THE BLACK HOLE INFORMATION PARADOX

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Περίληψη

Σε δύο πολύ σημαντικές δημοσιεύσεις ο Hawking κατάφερε να αποδείξει οτι οι μαύρες τρύπες εκπέμπουν ακτινοβολία, η οποία πια ονομάζεται "Ακτινοβολία Hawking" , και ότι η εξαύλωση μιας μαύρης τρύπας οδηγεί σε μη αναστρέψιμη απώλεια Πληροφορίας. Το πρόβλημα έμεινε από τότε ως το "Παράδοξο της Πληροφορίας" και εδώ και 35 χρόνια καμία πλήρως ικανοποιητική λύση δεν έχει προταθεί. Σκοπός αυτής της πτυχιακής εργασίας είναι να παραθέσει μια περιγραφή της διαδικασίας με την οποία χάνεται η πληροφορία κατά την εξαΰλωση μιας μαύρης τρύπας. Για το σκοπό αυτό, θα κάνουμε αρχικά μια εισαγωγή στα απαραίτητα μαθηματικά εργαλεία της Κβαντικής Θεωρίας Πεδίου. Έπειτα, εφαρμόζοντας όσα μάθαμε σε καμπυλωμένους χωροχρόνους, θα συνάγουμε τη θερμοκρασία Hawking για μια μαύρη τρύπα. Σημαντικές έννοιες από τη θεωρία Κβαντικής Πληροφορίας θα παρουσιαστούν σε αυτό το σημείο. Τέλος, χρησιμοποιώντας την "εικονική προσέγγιση" για τη δημιουργία σωματιδίων κοντά στον ορίζοντα γεγονότων, θα δείξουμε ότι η τελική κβαντική κατάσταση του συστήματος είναι mixed, αποδεικνύοντας έτσι ότι η Πληροφορία χάνεται. Συμπερασματικά, σημαντικές ιδέες που γεννήθηκαν μέσα από την προσπάθεια επίλυσης του παραδόξου, όπως η αρχή της Black Hole Complementarity, θα αναφερθούν.

ABSTRACT

In two very important papers, Hawking was able to show that Black Holes emit radiation now referred as “Hawking Radiation”, and that the evaporation of a Black Hole leads to irreversible Information Loss. The problem was named as the “Information Paradox” and for over 35 years no satisfactory solution has been suggested. The goal of this thesis is to provide a description of the Information Loss process in Black Hole Evaporation. For this purpose, we will first introduce the necessary mathematical background from Quantum Field Theory. Then, by applying the rules of Quantum Field Theory in curved spacetime ,we will derive the Hawking Temperature for a Black Hole. Important notions and ideas from Quantum Information will also be presented at this stage. Finally, using an “eikonical approximation” of how particles are created near the Black Hole Horizon, we will show that the final state of the system is mixed , hence proving Information Loss. As a conclusion , important ideas that came forth in order to to make sense of the situation, like Black Hole Complementarity, will also be mentioned.

Contents

Chapter 1

Quantum Field Theory for Spinless Particles

In this chapter we will present a brief introduction of Quantum Field Theory on Minkowski space-time. Note however that we will only refer to spinless particles, which means we will not examine the case of quantizing the Dirac Field, but only the Klein - Gordon Field. Although the following presentation will not be by any means complete, it will be extremely useful as a guide to the more complex Quantum Field Theory for curved space-time , which is the essential tool we need to derive the Hawking Radiation.

1.1 Scalar Field

Let us consider a relativistic scalar field defined at all points $(t, \overrightarrow{x}) = (t, x, y, z) = \left(x^0, x^1, x^2, x^3\right)$ of a 4^{th}dimensional Minkowski space-time. Let us suppose that the field Φ satisfies the Klein - Gordon field equation : $\left(\Box + m^2\right)\Phi = 0$ where $\Box \equiv \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\eta^{\mu\nu}$ is the Minkowskian metric tensor [1] . The Lagrangian density that gives us the Klein- Gordon equation could be the following :

$$\mathcal{L} = \frac{1}{2}\left(\eta^{ab}\frac{\partial\Phi}{\partial x^a}\frac{\partial\Phi}{\partial x^b} - m^2\Phi^2\right)$$

[1] $\eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

The action is :

$$S = \int \mathcal{L} d^4x$$

So, from the principle of least action we have $\delta S = 0$ which gives us the usual Euler- Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \frac{\partial}{\partial x^a} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Phi}{\partial x^a}\right)} \Longrightarrow$$

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \frac{\partial}{\partial x^a} \frac{\partial \mathcal{L}}{\partial \Phi_{;a}} \Longrightarrow$$

$$-m^2\Phi = \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Phi}{\partial t}\right)} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Phi}{\partial x}\right)} + \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Phi}{\partial y}\right)} + \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Phi}{\partial z}\right)} \Longrightarrow$$

$$-m^2\Phi = \frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial z^2} \Longrightarrow$$

$$\left(\Box + m^2\right) \Phi = 0$$

The general real solution to the above equation is of the form :

$$\Phi_{(\vec{x},t)} = \sum_k A_k f_{k(\vec{x},t)} + A_k^\star f_{k(\vec{x},t)}^\star$$

where $\quad f_{k(\vec{x},t)} \propto e^{\imath\left(\vec{k}\vec{x} - \omega t\right)}$ and $\quad \omega = \sqrt{\left(\vec{k}^2 + m^2\right)}$

Now, it is useful to normalize the solution to a box of length L. After normalization the general form will be:

$$\Phi_{(\vec{x},t)} = \sum_{\vec{k}} a_{\vec{k}} \frac{e^{\imath\left(\vec{k}\vec{x} - \omega t\right)}}{\sqrt{L^3}\sqrt{2\omega\left(2\pi\right)^3}} + a_{\vec{k}}^\star \frac{e^{-\imath\left(\vec{k}\vec{x} - \omega t\right)}}{\sqrt{L^3}\sqrt{2\omega\left(2\pi\right)^3}}$$

1.2 Quantization of the Scalar field

The mathematical recipe for quantizing a field $\Phi = \Phi_{(\vec{x},t)}$ is to treat the field Φ as an operator which obeys the following equal time commutation relations:[2]

$$\left[\Phi_{(\vec{x},t)}, \Phi_{(\vec{y},t)}\right] = 0$$

$$\left[\Pi_{(\vec{x},t)}, \Pi_{(\vec{y},t)}\right] = 0$$

$$\left[\Phi_{(\vec{x},t)}, \Pi_{(\vec{y},t)}\right] = \imath\delta^3_{(\vec{x}-\vec{y})}$$

Where $\Pi_{(\vec{x},t)}$ is the canonically conjugate momentum of the field , defined as:

$$\Pi_{(\vec{x},t)} = \frac{\partial \mathcal{L}}{\partial \Phi_{;t}}$$

Note, that the previous commutation relations are in close analogy to the commutation relations of the operators $\hat{\boldsymbol{x}}$,$\hat{\boldsymbol{p}}$ in classical quantum mechanics. Since our Lagrangian[3] is $\mathcal{L} = \frac{1}{2}\left(\eta^{ab}\frac{\partial \Phi}{\partial x^a}\frac{\partial \Phi}{\partial x^b} - m^2\Phi^2\right)$ then we get that :

$$\Pi_{(\vec{x},t)} = \frac{\partial \Phi}{\partial t}$$

The solution of the Klein- Gordon equation that we wrote previously can also be used here:

$$\Phi_{(\vec{x},t)} = \sum_{\vec{k}} \boldsymbol{a}_{\vec{k}} \frac{e^{\imath\left(\vec{k}\vec{x}-\omega t\right)}}{\sqrt{L^3}\sqrt{2\omega\left(2\pi\right)^3}} + \boldsymbol{a}^{\dagger}_{\vec{k}} \frac{e^{-\imath\left(\vec{k}\vec{x}-\omega t\right)}}{\sqrt{L^3}\sqrt{2\omega\left(2\pi\right)^3}}$$

where $\boldsymbol{a}_{\vec{k}}$, $\boldsymbol{a}^{\dagger}_{\vec{k}}$ instead of being complex numbers ,will now be operators. So $\Phi_{(\vec{x},t)}$ is now an operator instead of a function as we have already said. The operators $\boldsymbol{a}_{\vec{k}}$, $\boldsymbol{a}^{\dagger}_{\vec{k}}$ will satisfy the following commutation relations (analogous to the commutation relations of Φ and Π).

$$\left[\boldsymbol{a}_{\vec{k}}, \boldsymbol{a}_{\vec{k}'}\right] = 0$$

$$\left[\boldsymbol{a}^{\dagger}_{\vec{k}}, \boldsymbol{a}^{\dagger}_{\vec{k}'}\right] = 0$$

[2]We assume at this point that the process of quantization is familiar to the reader so that only a brief summary will be presented

[3]Every time we say Lagrangian we will mean Lagrangian density from now on

$$\left[\boldsymbol{a}_{\overrightarrow{\boldsymbol{k}}}, \boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}\boldsymbol{\prime}}\right] = \delta_{\overrightarrow{k}\,\overrightarrow{k}'}$$

The operators $\boldsymbol{a}_{\overrightarrow{\boldsymbol{k}}}$, $\boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}$ are the usual annihilation and creation operators correspondingly.

1.3 Heisenberg Picture and Fock states

It is very convenient to use the Heisenberg picture in our treatment, where every quantum state spans a Hilbert space and remains time- independent[4]. A natural basis in this Hilbert space is the Fock representation. In the Fock representation every normalized basis ket vector can be constructed from the $|0\rangle$ ket or the " vacuum " ket which is defined as :

$$\boldsymbol{a}_{\overrightarrow{\boldsymbol{k}}}\,|0\rangle = 0, \quad for\,all\,\overrightarrow{k}$$

The above equation defines the "vacuum state" which is the state where all the annihilation operators that operate on the Hilbert vector state give zero. Now as we have mentioned $\boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}$ is called the creation operator. This means that if we operate on $\boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}$ on the vacuum state $|0\rangle$ we get a one particle state with momentum $\overrightarrow{k}$, symbolized as $\left|1_{\overrightarrow{k}}\right\rangle$:

$$\left|1_{\overrightarrow{k}}\right\rangle = \boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}\,|0\rangle$$

More generally a many particle state is symbolized as $\left|n^1_{\overrightarrow{k_1}}, n^2_{\overrightarrow{k_2}}, \cdots, n^j_{\overrightarrow{k_j}}\right\rangle$. In this state we have n^1 particles with $\overrightarrow{k_1}$ momentum, n^2 particles with $\overrightarrow{k_2}$ momentum, etc.. It can be shown that the annihilation and creation operations satisfy the following equations:

$$\boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}\left|n_{\overrightarrow{k}}\right\rangle = \sqrt{n+1}\left|(n+1)_{\overrightarrow{k}}\right\rangle$$

$$\boldsymbol{a}_{\overrightarrow{\boldsymbol{k}}}\left|n_{\overrightarrow{k}}\right\rangle = \sqrt{n}\left|(n-1)_{\overrightarrow{k}}\right\rangle$$

One can easily verify that the operator $\boldsymbol{N} = \boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}\boldsymbol{a}_{\overrightarrow{\boldsymbol{k}}}$ has eigenvalues the number of quanta with momentum $\overrightarrow{k}$:

$$N\left|n_{\overrightarrow{k}}\right\rangle = \boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}\left(\boldsymbol{a}_{\overrightarrow{\boldsymbol{k}}}\left|n_{\overrightarrow{k}}\right\rangle\right) = \boldsymbol{a}^{\dagger}_{\overrightarrow{\boldsymbol{k}}}\sqrt{n}\left|(n-1)_{\overrightarrow{k}}\right\rangle = n\left|n_{\overrightarrow{k}}\right\rangle$$

[4]The time dependency in the Heisenberg picture is carried over to the operators

1.4 Hamiltonian and Momentum Operator

In order to find the Hamiltonian and momentum operators one can use the stress- energy momentum tensor of the field defined as:

$$T_{ab} = \Phi_{;a}\Phi_{;b} - \frac{1}{2}\eta_{ab}\eta^{cd}\Phi_{;c}\Phi_{;d} + \frac{1}{2}m^2\Phi^2\eta_{ab}$$

