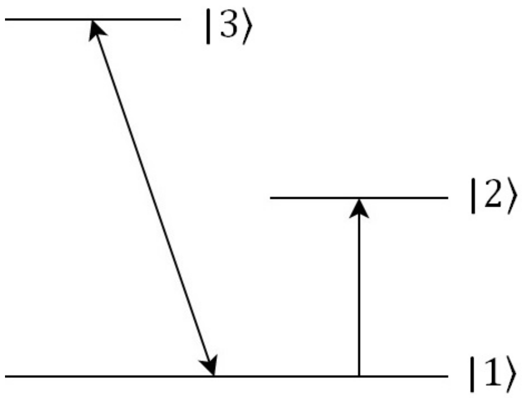

Figure 4.1: The Three Level System (also referred as the V-configuration)

of finding the system in level-2 can be formally calculated. [7]

Since an atomic system cannot be a "pure" two level (or three level) system, as it has an infinite amount of possible excited levels, we must take into consideration the *weak field approximation*, which means that the laser beam must operate in lower frequencies (for example in radio frequencies). The Hamiltonian in that case is written as

$$\mathcal{H}(t) = \frac{1}{2}E_0\sigma_z + V_0\sigma_x\cos(\omega t) \tag{4.1}$$

where $E_0 = E_2 - E_1$, V_0 is the strength of the interaction and ω the field frequency. Next, we write the superposition state

$$|\psi(t)\rangle = A(t)|1\rangle + B(t)|2\rangle \tag{4.2}$$

and plug it in the Scrödinger Equation, so

$$i\hbar[\dot{A}(t)|1\rangle + \dot{B}(t)|2\rangle] = \left[\frac{1}{2}E_0\sigma_z + V_0\sigma_x\cos(\omega t)\right][A(t)|1\rangle + B(t)|2\rangle] \tag{4.3}$$

or by multiplying by $\langle 1|$,

$$i\hbar\dot{A}(t) = \frac{1}{2}E_0A(t)\langle 1|\sigma_z|1\rangle + V_0\cos(\omega t)B(t)\langle 1|\sigma_x|1\rangle \tag{4.4}$$

and denoting $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, $|2\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ the above yields

$$i\hbar\dot{A}(t) = -\frac{1}{2}E_0 A(t) + V_0 \cos(\omega t) B(t) \tag{4.5}$$

and by taking the $\langle 2|$ product instead,

$$i\hbar\dot{B}(t) = \frac{1}{2}E_0 A(t) + V_0 \cos(\omega t) A(t) \tag{4.6}$$

by setting new amplitudes

$$A(t) = e^{iE_0 t/2\hbar} a(t) \tag{4.7}$$

$$B(t) = e^{-iE_0 t/2\hbar} b(t) \tag{4.8}$$

the equations can be written in the form

$$i\hbar\dot{a}(t) = \frac{V_0}{2}(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t})b(t) \tag{4.9}$$

$$i\hbar\dot{b}(t) = \frac{V_0}{2}(e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t})a(t) \tag{4.10}$$

where $\omega_0 = E_0/\hbar$. In order to solve the above differential equations, one must apply the *rotating wave approximation*, which means that the slower varying exponential term $e^{i(\omega-\omega_0)}$ is ruled out. With that in mind, for the precise resonance case ($\omega = \omega_0$) the equations become

$$i\hbar\dot{a}(t) = \frac{V_0}{2}b(t) \tag{4.11}$$

$$i\hbar\dot{b}(t) = \frac{V_0}{2}a(t) \tag{4.12}$$

combining the above, it is easy to see that

$$\ddot{a}(t) + \left(\frac{\Omega}{2}\right)^2 a(t) = 0 \tag{4.13}$$

in which the Rabi frequency is defined as $\Omega = V_0/\hbar$. Solving the above,

$$a(t) = c_1 \cos(\Omega t/2) + c_2 \sin(\Omega t/2) \tag{4.14}$$

and

$$b(t) = \frac{2i\hbar}{V_0}\dot{a}(t) = i[-c_1 \sin(\Omega t/2)\cos(\Omega t/2) + c_2 \cos(\Omega t/2)] \tag{4.15}$$

substituting the results to the original superposition relation, we are left with the expression

$$|\psi(t)\rangle = e^{i\omega_0 t}[c_1 \cos(\Omega t/2) + c_2 \sin(\Omega t/2)]\,|1\rangle + ie^{-i_0 t/2}[-c_1 \sin(\Omega t/2) + c_2 \cos(\Omega t/2)] \tag{4.16}$$

applying the initial condition $|\psi(0)\rangle = |1\rangle$, we find $c_1 = 1, c_2 = 0$ and

$$|\psi(t)\rangle = e^{i\omega_0 t} \cos(\Omega t/2)\,|1\rangle - ie^{-i_0 t/2} \sin(\Omega t/2)\,|2\rangle \tag{4.17}$$