So from the above definition one obtains the Hamiltonian density :

$$T_{tt} = \frac{1}{2}\left[(\partial_t\Phi)^2 + (\partial_x\Phi)^2 + (\partial_y\Phi)^2 + (\partial_z\Phi)^2 + m^2\Phi^2\right]$$

and momentum density :

$$T_{ti} = \partial_t\Phi\partial_i\Phi \qquad i = x, y, z$$

The Hamiltonian[5] is equal to :

$$H = \int T_{tt}d^3x = \frac{1}{2}\sum_{\vec{k}}\left(\boldsymbol{a}^\dagger_{\vec{\boldsymbol{k}}}\boldsymbol{a}_{\vec{\boldsymbol{k}}} + \boldsymbol{a}_{\vec{\boldsymbol{k}}}\boldsymbol{a}^\dagger_{\vec{\boldsymbol{k}}}\right)\omega = \sum_{\vec{k}}\left(\boldsymbol{a}^\dagger_{\vec{\boldsymbol{k}}}\boldsymbol{a}_{\vec{\boldsymbol{k}}} + \frac{1}{2}\right)\omega$$

Similarly the momentum is equal to:

$$P_i = \int T_{ti}d^3x = \sum \boldsymbol{a}^\dagger_{\vec{\boldsymbol{k}}}\boldsymbol{a}_{\vec{\boldsymbol{k}}}\vec{k}$$

1.5 Energy and Momentum of the Vacuum State

Since we have shown the momentum and energy operators, let us compute the momentum and energy of the vacuum state. For the momentum the answer is trivial since:

$$\langle 0|\, P\, |0\rangle = \langle 0| \sum_{\vec{k}} \boldsymbol{a}^\dagger_{\vec{\boldsymbol{k}}}\boldsymbol{a}_{\vec{\boldsymbol{k}}}\vec{k}\, |0\rangle = 0$$

So we see that the vacuum state carries zero momentum as expected. However if we try to compute the energy of the vacuum state we get[6] :

$$\langle 0|\, H\, |0\rangle = \langle 0|\, 0\rangle \sum_{\vec{k}}\frac{1}{2}\omega = \sum_{\vec{k}}\frac{1}{2}\omega$$

[5] We use the commutation relations for $\boldsymbol{a}^\dagger_{\vec{\boldsymbol{k}}}$, $\boldsymbol{a}_{\vec{\boldsymbol{k}}}$

[6] We use the orthonormality relation $\langle 0|\, 0\rangle = 1$

In the above summation there is no upper bound in the magnitude of $\overrightarrow{k}$, so the energy comes out infinite. In flat space-time physics , this infinity of the energy density poses no serious threat because we only measure differences of energies and not total energies. Therefore , we can make the energy density of the vacuum zero by throwing away the $\sum_{\overrightarrow{k}} \frac{1}{2}\omega$ that appears in the Hamiltonian which is the source of the infinity. It is important to note however that when we include gravity in the picture things become more complicate. The energy momentum tensor in Einsteins Field Equations "sees" all energies and not only differences of energies. The vacuum energy divergence is therefore not so easily ignored when we consider curved space-time geometries.

Chapter 2

Hawking Radiation

In this chapter we will examine the notion of “particle” as it appears in Quantum Field Theory. We will then examine the classical notion of the particle notion and the very interesting Unruh effect. Finally, we will derive the very important result Hawking discovered, namely Hawking Radiation .

2.1 The classical notion of Particle and the Unruh Effect

Hawking Radiation is based on the observation that the vacuum state for one observer appears as a non vacuum state for another observer. The notion of particle becomes relative in the sense that it is observer -dependent. This is also true in Flat space-time. Although inertial observers will agree on the same vacuum state, non-inertial or accelerated observers will generally see the same state as filled with particles. This is the famous Unruh Effect , which we will examine more thoroughly shortly.

Since both the Hawking and the Unruh effects are quantum mechanical one would expect that the notion of particle becomes observer -dependent only when we take into account Quantum Field Theory. On the other hand it is interesting to notice that the notion of particle becomes blurred even when we try to combine classical electromagnetic theory with the Equivalence Principle. How exactly does this happen will become clear with the following thought experiment:

Imagine an non- inertial accelerated observer with constant proper acceleration. Now suppose that the observer has a charged sphere and an electromagnetic radiation detector that accelerates with him. Using Maxwell equations one would expect that the charged sphere should radiate electromagnetic radiation (since it is accelerating) so the accelerated detector should pick up an energy flux. So far so good. Let us now examine the point of view of the accelerated observer using the equivalence principle. From his point of view he cannot decide whether he is accelerating or is in a uniform gravitation field e.g the earths gravitation field for small space-time volume. He is feeling the weight of the charged sphere and the detector due to the gravitation field. But as we have already found out with another argument the detector should detect an energy flux which means that stationary charges inside a gravitational field must radiate. This is a rather uncomfortable result since there is not at all clear how a stationary state like that could produce a constant energy flux , which in turn would mean a flux o photons passing through the detector.

To sum up, if we assume the following two propositions :

1. Equivalence Principle and Maxwell equations of Electromagnetism.

2. The number of particles that a detector detects is independent of the motion of the detector.

we deduce a rather absurd result. Obviously we have to abandon at least one of our assumptions and the best candidate is the assumption that the number of particles a detector detects does not depend on its motion.[1] We have therefore seen that even in classical physics there were clear hints that the number of particles a detector sees is not invariant ,but depends on its motion.

2.2 Unruh effect

We come back now to the realm of quantum field theory in which the notion of particle becomes clearly observer dependent. First we will show that an accelerated observer will see a bath of radiation whereas all inertial observers will see the same state as the vacuum state or no particle state. This the famous Unruh effect [1]. Let us first consider for simplicity a two dimensional flat space-time with the metric[2]:

$$ds^2 = dt^2 - dx^2 \tag{2.2.1}$$

We will define the coordinates $\overline{u}$, $\overline{v}$ by :

$$\overline{u} = t - x$$

$$\overline{v} = t + x$$

The line element then becomes :

[1] A detailed explanation of the charge in a gravitation field problem can be found here http://link.springer.com/article/10.1023%2FA%3A1018821619763

[2] c=1

$$ds^2 = dt^2 - dx^2 = d\overline{u}d\overline{v} \tag{2.2.2}$$

Now , consider the following coordinate transformation :

$$t = a^{-1}e^{a\xi}sinh\,(a\eta) \tag{2.2.3}$$

$$x = a^{-1}e^{a\xi}cosh\,(a\eta) \tag{2.2.4}$$

with $a = constant > 0$, $-\infty < (\eta, \xi) < \infty$.

Lets calculate $\overline{u}$, $\overline{v}$ as functions of η, ξ :

$$\overline{u} = t - x = a^{-1}e^{a\xi}\left[sinh\,(a\eta) - cosh\,(a\eta)\right] = a^{-1}e^{a\xi}\left[\frac{e^{a\eta} - e^{-a\eta}}{2} - \frac{e^{a\eta} + e^{-a\eta}}{2}\right] = a^{-1}e^{a\xi}\left(-e^{-a\eta}\right) = -a^{-1}e^{-a(\eta-\xi)}$$

$$\overline{v} = t + x = a^{-1}e^{a\xi}\left[sinh\,(a\eta) + cosh\,(a\eta)\right] = a^{-1}e^{a\xi}\left[\frac{e^{a\eta} - e^{-a\eta}}{2} + \frac{e^{a\eta} + e^{-a\eta}}{2}\right] = a^{-1}e^{a(\eta+\xi)}$$

Defining :

$$u = \eta - \xi$$

$$v = \eta + \xi$$

We get :

$$\overline{u} = -a^{-1}e^{-au}$$

$$\overline{v} = a^{-1}e^{av}$$

So the line element (2.2.2) becomes :

$$ds^2 = d\overline{u}d\overline{v} = -a^{-1}\left(-a\right)e^{-au}du\left(a^{-1}a\right)e^{av}dv \;\Rightarrow$$

$$ds^2 = e^{a(v-u)}dudv = e^{2a\xi}dudv = e^{2a\xi}\left(d\eta^2 - d\xi^2\right)$$

If we examine the transformation relations (2.2.3), (2.3.4) we can see that the coordinates (η, ξ) cover only the portion $x > |t|$ of the Minkowski space-time, called the Rindler wedge. We can also notice that lines of constant η are straight lines since :

$$\frac{t}{x} = \frac{a^{-1}e^{a\xi}sinh\,(a\eta)}{a^{-1}e^{a\xi}cosh\,(a\eta)} = constant \;\Rightarrow$$

$$t \propto x$$

On the other hand lines of constant ξ are hyperbola[3] :

$$x^2 - t^2 = a^{-2}e^{2a\xi}cosh^2\,(a\eta) - a^{-2}e^{2a\xi}sinh^2\,(a\eta) =$$

$$= a^{-2}e^{2a\xi}\left(cosh^2\,(a\eta) - sinh^2\,(a\eta)\right)$$
$$= a^{-2}e^{2a\xi} = constant$$

Lines of constant ξ represent the world lines of uniformly accelerated observers with proper acceleration ϖ :

$$\varpi = \left(a^{-2}e^{2a\xi}\right)^{-\frac{1}{2}} = ae^{-a\xi}$$

All the hyperbola approach the null rays $\overline{u} = 0$, $\overline{v} = 0$ asymptotically , which means that the accelerated observers approach the speed of light as $\eta \to \pm\infty$. The coordinate η is like the time coordinate for every accelerated observer with the exact relation (since $d\xi = 0$) :

$$d\tau^2 = e^{2a\xi}d\eta^2 \;\Rightarrow$$

$$d\tau = e^{a\xi}d\eta \;\Rightarrow$$

$$\tau = e^{a\xi}\eta$$

[3]We will use the known equation : $cosh^2\,(x) - sinh^2\,(x) = 1$

In the following figure we depict lines of constant η and ξ :

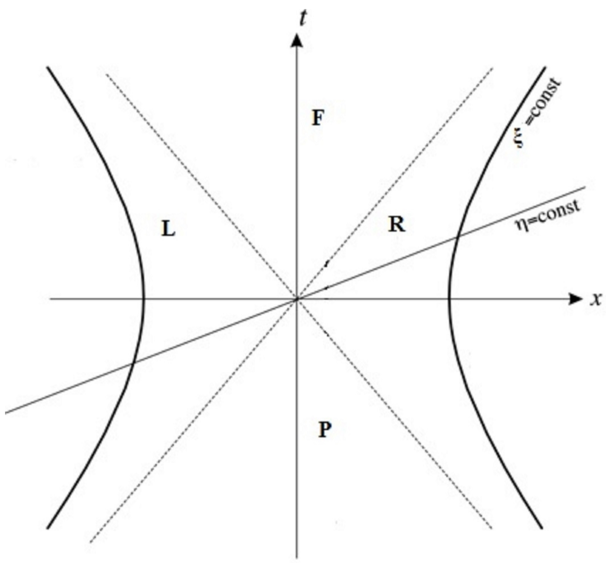

Figure 2.2.1: Rindler's Wedge in Minkowski Spacetime

It is important and quite interesting to understand the causal structure of the Rindler space-time. First, we can construct a second wedge $x < -|t|$ by changing the sign in the transformation relations (2.2.3), (2.2.4) . The $x > |t|$ is called the R region, and the $x < -|t|$ region, the L region. The remaining future and past regions of space-time are called F, and P correspondingly. Since every event in the L region can be connected with events in R region only through a space-like slice, these two regions are like two disjoint universes. Also events in F can't influence events in R, L so that the null rays $t = x$, $t = -x$ act as event horizons.

2.2.1 Field Quantization

Now let us consider the usual quantization of a massless scalar field Φ in two- dimensional Minkowski space time. The wave equation is :

$$\Box\Phi \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\Phi = \frac{\partial^2\Phi}{\partial\overline{u}\partial\overline{v}} = 0 \tag{2.2.5}$$

The above equation possesses the standard orthonormal mode solutions :

$$\overline{u_k} = (4\pi\omega)^{-\frac{1}{2}}\, e^{\imath k\xi - i\eta}$$

with $\omega = |k| > 0$ and $-\infty < k < \infty$.