Finally we are able to calculate the probability of finding the system in level-1 (P_1) and in level-2 (P_2)

$$P_1(t) = \cos^2(\Omega t/2) = \frac{1}{2} + \frac{1}{2}\cos(\Omega t) \tag{4.18}$$

$$P_2(t) = \sin^2(\Omega t/2) = \frac{1}{2} - \frac{1}{2}\cos(\Omega t) \tag{4.19}$$

this result means that the system oscillates between level-1 and level-2 during the application of a short frequency signal. We can check that for a very short pulse the probabilities become

$$P_1(t) \approx 1 \tag{4.20}$$

$$P_2(t) \approx \frac{\Omega^2}{4} t^2 \tag{4.21}$$

The quadratic time relation of P_2 stands in agreement with the Zeno effect condition that was discussed in the previous section, providing a good indication that we are following the correct thought process.

It is also important to note that by selecting the correct pulse duration, the system can be found with full certainty in either of the levels. As mentioned in this case, the selected duration of π/Ω puts the system after the π-pulse application in the $|2\rangle$ state. After the π-pulse application the system remains in the $|2\rangle$ state, as the probability for spontaneous decay to the ground state is considered negligible.

The role of the third level in this setup, is to make us able to determine the state of the system *during* the duration of the π-pulse. The transition $1 \to 3$ is done by applying different laser radiation of higher frequencies (optical transition). Additionally, the $3 \rightarrow 1$ transition is strongly allowed and is the only allowed decay of the system in the $|3\rangle$ state.

In other words, in the proposed single-atom system, the optical transition is achieved by a single-photon resonance radiation, that is absorbed by the $|1\rangle$ state system for the $1 \rightarrow 3$ transition, and then re-emitted by the instand $3 \rightarrow 1$ decay. The result is a photon scattering that can be experimentally observed, and when it does, can confirm that the system is indeed in the ground state. If the system is at the $|2\rangle$ state during the resonance radiation, there will be no $2 \rightarrow 3$ transition, so no scattering will occur.

In conclusion, the optical pulse is in fact an efficient way to measure the state of the initial two level system. In theory, if the optical pulses become frequent enough, then the QZE will manifest, prohibiting the $1 \rightarrow 2$ transition and despite the constant π-pulse application, the resonance photons will always scatter.

The expected result can also be theoretically verified by applying the *Bloch Sphere* representation in the two level system. As shown in Section 1 the state evolution of a system can be described by a Bloch vector $\vec{s}$ with the geometrical components

$$s_x = \rho_{12} + \rho_{21} \quad , \quad s_y = i\left(\rho_{21} - \rho_{12}\right) \quad , \quad s_z = \rho_{22} - \rho_{11} \tag{4.22}$$

that define the initial state (ground state) as a downward vector projected in the z-axis and the excited state as an upward vector projected in the same axis

$$|1\rangle = (0, 0, -1) \quad , \quad |2\rangle = (0, 0, 1)$$

as we already deduced from the Eq. (4.18) and Eq. (4.19) the state of the system oscillates between $|1\rangle$ and $|2\rangle$ with a frequency of $\omega = \Omega$, *when no measurements are performed.* In the Bloch sphere that can be geometrically presented as the circular motion of the state vector, as written in Eq. (4.17), with half-period of π/Ω (starting at (0,0,-1)) in the z-y plane, which means that

$$s_z = -\cos(\Omega t) \tag{4.23}$$

$$s_y = \sin(\Omega t) \tag{4.24}$$

$$s_x = 0 \tag{4.25}$$

Now, as mentioned before in Section 2, at the moment of measurement the coherent components of the density matrix fall quickly to zero[1]. In this case, the measurement

[1]For this proposal, the use of the *projection postulate* is necessary, which follows that a reduction of the wave packet happens during the optical transition.

performed by the optical pulse will diminish the ρ_{12} and ρ_{21} parts of Eq. (4.22) and leave the diagonal parts unchanged, which will evidently project the state vector at the z-axis the moment of measurement

$$\vec{s}_{first\ measurement} = (0, 0, -cos(\Omega t))$$

after the (first) measurement the state vector will continue the evolution as before, with a circular motion in the Bloch sphere that is, albeit with a reduced magnitude by a factor of $\cos(\Omega t)$.