The field Φ can be expanded as :

$$\Phi = \sum_{k=-\infty,\infty} \left(\boldsymbol{a_k} \overline{u_k} + \boldsymbol{a_k^\dagger} \overline{u_k}^\star \right)$$

where $\boldsymbol{a_k}$ is the usual annihilation operator and $\boldsymbol{a_k^\dagger}$ the creation operator. Let us now solve the wave equation (2.2.5) in Rindler coordinates (η, ξ) :

$$\Box\Phi \equiv e^{-2a\xi} \left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2} \right) \Phi = e^{-2a\xi} \frac{\partial^2\Phi}{\partial u \partial v} = 0$$

This equation has the same form as before and has solutions :

$$u_k = (4\pi\omega)^{-\frac{1}{2}}\, e^{\imath k\xi \pm i\eta}, \quad \omega = |k| > 0,\, -\infty < k < \infty$$

The plus sign applies in region L and the minus in region R. The change of sign is due to the fact that we have different transformation relations for the two wedges R, L .

Let us now define the functions :

$$R_{u_k} = \begin{cases} (4\pi\omega)^{-\frac{1}{2}}\, e^{\imath k\xi - i\eta} & in\ R \\ 0 & in\ L \end{cases}$$

$$L_{u_k} = \begin{cases} (4\pi\omega)^{-\frac{1}{2}}\, e^{\imath k\xi + i\eta} & in\ L \\ 0 & in\ R \end{cases}$$

Although each set is complete for either region R or L neither one is complete in all Minkowski space-time. However, both sets can be analytically continued into regions F an P and can be as a good basis as any other complete basis. We can thus write the field Φ in the new basis :

$$\Phi = \sum_{k=-\infty,\infty} \left(\boldsymbol{b_k^{(1)}} L_{u_k} + \boldsymbol{b_k^{\dagger(1)}} L_{u_k^\star} + \boldsymbol{b_k^{(2)}} R_{u_k} + \boldsymbol{b_k^{\dagger(2)}} R_{u_k^\star} \right)$$

In this basis we can define the vacuum state or Rindler vacuum as :

$$\boldsymbol{b_k^{(1)}} \left|0_R\right\rangle = \boldsymbol{b_k^{(2)}} \left|0_R\right\rangle = 0$$

The question that remains at this point is whether the vacuum state as defined for inertial observers, remains the vacuum state for Rindler observers. One can simply argue that since R_{u_k} do not go smoothly to L_{u_k} from the R region to the L region, and since the Minkowski modes are analytical everywhere, then both operators $\boldsymbol{b_k^{(1)}}$, $\boldsymbol{b_k^{(2)}}$ are linear combinations of the annihilation and creation operators $\boldsymbol{a_k}$, $\boldsymbol{a_k^{\dagger}}$. This means that the vacuum state for an inertial observer is not the vacuum state for the Rindler observer :

$$\boldsymbol{b_k^{(1,2)}} |0\rangle \neq 0$$

Now consider an observer with world line $\xi = constant$. If such an observer is on the $x > |t|$ part of space-time, then he is a Rindler observer and will detect particles determined by the number operator $\boldsymbol{b_k^{\dagger(1)} b_k^{(1)}}$.[4]

If the state is $|0\rangle$ which is the vacuum state for inertial observers then the Rindler observer will detect :

$$\langle 0| \boldsymbol{b_k^{\dagger(1)} b_k^{(1)}} |0\rangle = \left(e^{\frac{2\pi\omega}{a}} - 1\right)^{-1}$$

particles in mode k.[5] This is precisely the Planck spectrum for radiation at temperature :

$$T_0 = \frac{a}{2\pi k_B}$$

So we have found out that Rindler observers will detect a thermal bath of particles in flat space-time. By the same logic one could see an analogous phenomenon, that inertial observers will detect particles in curved space-time.[6] This is the famous Hawking radiation which we will explain shortly after.

2.3 Hawking Radiation

In this section we will derive Hawking radiation following Hawking's original derivation [2]. As Hawking explained in his 1975 paper, we still don't have a complete theory of quantum gravity , so one has the problem of combining the metric which obeys Einstein's Equations and the matter fields which are treated quantum- mechanically. Such a treatment is obviously incomplete but should be a very good approximation for all purposes except near space-time singularities where

[4] The number operator for the L region is $\boldsymbol{b_k^{\dagger(2)} b_k^{(2)}}$

[5] For full derivation see Page 115 of [N._D._Birrell,_P._C._W._Davies]_Quantum_Fields_in_Curved_Spacetime

[6] This is not so obvious but will be shown when we will derive Hawking radiation

the curvature becomes very big. The procedure we will follow in order to derive Hawking radiation is the following:

First, the scalar field Φ will obey the same equation of motion as in flat space-time, except that the Minkowski metric will be replaced by a classical space-time metric g_{ab} obeying Einstein's Equations. As we have already shown in flat space-time, there is a standard procedure of defining the vacuum state for all inertial observers. The massless Hermitian scalar field obeys the equation:

$$\Phi_{;ab}\eta^{ab} = 0$$

and one expresses Φ as:

$$\Phi = \sum_i \left(\boldsymbol{a_i} f_i + \boldsymbol{a_i^\dagger} \overline{f_i} \right)$$

the $\{f_i\}$ form a complete basis and contain only positive frequencies with respect to the usual Minkowski time coordinate. This guarantees that the vacuum state defined by :

$$\boldsymbol{a_i} \left|0\right\rangle = 0, \qquad for\, all\, i$$

is the same for all inertial observers. On the other hand we have also seen that Rindler observers will not agree on the same vacuum (Unruh effect). Following the same recipe in curved space-time, the Hermitian scalar field operator Φ obeys the wave equation :

$$\Phi_{;ab}g^{ab} = 0$$

However one can not decompose Φ into it's positive and negative frequency parts as positive and negative frequencies have no invariant meaning in curved space-time. Therefore if one tries to decompose Φ into it's positive an negative frequencies in one region of space-time it will in general not be true for another region of space-time. Let us suppose on the other hand that we have a time dependent metric which at some region is flat or asymptotically flat. Then the natural choice of $\{f_i\}$ is that they should contain only positive frequencies with respect to the Minkowski time coordinate. This gives us a definition of the vacuum state. If at later times the flat region is followed by a curved region and then a final flat region , the basis which contains only positive frequencies in the initial flat region will not be the same as the basis which contains only positive frequencies in the final flat region. This means that if we start with a vacuum state , it will not remain a vacuum at later times. This is the line of thinking Hawking followed in his original paper in order to show that a time dependent gravitational field will create particles.

2.3.1 Carter- Penrose Diagrams

Before moving on to the derivation of Hawking radiation we will introduce a new way of representing space-time which will be extremely useful for our derivation. The idea is to do a conformal transformation of coordinates which brings the entire manifold into a compact region so that we can fit the whole space-time on a piece of paper. The result of this procedure is the Penrose- Carter Diagram.

2.3.1.1 Minkowski Space-time

Let us first begin with Minkowski space and after that to the Schwarzschild metric. The Minkowski metric[7] in polar coordinates is :

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

The range of t, r coordinates is :

$$-\infty < t < \infty, \quad 0 \leq r < \infty$$

The first step is to switch to null coordinates :

$$u = \frac{1}{2}(t + r)$$

$$v = \frac{1}{2}(t - r)$$

with corresponding ranges : $-\infty < u < \infty\,, \quad -\infty < v < \infty\,, \quad v \leq u$

Each point in the following picture represents a 2- sphere of radius $r = u - v$:

[7]We use the $(-,+++)$metric which is convention in General Relativity

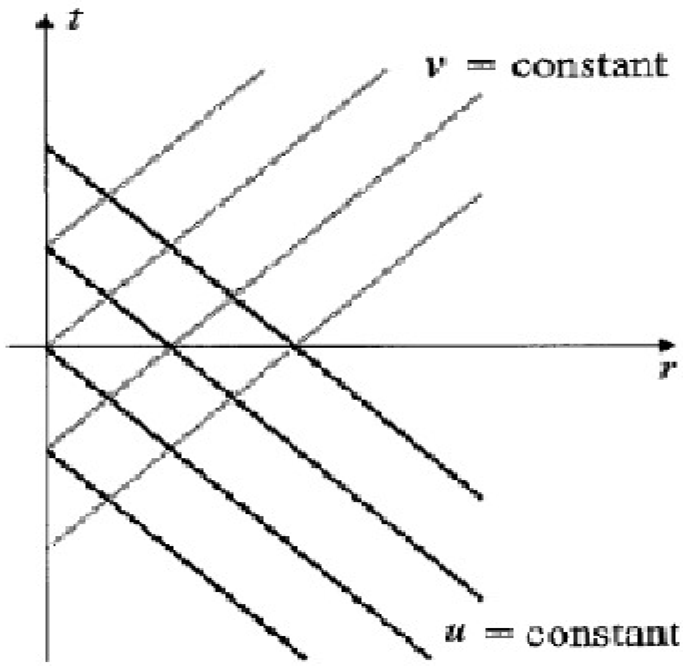

Figure 2.3.1: Null coordinates in Flat Spacetime

The metric in the null coordinates is :

$$ds^2 = -4dudv + (u-v)^2 \, d\Omega^2$$

Now, in order to compact an infinite area to a finite region we make the following transformation:

$$U = arctan\,(u)$$

$$V = arctan\,(v)$$

The ranges are now :

$$-\frac{\pi}{2} < U < \frac{\pi}{2}$$

$$-\frac{\pi}{2} < V < \frac{\pi}{2}$$

$$V \leq U$$

It is fairly easy to get the metric in the U, V coordinates :

$$u = tanU \;\Rightarrow\; du = \frac{1}{cos^2U}\,dU$$

$$v = tanV \;\Rightarrow\; dv = \frac{1}{cos^2V}\,dV$$

$$(u-v)^2 = (tanU - tanV)^2 = \frac{(sinU\,cosV - cosU\,cosV)^2}{cos^2U\,cos^2V}$$

So the metric becomes :

$$ds^2 = \frac{1}{cos^2U\,cos^2V}\;(-4dUdV) + sin^2\,(U-V)\,d\Omega^2$$

The final transformation is :

$$\eta = U + V$$

$$\chi = U - V$$

with ranges :

$$-\pi < \eta < \pi\,,\;\; 0 \leq \chi < \pi$$

Now the metric becomes :

- $$cos\eta + cos\chi = cos\,(U+V) + cos\,(U-V) = cosU cosV - sinU sinV + cosU cosV - sinU sinV = 2cosU cosV \Rightarrow$$

$$\Rightarrow \frac{1}{cos^2U\,cos^2V} = \frac{1}{\left(\frac{cos\eta + cos\chi}{2}\right)^2}$$

- $$-4dUdV = -\left(d\eta^2 - d\chi^2\right) = d\eta^2 - d\chi^2$$

- $$sin^2\left(U - V\right) = sin^2\chi$$

$$ds^2 = \frac{1}{\left(\frac{cos\eta + cos\chi}{2}\right)^2}\left(-d\eta^2 + d\chi^2 + sin^2\chi d\Omega^2\right)$$

Using the transformation relations we can plot a space-time diagram where η is in the vertical axis and χ the horizontal axis.