Now, if a series of n measurements are performed in succession, each one at time $t = \frac{\pi}{n\Omega}$, the final state vector after the nth measurement will be

$$\vec{s}_n = (0, 0, -\cos^n(\pi/n)) \tag{4.26}$$

combining that result with Eq. (4.22) and substituting the matrix elements with the probabilities

$$\rho_{ii} = \langle i|\psi\rangle \langle\psi|i\rangle = P_i$$

we are left with

$$P_2\Big(t = \frac{\pi}{\Omega}\Big) = \frac{1}{2}(1 - \cos^n(\pi/n)) \approx \frac{1}{2}\Big(1 - e^{-\frac{\pi^2}{2n}}\Big) + \mathcal{O}\Big(\frac{1}{n^3}\Big) \xrightarrow{n\to\infty} 0 \tag{4.27}$$

which indeed shows that under infinite measurements the $|1\rangle$ state will never make the $1 \to 2$ transition.

4.2 The IHBW Experiment

For the actual experiment, *Itano et al.* [2] used an almost identical setup as the one proposed in the previous section. They confined a small amount of $^9\mathrm{Be}^+$ (around 5000 ions) in a *Penning Trap.*[6] The three energy levels used where the two hyperfine energy levels of the $2s^2S_{1/2}$ ground state

$$|m_I, m_J\rangle = \left|\frac{3}{2}, \frac{1}{2}\right\rangle \text{ and } \left|\frac{1}{2}, \frac{1}{2}\right\rangle$$

[6] This device uses a homogeneous axial magnetic field and an inhomogeneous quadrupole electric field in order to store charged particles and ions for precision measurements.Recently these traps have been used in the physical realization of quantum computation and quantum information processing by trapping qubits. *[Source: "Penning Trap | ALPHA Experiment". alpha.web.cern.ch."]*

as $|1\rangle$ and $|2\rangle$ respectively, seperated by a 0.8194 T magnetic field and the sub-level of the $2p^2P_{3/2}$ state as level 3

$$|3\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle$$

Initially, the system is prepared at the $|1\rangle$ state by optical pumping with a 313-nm laser generated radiation for about 5s. After the preparation, the laser is turned

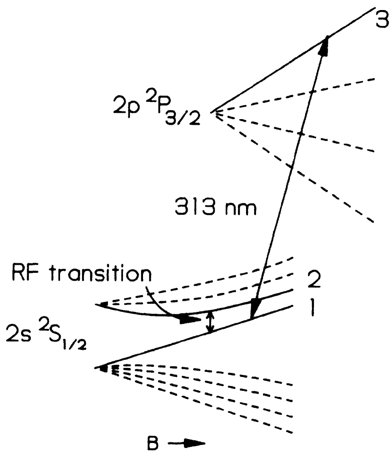

Figure 4.2: Diagram of the energy levels of $^9Be^+$ in a magnetic field B. [2]

off and the π-pulse is applied (320.7 MHz r.f. field with a duration of 256 ms, and adjusted wavelength and amplitude to achieve the on-resonance of the specific system). During that time the system is making the $1 \rightarrow 2$ transition, and in the 256 ms mark, the whole population is exactly at the $|2\rangle$ state.

The procedure is then repeated, with the additional application of a series of n very short pulses (2.4 ms duration with intermediate delay of (T/n-1.3) ms) with 313 nm wavelength, that induced the fluorescence between levels 1 and 3.

After that, the r.f. pulse was turned off and the 313 nm radiation was turned on in order to prepare the system once again and calibrate the the signal, by counting the detected photons and comparing them with the expected amount of ions at a given time[7], as calculated by Eq. (4.18). Note that the ions were continuously supercooled by a 280 nm laser beam (which does not interact with the system in any other way).