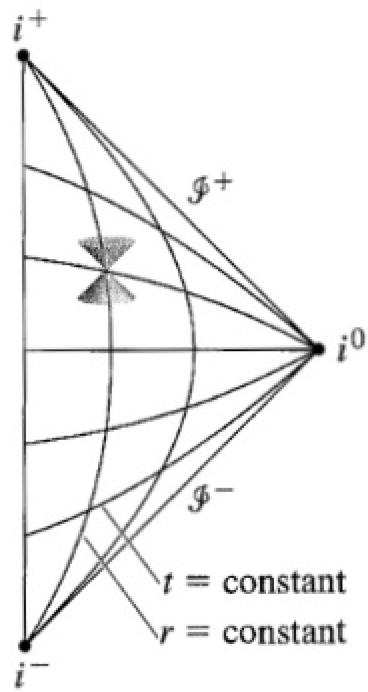

Figure 2.3.2: Penrose Diagram for Minkowski Spacetime

$\imath^+ =$ future timelike infinity

$\imath^0 =$ spatial infinity

$\imath^- =$ past timelike infinity

$\Im^+ =$ future null infinity

$\Im^- =$ past null infinity

The whole Minkowski space-time is depicted to a finite triangle region as shown above. Spacelike curves with $t = constant$ and timelike curves with $r = constant$ are also depicted on the diagram. The reason why these curves are the way they are , can be calculated using the transformation relations :

Spacelike $t = constant$ **curves**

$$u + v = t \Rightarrow tanU + tanV = t \Rightarrow$$

$$tan\left(\frac{\eta+\chi}{2}\right) + tan\left(\frac{\eta-\chi}{2}\right) = t \Rightarrow$$

$$\frac{sin\left(\frac{\eta+\chi}{2}\right)}{cos\left(\frac{\eta+\chi}{2}\right)} + \frac{sin\left(\frac{\eta-\chi}{2}\right)}{cos\left(\frac{\eta-\chi}{2}\right)} = t \Rightarrow$$

$$\frac{sin\left(\frac{\eta+\chi}{2}\right)cos\left(\frac{\eta-\chi}{2}\right) + sin\left(\frac{\eta-\chi}{2}\right)cos\left(\frac{\eta+\chi}{2}\right)}{cos\left(\frac{\eta+\chi}{2}\right)cos\left(\frac{\eta-\chi}{2}\right)} = t \Rightarrow$$

$$\frac{sin\left(\frac{\eta+\chi}{2} + \frac{\eta-\chi}{2}\right)}{cos\left(\frac{\eta+\chi}{2}\right)cos\left(\frac{\eta-\chi}{2}\right)} = t \Rightarrow$$

$$\frac{sin\left(\eta\right)}{\left(\frac{cos\eta+cos\chi}{2}\right)} = t \Rightarrow$$

[8]

$$2sin\eta = t\left(cos\eta + cos\chi\right)$$

This is the equation of the $t = constant$ curve. As we can see for $\eta = 0$ we get $\chi = \pi$ which can be verified on the graph.

Timelike $r = constant$ **curves**

The derivation is quite similar :

$$u - v = r \Rightarrow tanU - tanV = r \Rightarrow$$

[8] We use the identity : $cos\theta cos\varphi = \frac{cos(\theta-\varphi)+cos(\theta+\varphi)}{2}$

$$tan\left(\frac{\eta+\chi}{2}\right)-tan\left(\frac{\eta-\chi}{2}\right)=r\ \Rightarrow$$

$$\frac{sin\left(\frac{\eta+\chi}{2}\right)}{cos\left(\frac{\eta+\chi}{2}\right)}-\frac{sin\left(\frac{\eta-\chi}{2}\right)}{cos\left(\frac{\eta-\chi}{2}\right)}=r\ \Rightarrow$$

$$\frac{sin\left(\frac{\eta+\chi}{2}\right)cos\left(\frac{\eta-\chi}{2}\right)-sin\left(\frac{\eta-\chi}{2}\right)cos\left(\frac{\eta+\chi}{2}\right)}{cos\left(\frac{\eta+\chi}{2}\right)cos\left(\frac{\eta-\chi}{2}\right)}=r\ \Rightarrow$$

$$\frac{sin\left(\frac{\eta+\chi}{2}-\frac{\eta-\chi}{2}\right)}{cos\left(\frac{\eta+\chi}{2}\right)cos\left(\frac{\eta-\chi}{2}\right)}=r\ \Rightarrow$$

$$\frac{sin\left(\chi\right)}{\left(\frac{cos\eta+cos\chi}{2}\right)}=r\ \Rightarrow$$

$$rcos\eta=2sin\chi-rcos\chi$$

There are several important features one should point out in the Penrose diagram for Minkowski space-time. First, radial null geodesics are lines at $\pm45^0$ in the diagram. This is extremely useful because we can see the causal relation of every point in space-time easily. All timelike geodesics begin at $\imath^-$ and end at $\imath^+$. All null geodesics begin at $\Im^-$ and end at $\Im+$. Finally, $\imath^0$ is the asymptotic point of all spacelike geodesics.

2.3.1.2 Schwarzschild Spacetime

Constructing the Penrose diagram for the Schwarzschild metric is very similar to the Minkowski space-time, transforming the Schwarzschild coordinates to some new ones. The Schwarzschild metric is[9]:

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2d\Omega^2$$

The first coordinate transformation we will use is :

For $r>2M$

$$U=\left(\frac{r}{2M}-1\right)^{\frac{1}{2}}e^{\frac{r}{4M}}cosh\left(\frac{t}{4M}\right)$$

$$V=\left(\frac{r}{2M}-1\right)^{\frac{1}{2}}e^{\frac{r}{4M}}sinh\left(\frac{t}{4M}\right)$$

[9] $(c=G=\hbar=1)$

For $r \leq 2M$

$$U = \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} sinh\left(\frac{t}{4M}\right)$$

$$V = \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} cosh\left(\frac{t}{4M}\right)$$

From the above equations we can verify that :

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = U^2 - V^2 \tag{2.3.1}$$

$$tan\left(\frac{t}{4M}\right) = \frac{V}{U}\ , \quad r > 2M \tag{2.3.2}$$

$$tan\left(\frac{t}{4M}\right) = \frac{U}{V}\ , \quad r \leq 2M \tag{2.3.3}$$

The metric in the new coordinates (U, V) id not very hard to find but requires a certain amount of computation :

$$ds^2 = g_{UU} dU^2 + g_{VV} dV^2 + r^2 d\Omega^2$$

$g_{ab}$is a second rank tensor and transforms as :

$$g_{ab}^{'} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd} \Rightarrow$$

$$g_{UU} = \left(\frac{\partial t}{\partial U}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial U}\right)^2 g_{rr}$$

Using equation (2.3.1) we have :

$$\left(\frac{1}{2M} e^{\frac{r}{2M}} + \frac{1}{2M}\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}}\right)\left(\frac{\partial r}{\partial U}\right) = 2U \Rightarrow$$

$$\frac{1}{2M} e^{\frac{r}{2M}} \frac{r}{2M}\left(\frac{\partial r}{\partial U}\right) = 2U \Rightarrow$$

$$\frac{r e^{\frac{r}{2M}}}{4M^2}\left(\frac{\partial r}{\partial U}\right) = 2U \Rightarrow$$

$$\frac{\partial r}{\partial U} = \frac{8M^2 U}{r e^{\frac{r}{2M}}}$$

Using equation (2.3.2) [10] we have :

$$\frac{1}{cosh^2\left(\frac{t}{4M}\right)}\frac{1}{4M}\left(\frac{\partial t}{\partial U}\right) = -\frac{V}{U^2} \Rightarrow$$

$$\frac{\partial t}{\partial U} = -\frac{4MV}{U^2}cosh^2\left(\frac{t}{4M}\right)$$

g_{UU} is then equal to :

$$g_{UU} = \left(\frac{\partial t}{\partial U}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial U}\right)^2 g_{rr} \Rightarrow$$

$$g_{UU} = \frac{16M^2V^2}{U^4}cosh^4\left(\frac{t}{4M}\right)\left[-\left(1-\frac{2M}{r}\right)\right] + \frac{64M^4U^2}{r^2e^{\frac{r}{M}}}\left(1-\frac{2M}{r}\right)^{-1} \Rightarrow$$

$$g_{UU} = \frac{16M^2V^2}{U^4}\left(\frac{1}{1-tanh^2\left(\frac{t}{4M}\right)}\right)^2\left[-\left(1-\frac{2M}{r}\right)\right] + \frac{64M^4U^2}{re^{\frac{r}{M}}\left(r-2M\right)} \Rightarrow$$

$$g_{UU} = \frac{16M^2V^2}{U^4}\left(\frac{1}{1-\frac{V^2}{U^2}}\right)^2\left[-\left(1-\frac{2M}{r}\right)\right] + \frac{64M^4U^2}{re^{\frac{r}{M}}\left(r-2M\right)} \Rightarrow$$

$$g_{UU} = -\frac{16M^2V^2}{r\left(U^2-V^2\right)}\left(r-2M\right) + \frac{64M^4}{re^{\frac{r}{M}}\left(r-2M\right)}U^2 \Rightarrow$$

$$g_{UU} = -\frac{16M^2V^2 4M^2}{r\left(r-2M\right)e^{\frac{r}{M}}} + \frac{64M^4}{re^{\frac{r}{M}}\left(r-2M\right)}U^2 = \frac{64M^4}{re^{\frac{r}{M}}\left(r-2M\right)}\left(U^2-V^2\right) \Rightarrow$$

$$g_{UU} = \frac{64M^4}{re^{\frac{r}{M}}\left(r-2M\right)}\left(\frac{r}{2M}-1\right)e^{\frac{r}{2M}} = \frac{32M^3}{re^{\frac{r}{2M}}} \Rightarrow$$

$$g_{UU} = \frac{32M^3}{r}e^{-\frac{r}{2M}}$$

The same calculation can be done for g_{VV} :

$$g_{VV} = -\frac{32M^3}{r}e^{-\frac{r}{2M}}$$

Finally the metric is :

$$ds^2 = \frac{32M^3}{r}e^{-\frac{r}{2M}}\left(-dV^2+dU^2\right) + r^2d\Omega^2$$

[10] We use only the $r > 2M$ case here but the result is valid for all r

The (U, V) coordinates are called Kruskal- Szekeres coordinates. The second transformation is:

$$u = U + V$$

$$v = V - U$$

In the (u, v) coordinates light rays move at constant u or v. The final transformations relation that will bring the whole space-time to a finite diagram are :

$$u^{'} \equiv arctan\,(u) \equiv V^{'} - U^{'}$$

$$v^{'} \equiv arctan\,(v) \equiv V^{'} + U^{'}$$

The diagram of the Schwarzschild solution in $\left(V^{'}, U^{'}\right)$ coordinates is the Penrose Diagram and is shown in the next page.

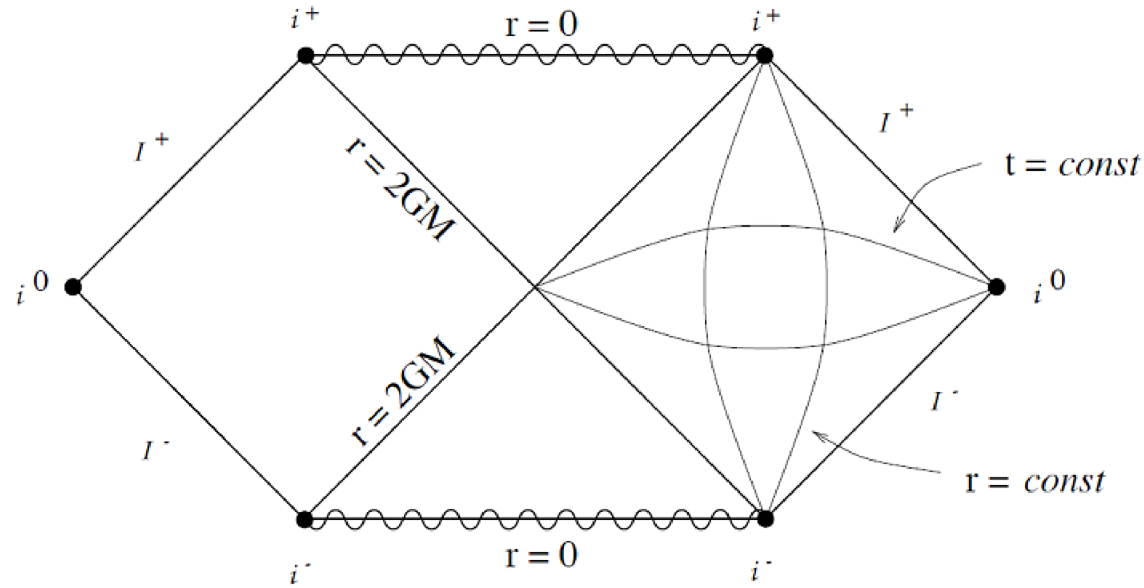

Figure 2.3.3: Penrose Diagram for a Schwarzschild Spacetime

In the above Penrose Diagram for the Schwarzschild space-time the $r = 0$ singularity is depicted as a straight horizontal line with $V^{'} = \frac{\pi}{4}$ and $-\frac{\pi}{4} \leq U^{'} \leq \frac{\pi}{4}$. The horizon $r = 2M$ of the Black Hole is the $V^{'} = U^{'}$ line at 45^0 angle.

2.3.2 Collapsing Shell of Mass

Having described the Penrose Diagram for both Minkowski and Schwarzschild space time we can now move on to derive Hawking Radiation. For simplicity we will imagine a spherical thin shell of Mass, collapsing to form a black hole. In order to understand how a black hole can radiate particles we must consider the time- dependent phase of the collapsing shell and not just the stationary state after the black hole has formed. There is a really important theorem in General Relativity called the no-hair theorem which states that every asymmetric state of collapse will settle down fairly quickly to a stationary solution. For example, if the initial collapsing mass was not spherically symmetric but had a “bulge” then it would radiate away its asymmetry through gravity waves and would settle down to the stationary Schwarzschild or Newmann- Kerr solution. In our treatment (as in Hawking's 1975 paper) we will consider a non- rotating and uncharged shell of mass collapsing to form the black hole. The stationary solution for such case is the Schwarzschild metric.

Isaac Newton had the intuition and later managed to prove that the field outside a spherically symmetric mass is the same as if the whole mass was at the center of the sphere. A similar theorem exists in General relativity. In the case of the thin collapsing shell of mass, the theorem states that the space-time metric outside the shell is described by the Schwarzschild metric and inside the shell the metric is that of flat space-time[11]. This is the reason why we previously needed to analyze the Penrose diagrams for both Minkowski and Schwarzschild space-time. As we see both are needed for the collapsing shell of mass.

The Penrose diagram for the collapsing shell is presented in the figure (2.3.4) :

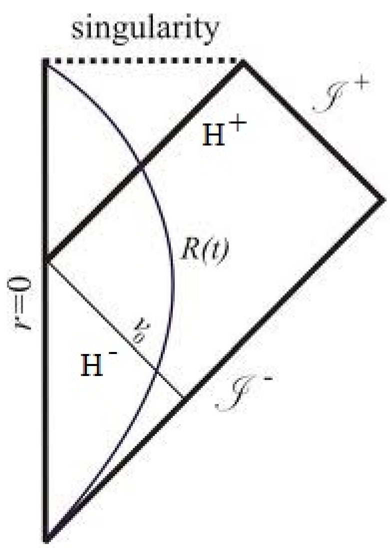

Figure 2.3.4: Penrose Diagram for the collapsing Shell

The timelike geodesic of the thin shell is described by the $R_{(t)}$ curve in the above diagram. The portion of the diagram outside the curve is described by the Schwarzschild metric and inside by the Minkowski metric as we have mentioned.

[11] In fact we could take a more realistic case of a normal collapsing star so the metric would not be flat inside. Whatever the inside metric is, it doesn't matter in the derivation.

2.3.3 Quantum Field Theory in the Collapsing Shell Spacetime

In this space-time we consider for simplicity a massless Hermitian field operator Φ, obeying the wave equation :

$$g^{ab}\nabla_a\Phi\nabla_b\Phi = 0$$

The operator Φ can be expressed in the basis :

$$\Phi = \sum_i \left\{ f_i \mathbf{a_i} + f_i^\star \mathbf{a_i^\dagger} \right\}$$

The solutions $\{f_i\}$ can be chosen so that on the past null infinity $\mathcal{L}^-$ they form a complete basis which satisfy the following orthonormality conditions :

$$(f_i, f_j) = \delta_{ij}\ ,\ \left(f_i, f_j^\star\right) = 0 \tag{2.3.4}$$

with the (f,h) operation defined as :

$$(f,h) = \imath \int d^3\Sigma \left(f\left(\partial_t h^\star\right) - \left(\partial_t f\right) h^\star\right) \tag{2.3.5}$$

where the integral at (2.3.5) is taken over a Cauchy surface.

The natural interpretation of the operators $\mathbf{a_i}$, $\mathbf{a_i^\dagger}$ is as annihilation and creation operators for the ingoing particles. The field Φ in every space-time point outside the horizon can be determined in two ways :

- The first one is by having the data of the field on the past null infinity $\mathcal{L}^-$ and going forward in time.
- The second one is by having the data of the field on both the future null infinity $\mathcal{L}^+$ and the horizon , and going backwards in time.

So we see that the field Φ can be written with two forms :

$$\Phi = \sum_i \left\{ f_i \mathbf{a_i} + f_i^\star \mathbf{a_i^\dagger} \right\} \tag{2.3.6}$$

$$\Phi = \sum_i \left\{ p_i \mathbf{b_i} + p_i^\star \mathbf{b_i^\dagger} + q_i \mathbf{c_i} + q_i^\star \mathbf{c_i^\dagger} \right\} \tag{2.3.7}$$

Here the $\{p_i\}$ solutions of the wave equation are purely outgoing (zero Cauchy data on the event horizon) , and the $\{q_i\}$ are solutions which contain no outgoing component (zero Cauchy data on $\mathcal{L}^+$). The $\{p_i\}$ and $\{q_i\}$ solutions satisfy the orhonormality conditions similar to the $\{f_i\}$, with the only difference that the surface of integration of the integral $(f,h) = \imath \int d^3\Sigma \left(f\left(\partial_t h^\star\right) - \left(\partial_t f\right) h^\star\right)$ is taken to be $\mathcal{L}^+$ and the event horizon respectively.

So we have :

$$(p_i, p_j) = \delta_{ij} \qquad and \qquad (q_i, q_j) = \delta_{ij}$$

$$\left(p_i, p_j^\star\right) = \left(p_i^\star, p_j\right) = 0 \qquad \left(p_i^\star, p_j^\star\right) = \left(q_i^\star, q_j^\star\right) = -\delta_{ij} \qquad \left(p_i, q_j\right) = \left(p_i, q_j^\star\right) = \left(p_i^\star, q_j\right) = 0$$

In analogy to the interpretation of the operators $\mathbf{a_i}$, $\mathbf{a_i^\dagger}$ as annihilation and creation operators, the operators $\mathbf{b_i}$, $\mathbf{b_i^\dagger}$ can be interpreted as the annihilation and creation operators for outgoing particles (particles on the $\mathcal{L}^+$ surface) . We will not bother about the interpretation of the $\mathbf{c_i}$, $\mathbf{c_i^\dagger}$ operators because the choice of $\{q_i\}$ does not affect the calculation of the emission of particles to $\mathcal{L}^+$.

Since the massless field at any spacetime point can be determined by their data on $\mathcal{L}^-$ then the $\{p_i\}$ and $\{q_i\}$ solutions can be written as linear combination of the $\{f_i\}$ basis. So we have :

$$p_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^\star\right)$$

$$q_i = \sum_j \left(\gamma_{ij} f_j + \epsilon_{ij} f_j^\star\right)$$

Using the definition of the product (2.3.5) , we can find the formula's for α_{ij} , β_{ij} :

$$(p_i, f_k) = \sum_j \left(\alpha_{ij}\left(f_j, f_k\right) + \beta_{ij}\left(f_j^\star, f_k\right)\right) \Rightarrow$$

$$(p_i, f_k) = \sum_j \left(\alpha_{ij}\delta_{jk} + \beta_{ij}\cdot 0\right) \Rightarrow$$

$$\alpha_{ik} = (p_i, f_k) \Rightarrow$$

$$\alpha_{ij} = (p_i, f_j)$$

Now the same for the β_{ij} :

$$(p_i, f_k^\star) = \sum_j \left(\alpha_{ij}\left(f_j, f_k^\star\right) + \beta_{ij}\left(f_j^\star, f_k^\star\right)\right) \Rightarrow$$

$$(p_i, f_k^\star) = \sum_j (\alpha_{ij} \cdot 0 + \beta_{ij}(-\delta_{jk})) \Rightarrow$$

$$\beta_{ik} = -(p_i, f_k^\star) \Rightarrow$$

$$\beta_{ij} = -\left(p_i, f_j^\star\right)$$

Some other very important relations we can find using the normalization conditions of the $\{p_i\}$ and $\{q_i\}$ solutions are :

$$(p_i, p_i) = 1 \Rightarrow$$

$$\left(\sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^\star\right), \sum_k (\alpha_{ik} f_k + \beta_{ik} f_k^\star)\right) = 1 \Rightarrow$$

$$\left(\sum_{j,k} \left(\alpha_{ij}\alpha_{ik}^\star (f_i, f_k) + \alpha_{ij}\beta_{ik}(f_j, f_k^\star) + \beta_{ij} a_{ik}^\star \left(f_j^\star, f_k\right) + \beta_{ij}\beta_{ik}^\star \left(f_j^\star, f_k^\star\right)\right) = 1\right) \Rightarrow$$

$$\sum_j \alpha_{ij}\alpha_{ij}^\star - \beta_{ij}\beta_{ij}^\star = 1 \tag{2.3.8}$$

$$(p_i, p_i^\star) = 0 \Rightarrow$$

$$\left(\sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^\star\right), \sum_k (\alpha_{ik}^\star f_k^\star + \beta_{ik}^\star f_k)\right) = 0 \Rightarrow$$

$$\sum_{j,k} \left(\alpha_{ij}\alpha_{ik} (f_i, f_k^\star) + \alpha_{ij}\beta_{ik}(f_j, f_k) + \beta_{ij} a_{ik}^\star \left(f_j^\star, f_k^\star\right) + \beta_{ij}\beta_{ik} \left(f_j^\star, f_k\right)\right) = 0 \Rightarrow$$

$$\sum_j \alpha_{ij}\beta_{ij} - \beta_{ij} a_{ij} = 0 \tag{2.3.9}$$

Equations (2.3.8) and (2.3.9) are very important and will be used later . Having found the formula's for α_{ij} , β_{ij} we can now move on to find the relations between the operators $\mathbf{b_i}$ and $\mathbf{a_i}$, $\mathbf{a_i^\dagger}$. Using the two forms of the field operator we have :

$$\Phi = \sum_i \left\{ f_i \mathbf{a_i} + f_i^\star \mathbf{a_i^\dagger} \right\} = \sum_i \left\{ p_i \mathbf{b_i} + p_i^\star \mathbf{b_i^\dagger} + q_i \mathbf{c_i} + q_i^\star \mathbf{c_i^\dagger} \right\}$$

Taking the product of p_k of each side of the above equation we get :

$$\sum_i \left\{ (f_i, p_k)\, \mathbf{a_i} + (f_i^\star, p_k)\, \mathbf{a_i^\dagger} \right\} = \sum_i \left\{ (p_i, p_k)\, \mathbf{b_i} + (p_i^\star, p_k)\, \mathbf{b_i^\dagger} + (q_i, p_k)\, \mathbf{c_i} + (q_i^\star, p_k)\, \mathbf{c_i^\dagger} \right\}$$

And using the orthonormality relations and the formulas of α_{ij} , β_{ij} we have :

$$\sum_i \left\{ (f_i, p_k)\, \mathbf{a_i} + (f_i^\star, p_k)\, \mathbf{a_i^\dagger} \right\} = \sum_i \left\{ \delta_{ik} \mathbf{b_i} + 0 \cdot \mathbf{b_i^\dagger} + 0 \cdot \mathbf{c_i} + 0 \cdot \mathbf{c_i^\dagger} \right\} \qquad \Rightarrow$$

$$\mathbf{b_i} = \sum_j \left\{ \alpha_{ij} \mathbf{a_j} - \beta_{ij}^\star \mathbf{a_j^\dagger} \right\} \tag{2.3.10}$$

with analogous procedure we find that :

$$\mathbf{b_i^\dagger} = \sum_j \left\{ -\beta_{ij} \mathbf{a_j} + \alpha_{ij} \mathbf{a_j^\dagger} \right\} \tag{2.3.11}$$