Then, as described in the previous Section, the r.f. pulse was turned on for the same duration of 256 ms and the 313 nm was applied at the same time in short pulses with the help of an electromagnetic shatter (Fig.4.3)

[7]Because in this experiment there is no single ion system, there is no 1-1 match between scattered photons and ions. It was estimated that about 72 photons scattered per ion per pulse.

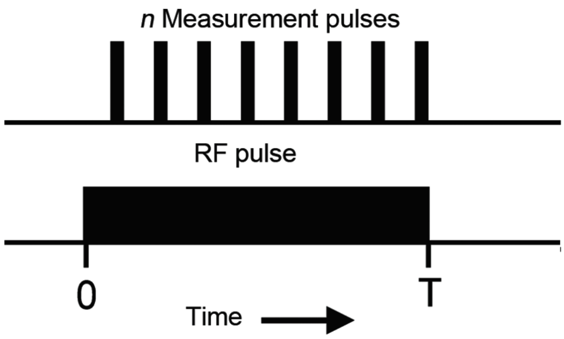

Figure 4.3: Timing of the radio frequency and optical fields applied to beryllium ions in the IHBW experiment. Short 'measurement' pulses resonant with the $1 \leftarrow 3$ transition interrupt the 1-to-2 transition driven by a single, long RF pulse. [11]

As stated in the previous chapter, shorter measurement pulses will in principle provide better results, because in our theoretical description we assumed *instant* and *definite* measurements. However this is not true in a realistic experimental procedure of this kind. The ions are confined in a cylinder (the Penning Trap) with 1000 μm height and $350 \mu m$ radius, while the beam is focused in a 50 μm radius. This means that if the pulse is not long enough, some ions may not get through the beam (since they are supercooled).

A detector monitored the scattered photons for each run. The results of Table are shown graphically in Fig. 4.4. Some corrections were applied to the theoretical predictions of Eq. (4.27) due to optical pumping, the finite measurement pulses and the fluctuations in the value of the Rabi frequency $\Omega = \pi/T$.

Table 4.1: Experimental Data for the $1 \rightarrow 2$ transition. [2]

n	Calculated	Corrections	Observed
1	1	0,995	0,995
2	0,5	0,497	0,5
4	0,375	0,351	0,335
8	0,234604979	0,201	0,194
16	0,13343328	0,095	0,103
32	0,071561516	0,034	0,013
64	0,037118617	0,006	-0,006

The general decrease of the probabilities and the good agreement with the experimental data, demonstrates the Quantum Zeno Effect.

Note 6. The data seem to fall a bit shorter that the predictions (even with the appropriate corrections), and the difference seems to increase with n. That is explained through one particular assumption that was made in the beggining, and

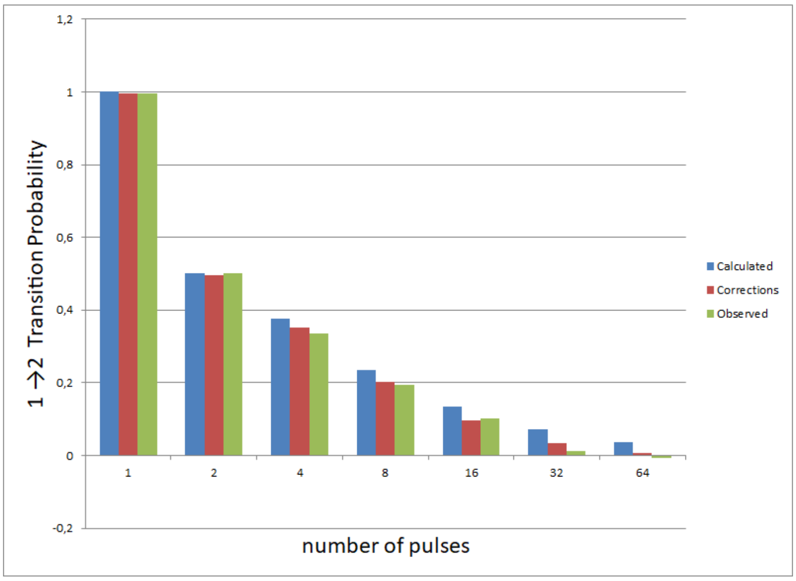

Figure 4.4: Probability of making the $1 \to 2$ transition as a function of n number of optical 'measurement' pulses.

that was the neglect-able spontaneous $2 \to 1$ transition. In reality that transition has a non zero probability, that is enhanced due to the optical pumping of the 313 nm beam. Itano et al. repeated the experiment but this time had an the initial state be at the level 2, in order to monitor the prohibition of the $2 \leftarrow 1$ transition with the same method. The final results had a bit higher probabilities for n=32 or 64, thus confirming the suspicion.