Equations (2.3.10) , (2.3.11) are called Bogoliubov transformations. Let us now define the initial vacuum state $|0_{in}\rangle$ which is the state containing no incoming particles. This means that there are no particles on the surface $\mathcal{L}^-$:

$$\mathbf{a_i} \,|0_{in}\rangle = 0 \qquad \forall\, i$$

The particle number operator of the " $\imath$ " mode for the ingoing particles is :

$$N_i = \mathbf{a_i^\dagger} \mathbf{a_i}$$

and gives for the vacuum state :

$$\langle 0_{in}|\, N_i \,|0_{in}\rangle = \langle 0_{in}|\, \mathbf{a_i^\dagger} \mathbf{a_i} \,|0_{in}\rangle = 0$$

However , the number operator for the outgoing particles is :

$$N_i^{'} = \mathbf{b_i^\dagger} \mathbf{b_i}$$

and the initial vacuum state will not appear to be a vacuum state to an observer at $\mathcal{L}^+$. The expectation of the number operator for this observer is :

$$\langle 0_{in}|\, N_i^{'} \,|0_{in}\rangle = \langle 0_{in}|\, \mathbf{b_i^\dagger} \mathbf{b_i} \,|0_{in}\rangle$$

and using equations (2.3.10) , (2.3.11) we have :

$$\langle 0_{in}| \mathbf{b_i^\dagger b_i} |0_{in}\rangle = \langle 0_{in}| \sum_j \left\{-\beta_{ij}\mathbf{a_j} + \alpha_{ij}\mathbf{a_j^\dagger}\right\} \sum_k \left\{\alpha_{ik}\mathbf{a_k} - \beta_{ik}^\star \mathbf{a_k^\dagger}\right\} |0_{in}\rangle =$$

$$= \langle 0_{in}| \sum_{j,k} \left(-\beta_{ij} a_{ik}^\star \mathbf{a_j a_k} + \beta_{ij}\beta_{ik}^\star \mathbf{a_j a_k^\dagger} + \alpha_{ij} a_{ik}^\star \mathbf{a_j^\dagger a_k} - \alpha_{ij}\beta_{ik}^\star \mathbf{a_j^\dagger a_k^\dagger}\right) |0_{in}\rangle =$$

$$= \langle 0_{in}| \sum_{j,k} \left(\beta_{ij}\beta_{ik}^\star \mathbf{a_j a_k^\dagger}\right) |0_{in}\rangle =$$

$$= \sum_{j,k} \beta_{ij}\beta_{ik}^\star \langle 0_{in}| \mathbf{a_j a_k^\dagger} |0_{in}\rangle =$$

$$= \sum_{j,k} \beta_{ij}\beta_{ik}^\star \delta_{jk} = \sum_j \beta_{ij}\beta_{ij}^\star = \sum_j |\beta_{ij}|^2$$

So finally we see that :

$$\langle 0_{in}| N_i^{'} |0_{in}\rangle = \langle 0_{in}| \mathbf{b_i^\dagger b_i} |0_{in}\rangle = \sum_j |\beta_{ij}|^2 \neq 0$$

In conclusion, the only thing that remains to be done is to calculate the coefficients β_{ij} for the collapsing shell , and we can determine the number of particles created by the gravitational field and emitted to $\mathcal{L}^+$.

2.3.4 Temperature of a Black hole

We move on now to the exact calculation of the crucial β_{ij} terms. In order to do this we have to solve the Klein - Gordon equation of the field Φ :

$$g^{ab}\nabla_a \Phi \nabla_b \Phi = 0 \qquad (2.3.12)$$

In order to solve the above equation we express Φ in the form[12] :

$$\Phi = \frac{1}{r} R_{\omega l(r)} \Upsilon_{lm(\theta,\phi)} e^{-\imath\omega t}$$

[12] Because the space-time metric in our case is spherically symmetric we can compose the solution in spherical harmonics

Substituting Φ in equation (2.3.12) we get :

$$-\frac{d^2R_{\omega l}}{dr_\star^2}+\left(V\left(r\right)-\omega^2\right)R_{\omega l}=0 \tag{2.3.13}$$

where :

$$V_{(r)}=\left(1-\frac{2M}{r}\right)\left(\frac{l\left(l+1\right)}{r^2}+\frac{2M}{r^3}\right)\quad and\quad r_\star=r+2Mlog\left|\frac{r}{2M}-1\right|$$

Although we can not solve equation (2.3.13) completely , we can find the asymptotic form of the solution at $t=\pm\infty$:

- $\mathbf{t}=-\infty$

For $t=-\infty$ we have $r=\infty$ for the incoming modes and $V_{(r)}$ becomes :

$$r\to\infty\ \Rightarrow\ V_{(r)}\to 0\quad\Rightarrow$$

$$-\frac{d^2R_{\omega l}}{dr_\star^2}+\left(0-\omega^2\right)R_{\omega l}=0\ \Rightarrow$$

$$\frac{d^2R_{\omega l}}{dr_\star^2}+\omega^2R_{\omega l}=0\ \Rightarrow$$

$$R_{\omega l}=e^{\pm\imath\omega r_\star}$$

So the asymptotic form of the solution for the ingoing modes[13] is :

$$f_\omega\sim\frac{1}{r}e^{-\imath\omega(r_\star+t)}=\frac{1}{r}e^{-\imath\omega v}$$

where v is the advanced coordinate defined as :

$$v=t+r+2Mlog\left|\frac{r}{2M}-1\right|$$

- $\mathbf{t}=+\infty$

For $t=+\infty$ there are actually two possibilities for the solution. The first one is that the waves go to infinity ($r=\infty$) and these waves are called p_ω . The second possibility is that the waves go to the singularity. These wave solutions are called q_ω .[14] Since p_ω's are the outgoing modes then :

[13] We drop the $l,\,m$ in our expression

[14] Remember that we have mentioned the p_ω's and q_ω's basis at equation (2.3.7) .

$$p_\omega \sim \frac{1}{r} e^{-\imath\omega(t-r_\star)} = \frac{1}{r} e^{-\imath\omega u}$$

where u is the retarded coordinate :

$$u = t - r - 2Mlog\left|\frac{r}{2M} - 1\right|$$

The q_ω's are left unspecified because only the outgoing modes actually interfere in the calculation. As we have mentioned before each solution p_ω can be expressed as an integral with respect to $\omega^{'}$ in the $\{f_{\omega'}\}$ basis[15] :

$$p_\omega = \int_0^\infty \left(a_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^\star_{\omega'}\right) d\omega^{'}$$

Remember that our initial purpose was to calculate the $\beta_{\omega\omega'}$ terms in order to find the expectation value of the number operator in $\mathcal{L}^+$. As Hawking points out in his 1975 paper, if we trace back the p_ω solution from $\mathcal{L}^+$ to $\mathcal{L}^-$, a part of p_ω which we call $p^{(1)}_\omega$ will be scattered by the potential $V_{(r)}$ in equation (2.3.13) . produced by the Schwarzschild metric. This scattering process does not change the frequency so it is not important in our case. The remainder part of p_ω , called $p^{(2)}_\omega$ will be partly scattered and partly reflected through the center and will eventually emerge to $\mathcal{L}^-$. This is the part of the p_ω solution that produces the interesting results, as Hawking argues in his paper .

Let us now consider for once more the Penrose diagram for the collapsing shell shown in the next page:

[15] Previously it was written in the summation form but now we will use the integral form

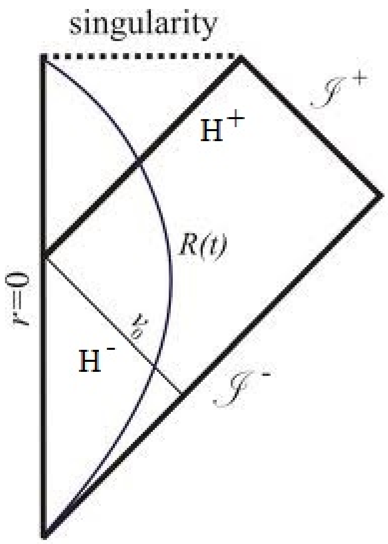

Figure 2.3.5: Penrose Diagram for the collapsing Shell

The null rays at $v = v_0$ shown in the diagram is the last inward propagating ray that reaches $\mathcal{L}^+$. Inward propagating rays with $v > v_0$ enter the horizon and fall to the singularity. Lets consider a region of space-time close to the horizon $r \to 2M$. Substituting to $u = t - r - 2Mlog\left|\frac{r}{2M} - 1\right|$ we see that $u \to \infty$ as $r \to 2M$. This means that the effective frequency of the wave $p_\omega^{(2)} \sim \frac{1}{r}e^{-\imath\omega u}$ is very big near the horizon, so it would propagate with very good approximation by geometric optics (following the null geodesic) , through the center of the body and out to $\mathcal{L}^-$.

So since we can use the geodesic curves as a good approximation, one can estimate the form of $p_\omega^{(2)}$ on $\mathcal{L}^-$ with incoming rays near $v = v_0$. This can be done by tracing back the null geodesic from $\mathcal{L}^+$ to $\mathcal{L}^-$ and keeping track of its affine distance with the the horizon H^+ and the incoming null ray $v = v_0$. The result which we will present with no proof is :

$$u = \frac{1}{\kappa} log\,(v_0 - v)$$

The above formula hold for small positive $v_0 - v$ and κ is the surface gravity , $\kappa = \frac{1}{4M}$ for the Schwarzschild black hole . Thus on $\mathcal{L}^-$ $p_\omega^{(2)}$ will be :

$$p_{\omega}^{(2)} \sim \begin{cases} 0 & v > v_0 \\ e^{\frac{\imath\omega}{\kappa} log(v_0 - v)} & v < v_0 \end{cases}$$

Having found the formula for $p_{\omega}^{(2)}$ it is fairly easy to calculate $\alpha_{\omega\omega'}$, $\beta_{\omega\omega'}$ by taking the Fourier transform :

$$\alpha_{\omega\omega'} = \int_{-\infty}^{v_0} e^{\frac{\imath\omega}{\kappa} log(v_0-v)} e^{\imath\omega' v} dv$$

$$\beta_{\omega\omega'} = \int_{-\infty}^{v_0} e^{\frac{\imath\omega}{\kappa} log(v_0-v)} e^{-\imath\omega' v} dv$$

we will calculate $\alpha_{\omega\omega'}$, $\beta_{\omega\omega'}$ for $v_0 = 0$ so that :

$$\alpha_{\omega\omega'} = \int_{-\infty}^{0} e^{\frac{\imath\omega}{\kappa} log(-v)} e^{\imath\omega' v} dv$$

$$\beta_{\omega\omega'} = \int_{-\infty}^{0} e^{\frac{\imath\omega}{\kappa} log(-v)} e^{-\imath\omega' v} dv$$

Lets name $\alpha_{\omega\omega'} = \Phi_{\omega(\omega')}$. We see from the above expressions that $\beta_{\omega\omega'} = \Phi_{\omega(-\omega')}$. Next we will show that :

$$\Phi_{\omega(-\omega')} = -e^{-\frac{\pi\omega}{\kappa}} \Phi_{\omega(\omega')} \qquad for\ \omega' > 0$$

Because we want to evaluate the integral $\Phi_{\omega(\omega')} = \int_{-\infty}^{0} e^{\frac{\imath\omega}{\kappa} log(-v)} e^{\imath\omega' v} dv = \int_{-\infty}^{0} Z_{(v)} dv$, we imagine that v is a complex number and integrate $\oint Z_{(v)} dv$ on a path in the complex v - plane .