4.3 QZE without the Projection Postulate

The way we derived the final formula of Eq. (4.27) for the transition probability, was by writing down the Bloch equations for the *two level* system, and include the third level as the "projective measurement device" that enables the use of the projection postulate, in order to zero the coherence terms of the $1 \to 2$ transition.

There is an alternative to that method [12], one that does not require the specific use of the projection postulate, but still concludes in the zeroing of the coherences. For this, instead of studying the two-level Bloch representation, we will include the third level formally into the total system, and write down the three level Bloch equations, which can be derived systematically from the interaction of a multilevel atom with the quantized electromagnetic field (with the use of the rotating wave approximation).

$$\dot{\rho}_{33} = -ia(\rho_{13} - \rho_{31}^*) - \gamma\rho_{33} \tag{4.28}$$

$$\dot{\rho}_{31} = ia(\rho_{11} - \rho_{33}) - i\beta\rho_{32} - \gamma\rho_{31} \tag{4.29}$$

$$\dot{\rho}_{22} = -i\beta(\rho_{21} - \rho_{21}^*) \tag{4.30}$$

$$\dot{\rho}_{21} = i\beta(\rho_{13} - \rho_{31}^*) - i\rho_{32}^* \tag{4.31}$$

$$\dot{\rho}_{11} = -(\dot{\rho_{33}} - \dot{\rho_{22}}) \tag{4.32}$$

$$\dot{\rho}_{32} = ia\rho_{21}^* - i\beta\rho_{31} - \frac{\gamma}{2}\rho_{32} \tag{4.33}$$

where a is the optical pulse frequency, β the r.f. frequency and γ the optical transition rate (the transition rate from level 2 has been set to zero). With the use of this formalism, one can study the equations without having to define which system acts as a measurement, or even assume ideal measurements that are needed for the projection postulate. The above can be reduced

$$\dot{\rho}_{33} = 2av_{31} - \gamma\rho_{33} \tag{4.34}$$

$$\dot{v}_{31} = a(\rho_{11} - \rho_{33}) - \beta u_{32} - \frac{\gamma}{2}v_{31} \tag{4.35}$$

$$\dot{\rho}_{22} = 2\beta v_{21} \tag{4.36}$$

$$\dot{v}_{21} = \beta(\rho_{11} - \rho_{22}) - au_{32} \tag{4.37}$$

$$\dot{\rho}_{32} = av_{21} + \beta v_{31} - \frac{\gamma}{2}u_{32} \tag{4.38}$$

where u_{ij} and v_{ij} are the real and imaginary parts of ρ_{ij} Before any measurements are applied, the evolution of the system is free: we set a=0 in the above equations and solve for

$$\begin{aligned} \dot{\rho}_{22} &= 2\beta v_{23} \\ \dot{v}_{21} &= \beta(1 - 2\rho_{22}) \end{aligned} \tag{4.39}$$

for initial state $|1\rangle$, we plot the form of the solution

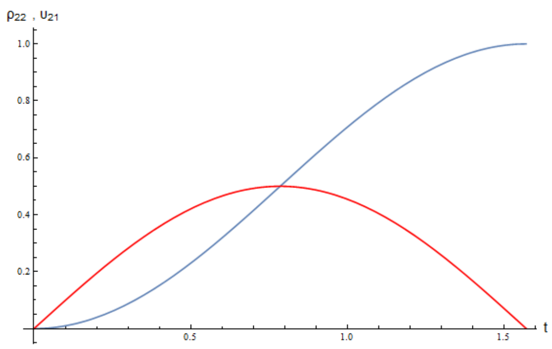

Figure 4.5: Free evolution of the two level system, no optical pulses applied (a=0). Blue line: ρ_{22} population; red line: coherence v_{21}. Units $\gamma = \beta = 1$.

As expected, the solution for the element ρ_{22} is in agreement with Eq. (4.19). The analytical solution, however, of the five independent differential equation system with four parameters (the matrix elements ρ_{ij} is not easily obtainable [13].

A numerical solution [12] is possible, by assuming that during the short optical pulse, the influence of the r.f. pulse is negligible, thus allowing us to set $\beta = 0$ during those time intervals.

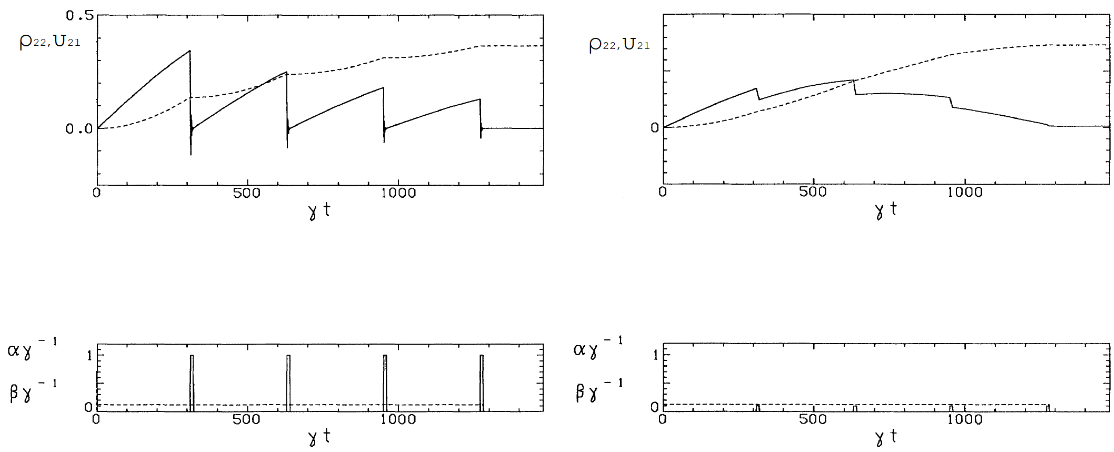

Figure 4.6: (**a**)Left: Temporal evolutionn of population ρ_{22} (dashed) and coherence υ_{21} (solid). Time is given on units γ^{-1} and the pulse duration is $20/\gamma$. (**b**)Right: as (**a**) but with a=0.1 [12].

The results of Fig. (4.6a) show that despite the fact that no assumptions about projective measurements are made, the coherences seem to drop during the optical pulse, and the increase transition probability (as shown in the free evolution) is slowed down. If the weak field approximation is taken for the optical field ($a\gamma^{-1}$ small) then the coherences will not drop to zero after the first optical pulse (Fig.(4.6b)) and the result will resemble more to the free evolution.

The results remind us of the discussion of Subsection 2.2.1, where we established that the loss of coherence is not not always instantaneous and depend on the "correlation" between the environment and the system. In this case the environment is considered the optical pulse that was previously referred as the measurement device, and the strong correlation depends on the field strength of the pulse, that determine the loss of coherence.

Chapter 5

Conclusion

The idea of slowing down a time evolution by going through infinite tasks, dates back in ancient times, with the form of the Zeno paradoxes. In the modern era, a theoretical quantum analogue was developed in the context of measurement theory, the Quantum Zeno Effect. An interesting aspect of the effect is that it can be formally explained through the Quantum Measurement Theory, and phenomenologically, without the use of any tools as the density matrix or the projection postulate.

Up until the late 20th century, the QZE could not be properly tested due to technological limitations, that made the manipulation of quantum systems a difficult task. Even a seemingly trivial application in the simplest possible three level system, is proven to be quite complicated in experimental terms and to hide many deeper theoretical concepts that surround the existence of the QZE: its description without the need of the projection postulate, arises questions about the *universality* of the QZE in the quantum theory.

In the recent years, the Quantum Zeno Effect is used through applications of Quantum Zeno Dynamics, i.e. the suppression of the transitions induced by the systems dynamics due to the QZE, in quantum information. In a broad sense, the QZE can be used to protect qubits from decoherence leakage to the environment, by driving the dynamics inside a designed region of the full space of accessible states, in which the noise cancels out [14]. Other examples include the control of quantum tunneling effects in ultracold lattice gases, due to constant photon imaging [15].

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