- $\omega' > 0$

For the the case where ω' is positive we choose the contour show below :

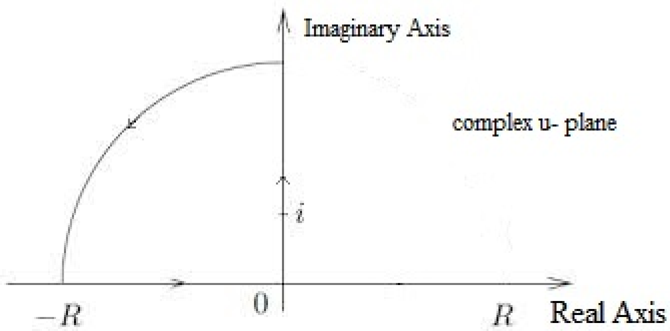

Figure 2.3.6: Contour in the v - plane for $\omega' > 0$

Since $Z_{(v)}$ is analytic inside the contour we have :

$$\oint Z_{(v)} dv = 0$$

Lets name the different parts of the contour :

C_1 = negative Real axis

C_2 = positive imaginary axis

C_3 = Quarter of a circle with radius R that goes to infinity

Since $R \longrightarrow \infty$ we get :

$$\oint Z_{(v)} dv = 0 \ \Rightarrow$$

$$\int_{C_1} Z_{(v)} dv + \int_{C_2} Z_{(v)} dv + \int_{C_3} Z_{(v)} dv = 0 \ \Rightarrow$$

$$\int_{C_1} Z_{(v)} dv = -\int_{C_2} Z_{(v)} dv \ \Rightarrow$$

$$\Phi_{\omega(\omega')} = -\int_{C_2} Z_{(v)} dv$$

To calculate the integral on the imaginary axis we change variables $v = \imath x, \quad x \in \mathbb{R},\ x > 0$:

$$\Phi_{\omega(\omega')} = -\int_{C_2} Z_{(v)} dv = -\imath \int_0^\infty e^{\frac{\imath\omega}{\kappa} log\left(xe^{-\imath\frac{\pi}{2}}\right)} e^{-\omega' x} dx \;\Rightarrow$$

$$\Phi_{\omega(\omega')} = -\imath e^{\left(\frac{\pi\omega}{2\kappa}\right)} \int_0^\infty e^{\frac{\imath\omega}{\kappa} log(x)} e^{-\omega' x} dx \qquad \omega' > 0$$

- $\omega' < 0$

If ω' is negative we choose the path of integration in the negative imaginary axis as shown below :

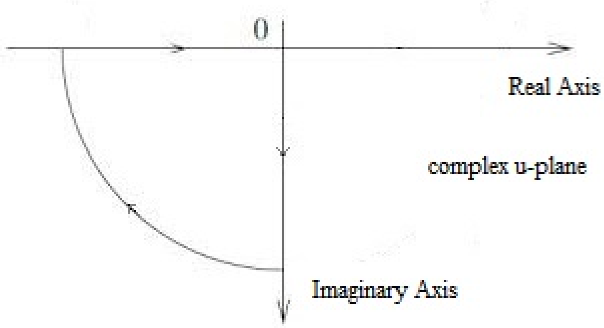

Figure 2.3.7: Contour in the v - plane for $\omega' < 0$

The only difference with before is that we have to integrate in the negative imaginary axis. So we change variables $v = -\imath x, \quad x \in \mathbb{R},\ x > 0$:

$$\Phi_{\omega(\omega')} = -\int_{C_2} Z_{(v)} dv = -\imath \int_0^\infty e^{\frac{\imath\omega}{\kappa} log\left(xe^{\imath\frac{\pi}{2}}\right)} e^{\omega' x} dx \;\Rightarrow$$

$$\Phi_{\omega(\omega')} = \imath e^{\left(-\frac{\pi\omega}{2\kappa}\right)} \int_0^\infty e^{\frac{\imath\omega}{\kappa} log(x)} e^{\omega' x} dx \qquad \omega' < 0$$

Hence we see that if $\omega^{'} > 0$:

$$\Phi_{\omega\left(\omega'\right)} = -\imath e^{\left(\frac{\pi\omega}{2\kappa}\right)} \int_0^\infty e^{\frac{\imath\omega}{\kappa} log(x)} e^{-\omega^{'} x} dx \tag{2.3.14}$$

$$\Phi_{\omega\left(-\omega'\right)} = \imath e^{\left(-\frac{\pi\omega}{2\kappa}\right)} \int_0^\infty e^{\frac{\imath\omega}{\kappa} log(x)} e^{-\omega^{'} x} dx \tag{2.3.15}$$

So it is pretty straightforward that if we divide equations (2.3.14) , (2.3.15) we get :

$$\Phi_{\omega\left(-\omega'\right)} = -e^{-\frac{\pi\omega}{\kappa}} \Phi_{\omega\left(\omega'\right)} \qquad \omega^{'} > 0 \tag{2.3.16}$$

which is what we wanted to prove. Having proved (2.3.16) we know the relation between $\alpha_{\omega\omega'}$ $\beta_{\omega\omega'}$ [16] :

$$\beta_{\omega\omega'} = -e^{-\frac{\pi\omega}{\kappa}} \alpha_{\omega\omega'} \Rightarrow \alpha_{ij} = -e^{\frac{\pi\omega_i}{\kappa}} \beta_{ij}$$

As the final step we will use the following formula that we have proved earlier [17] :

$$\sum_j \alpha_{ij}\alpha_{ij}^{\star} - \beta_{ij}\beta_{ij}^{\star} = 1 \Rightarrow$$

$$\sum_j \left(e^{\frac{2\pi\omega_i}{\kappa}} \beta_{ij}\beta_{ij}^{\star} - \beta_{ij}\beta_{ij}^{\star}\right) = 1 \Rightarrow$$

$$\sum_j \beta_{ij}\beta_{ij}^{\star} \left(e^{\frac{2\pi\omega_i}{\kappa}} - 1\right) = 1 \Rightarrow$$

$$\sum_j \beta_{ij}\beta_{ij}^{\star} = \frac{1}{e^{\frac{2\pi\omega_i}{\kappa}} - 1}$$

The above result gives us the expectation value of the number operator for outgoing particles :

$$\langle 0_{in}| N_i^{'} |0_{in}\rangle = \langle 0_{in}| \mathbf{b_i^{\dagger}} \mathbf{b_i} |0_{in}\rangle = \sum_j \beta_{ij}\beta_{ij}^{\star} = \frac{1}{e^{\frac{2\pi\omega_i}{\kappa}} - 1}$$

This is the Planck distribution for a black body radiation at temperature :

[16] Since $\alpha_{\omega\omega'} = \Phi_{\omega\left(\omega'\right)}$, $\beta_{\omega\omega'} = \Phi_{\omega\left(-\omega'\right)}$

[17] See equation (2.3.8)

$$T_H = \frac{\hbar\kappa}{2\pi}$$

T_H is called Hawking temperature . For the Schwarzschild black hole the surface gravity is $\kappa = \frac{1}{4M}$ so the Hawking temperature is :

$$T_H = \frac{\hbar\kappa}{8\pi M}$$

We will continue with some very important concepts from Quantum Information Theory.

Chapter 3

Quantum Information

In the previous chapter we derived the Hawking Radiation for a Schwarzschild Black hole, and showed that it is the same as the Planck distribution for a black body with temperature[1] $T_H = \frac{\hbar\kappa}{8\pi M}$. The fact that the radiation appears to be thermal poses some important questions. What happens to the information that fell into the black hole? Can someone reconstruct the state of the black hole by analyzing the Hawking Radiation? Our goal is to try answer these questions that analysis of Black Holes brought to the light. The problem generated by black Hole evaporation is known as the Black Hole Information Paradox. In this chapter we will present the necessary notions from Quantum Information Theory that will enable us to state the problem precisely.

3.1 Pure vs Mixed States

We will introduce now the notions of pure and mixed state in Quantum Mechanics which is extremely crucial if we want to understand the information Paradox

3.2 Pure State

A pure state of a quantum system is denoted by a vector (ket) $|\Psi\rangle$ with unit length (i.e $\langle\Psi|\ \Psi\rangle = 1$), in a complex Hilbert space $\mathbb{H}$. A pure vector state $|\phi\rangle$ has a dual vector $\langle\phi|$ (bra). The inner product for two vector states $|\phi\rangle$ and $|\psi\rangle$ is defined as :

$$Inner\ Product = (\phi, \psi) = \langle\phi|\ \psi\rangle$$

[1] G=c=1

3.2.1 Operators

Given the vectors and the dual vectors of the Hilbert space we can define operators that map vectors from the Hilbert space $\mathbb{H}$ to another vector in $\mathbb{H}$:

3.2.2 Identity Operator

Consider the complete basis $\{|n\rangle\}$ of $\mathbb{H}$, and the operator :

$$\hat{I} = \sum_n |n\rangle \langle n|$$

The above operator is called the identity operator, because if it operates on a ket $|\Psi\rangle$ it maps to the same ket again :

$$\hat{I} |\Psi\rangle = \sum_n |n\rangle \langle n| \Psi\rangle = |\Psi\rangle$$

The identity operator is very useful in proving propositions and will be used extensively.

3.2.3 Trace Operation

Let us define the trace operation of an arbitrary operator $\hat{A}$ as :

$$Tr\left(\hat{A}\right) = \sum_n \langle n| \hat{A} |n\rangle$$

where we sum over a set of basis vectors $\{|n\rangle\}$. The matrix elements of the operator $\hat{A}$ are :

$$A_{nm} = \langle n| \hat{A} |m\rangle$$

In the matrix representation the trace is simply the sum over all diagonal elements of the matrix A_{nm} .

3.2.3.1 Invariance of the Trace

A nice property of the trace that makes it extremely useful is that it does not depend on the choice of the basis vectors we use. Suppose we choose another basis $\{|k\rangle\}$. The trace in this basis is :

$$Tr\left(\hat{A}\right)_{basis\ \{|k\rangle\}} = \sum_k \langle k| \hat{A} |k\rangle$$

Applying the identity operator twice we have :

$$Tr\left(\hat{A}\right)_{basis\ \{|k\rangle\}} = \sum_k \sum_n \left(\langle k|\,|n\rangle \langle n|\,\hat{A}\,|n\rangle \langle n|\,|k\rangle\right) =$$

$$= \sum_k \sum_n \langle k|\ n\rangle \langle n|\,\hat{A}\,|n\rangle \langle n|\ k\rangle = \sum_k \sum_n \langle k|\ n\rangle \langle n|\ k\rangle \langle n|\,\hat{A}\,|n\rangle =$$

$$= \sum_n \langle n|\,\hat{A}\,|n\rangle = Tr\left(\hat{A}\right)_{basis\ \{|n\rangle\}} \Rightarrow$$

$$Tr\left(\hat{A}\right)_{basis\ \{|k\rangle\}} = Tr\left(\hat{A}\right)_{basis\ \{|n\rangle\}}$$

3.2.3.2 Density Operator

Let us now examine the following operator :

$$\hat{P}_\Psi = |\Psi\rangle \langle\Psi|$$

First we will be prove two properties of $\hat{P}_\Psi$:

- $\hat{P}_\Psi$ is Hermitian :

$$\left(\hat{P}_\Psi\right)^\dagger = (|\Psi\rangle \langle\Psi|)^\dagger = |\Psi\rangle \langle\Psi| = \hat{P}_\Psi$$

- Also[2] $\hat{P}{}^2_\Psi = \hat{P}_\Psi$:

$$\hat{P}{}^2_\Psi = \hat{P}_\Psi \hat{P}_\Psi = (|\Psi\rangle \langle\Psi|)(|\Psi\rangle \langle\Psi|) =$$

$$= |\Psi\rangle \langle\Psi|\ \Psi\rangle \langle\Psi| = |\Psi\rangle \langle\Psi| \Rightarrow$$

$$\hat{P}{}^2_\Psi = \hat{P}_\Psi$$

- $Tr\left(\hat{P}_\Psi\right) = 1$:

$$Tr\left(\hat{P}_\Psi\right) = \sum_n \langle n|\,\hat{P}_\Psi\,|n\rangle = \sum_n \langle n|\,|\Psi\rangle \langle\Psi|\,|n\rangle =$$

$$= \sum_n \langle\Psi|\,|n\rangle \langle n|\,|\Psi\rangle = \langle\Psi|\ \Psi\rangle = 1$$

[2]We assume that $\langle\Psi|\ \Psi\rangle = 1$

Expectation values of an arbitrary operator $\hat{O}$ can be expressed in terms of the operator $\hat{P}_{\Psi}$ instead of the state vector $|\Psi\rangle$. We have [3] :

$$\left\langle \hat{O} \right\rangle_{\Psi} = \langle\Psi| \hat{O} |\Psi\rangle \Rightarrow$$

$$\left\langle \hat{O} \right\rangle_{\Psi} = \sum_{n} \langle\Psi| n\rangle \langle n| \hat{O} |\Psi\rangle \Rightarrow$$

$$\left\langle \hat{O} \right\rangle_{\Psi} = \sum_{n} \langle n| \hat{O} |\Psi\rangle \langle\Psi| n\rangle \Rightarrow$$

$$\left\langle \hat{O} \right\rangle_{\Psi} = \sum_{n} \langle n| \hat{O}\hat{P}_{\Psi} |n\rangle \Rightarrow$$

$$\left\langle \hat{O} \right\rangle_{\Psi} = Tr\left(\hat{O}\hat{P}_{\Psi}\right) = Tr\left(\hat{P}_{\Psi}\hat{O}\right)$$

In the same manner we can express probabilities in terms of $\hat{P}_{\Psi}$:

$$|\langle\Phi| \Psi\rangle|^2 = \langle\Phi| \Psi\rangle \langle\Psi| \Phi\rangle = \sum_{n} \langle\Phi| n\rangle \langle n| \Psi\rangle \langle\Psi| \Phi\rangle =$$

$$= \sum_{n} \langle n| \Psi\rangle \langle\Psi| \Phi\rangle \langle\Phi| n\rangle = \sum_{n} \langle n| \hat{P}_{\Psi} \Phi\rangle \langle\Phi| n\rangle = Tr\left(\hat{P}_{\Psi} \Phi\rangle \langle\Phi|\right)$$

We see therefore that we can describe all the measurable quantities from the state $|\Psi\rangle$ using the operator $\hat{P}_{\Psi}$. We rename at this point the operator $\hat{P}_{\Psi}$ as $\hat{\rho}$:

$$\hat{\rho} = |\Psi\rangle \langle\Psi|$$

The above operator is now called the density operator or density matrix and describes completely a system in the state $|\Psi\rangle$.

Example. Lets imagine that the state of a quantum system is a superposition of spin up and down (in z axis) :

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$$

In the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ the density matrix is :

[3] In the last step we use the formula : $Tr\left(\hat{A}\hat{B}\right) = Tr\left(\hat{B}\hat{A}\right)$

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)\frac{1}{\sqrt{2}}\left(\langle\uparrow_z| + \langle\downarrow_z|\right) =$$

$$= \frac{1}{2}\left(|\uparrow_z\rangle\langle\uparrow_z| + |\uparrow_z\rangle\langle\downarrow_z| + |\downarrow_z\rangle\langle\uparrow_z| + |\downarrow_z\rangle\langle\downarrow_z|\right)$$

And in matrix form :

$$\rho = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We can easily verify that the trace of the matrix is :

$$Tr\left(\hat{\rho}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

3.3 Mixed State

We have seen that the density matrix $\hat{\rho} = |\Psi\rangle\langle\Psi|$ can describe all the physically observable properties in state $|\Psi\rangle$. The advantage of using the density matrix becomes clear when we introduce mixtures of pure states described by the density operator :

$$\hat{\rho} = \sum_{k=1}^{N} p_k |\Psi_k\rangle\langle\Psi_k|$$

where $\{|\Psi_k\rangle\}$ is some set of pure states , not necessarily orthogonal . Also the numbers p_k satisfy the relations [4]:

$$0 < p_k < 1 \ , \qquad \sum_{k=1}^{N} p_k = 1$$

The state of a system described by the previous density operator is called a Mixed State . We can prove now that that the expectation value of observables in a mixed state are probabilistic weighted averages of the expectation values of the pure states. Suppose we want to find the expectation value of the operator $\hat{O}$. So we have :

$$Tr\left(\hat{\rho}\hat{O}\right) = \sum_k p_k Tr\left(|\Psi_k\rangle\langle\Psi_k|\hat{O}\right) = \sum_k p_k \sum_n \langle n| |\Psi_k\rangle\langle\Psi_k|\hat{O}|n\rangle \ =$$

[4] From the following relation it is easy to prove that $Tr\left(\hat{\rho}\right) = 1$

$$= \sum_k p_k \sum_n \langle \Psi_k | \hat{O} | n \rangle \langle n | | \Psi_k \rangle = \sum_k p_k \langle \Psi_k | \hat{O} | \Psi_k \rangle \Rightarrow$$

$$Tr\left(\hat{\rho}\hat{O}\right) = \sum_k p_k \left\langle \hat{O} \right\rangle_k$$

We can view the mixed state as a statistical mixture of pure states $|\Psi_k\rangle$, each one with probability p_k .

There are many ways to determine whether a system is in a pure or mixed state . Some of them are presented below :

Pure State :

$$\hat{\rho}^2 = \hat{\rho} , \qquad Tr\left(\hat{\rho}^2\right) = 1 \tag{3.3.1}$$

Mixed State :

$$\hat{\rho}^2 \neq \hat{\rho} , \qquad Tr\left(\hat{\rho}^2\right) < 1 \tag{3.3.2}$$

3.3.1 Von Neumann Entropy

Another very important concept that we will use to determine whether the state of a system is pure or mixed is the Von Neumann Entropy , defined as :

$$S_{\left(\hat{\rho}\right)} = -Tr\left(\hat{\rho} log\left(\hat{\rho}\right)\right) \qquad \hat{\rho} : \textit{density matrix}$$

In order to calculate the entropy we first diagonalize the density matrix and then take the logarithm of the diagonal elements :

$$S_{\left(\hat{\rho}\right)} = -\sum_{k=1}^{d} \lambda_k log\left(\lambda_k\right)$$

where λ_k are the eigenvalues of the density matrix .

Using the Von Neumann Entropy we get another way to distinguish mixed from pure states:

Pure State :

$$S_{\left(\hat{\rho}\right)} = 0 \tag{3.3.3}$$

Mixed State :

$$S_{\left(\hat{\rho}\right)} > 0 \tag{3.3.4}$$

The entropy criterion is the one we will primarily use in order to show that information is lost in black hole evaporation .

3.4 Mixed States from Pure States - Reduced Density Matrix

In this section we will show how mixed states can arise from pure states when we consider only a subsystem of the whole quantum system. Consider systems A and B , each with a Hilbert space $\mathbb{H}_A$, $\mathbb{H}_B$. Let the state of the composite system be :

$$|\Psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$$

The state of the whole system is pure (by assumption) with density matrix [5]:

$$\rho_T = |\Psi\rangle \langle\Psi|$$

If we want to consider only the system A we cannot generally assign a pure state to it but we can assign a density matrix , called the Reduced Density Matrix [6] :

$$\rho_A \equiv \sum_j \langle j_B| \rho_T |j_B\rangle = \sum_j \langle j_B| (|\Psi\rangle \langle\Psi|) |j_B\rangle = Tr_B (\rho_T)$$

As we see, we basically take the partial trace of ρ_T over the basis of the system B . We "trace out" system B in order to obtain the reduced density matrix on A . If we want to find the expectation value of an operator $\hat{O}_A$ on the A system we calculate :

$$\left\langle \hat{O}_A \right\rangle = Tr\left(\rho_A \hat{O}_A\right)$$

Example. Let's assume that the state of the whole system is :

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|0_A\rangle |1_B\rangle + |1_A\rangle |0_B\rangle)$$

with basis $\{|1_A\rangle |1_B\rangle , |1_A\rangle |0_B\rangle , |0_A\rangle |1_B\rangle , |0_A\rangle |0_B\rangle\}$. First we will find the density matrix for the whole system ρ_T :

[5] Instead of $\hat{\rho}_T$ we will just write ρ_T .
[6] Dirac was the first to introduce this idea in 1930

$$\rho_T = |\Psi_{AB}\rangle \langle \Psi_{AB}| =$$

$$= \frac{1}{2}\left(|0_A\rangle |1_B\rangle \langle 0_A| \langle 1_B| \;+\; |0_A\rangle |1_B\rangle \langle 1_A| \langle 0_B| \;+\; |1_A\rangle |0_B\rangle \langle 0_A| \langle 1_B| \;+\; |1_A\rangle |0_B\rangle \langle 1_A| \langle 0_B|\right)$$

Then , in order to find the reduced density matrix ρ_A we trace over the basis of system B $\{|1_B\rangle, |0_B\rangle\}$:

$$\rho_A = \sum_j \langle j_B| \rho_T |j_B\rangle = \langle 1_B| \rho_T |1_B\rangle + \langle 0_B| \rho_T |0_B\rangle \;\Rightarrow$$

$$\rho_A = \frac{1}{2}\left(|0_A\rangle \langle 0_A| + |1_A\rangle \langle 1_A|\right)$$

and in matrix for [7] :

$$\rho_A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

we can verify that :

- $Tr\left(\rho_A\right) = \frac{1}{2} + \frac{1}{2} = 1$

- $\rho_A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \neq \rho_A$

- $Tr\left(\rho_A^2\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$

- $S_{(\rho_A)} = -\left(\frac{1}{4}log\left(\frac{1}{4}\right) + \frac{1}{4}log\left(\frac{1}{4}\right)\right) = \frac{1}{2}log\left(4\right) = log\left(2\right) > 0$

which means that the state of the system A is mixed , according to the criteria (3.3.1) , (3.3.2) , (3.3.3) , (3.3.4) . We conclude that a subsystem of a bigger system can be in a mixed state although the whole system is in a pure state .

[7] Basis $\{|1_A\rangle, |0_A\rangle\}$

3.5 Entropy of the Product of two States

Suppose we have the state $|\Psi\rangle$ that can be written as the tensor product of two states $|\Psi_1\rangle$, $|\Psi_2\rangle$:

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$$

Let's assume that the state $|\Psi_1\rangle$ has two subsystems A_1 and B_1 each with basis $\{|a_i^1\rangle\}$, $\{|b_i^1\rangle\}$. Similarly the state $|\Psi_2\rangle$ has two subsystems A_2 and B_2 , with basis $\{|a_i^2\rangle\}$, $\{|b_i^2\rangle\}$. For every subsystem we can find the corresponding density matrix by tracing over the basis of the other subsystem. For example the Reduced Density Matrix for the subsystem B_1 is :

$$\rho_{B_1} = \sum_i \langle a_i^1| \rho_1 |a_i^1\rangle \qquad where\ \rho_1 = |\Psi_1\rangle \langle\Psi_1|$$

The reduced density matrix for the subsystem B_1, B_2 is :

$$\rho_{B_1,B_2} = \sum_{i,j} \langle a_i^1| \langle a_j^2| \rho |a_i^1\rangle |a_j^2\rangle =$$

$$= \sum_{i,j} \langle a_i^1| \langle a_j^2| (|\Psi_1\rangle \otimes |\Psi_2\rangle \langle\Psi_1| \otimes \langle\Psi_2|) |a_i^1\rangle |a_j^2\rangle =$$

$$= \sum_i \langle a_i^1| (|\Psi_1\rangle \langle\Psi_1|) |a_i^1\rangle \otimes \sum_j \langle a_j^2| (|\Psi_2\rangle \langle\Psi_2|) |a_j^2\rangle \Rightarrow$$

$$\rho_{B_1,B_2} = \rho_{B_1} \otimes \rho_{B_2}$$

A very important property of the Von Neumann Entropy is called subadditivity and will be presented without proof :

$$If \qquad \rho = \rho_1 \otimes \rho_2 \quad then\ :$$

$$S_\rho = S_{\rho_1} + S_{\rho_2}$$

Both the relation $\rho_{B_1,B_2} = \rho_{B_1} \otimes \rho_{B_2}$ and the subadditivity property of the Von Neumann Entropy will be used in the next chapter.

3.6 Pure only goes to Pure

One of the fundamentals assumptions of Quantum Mechanics and Quantum field Theory is that the evolution of the vector state $|\Psi\rangle$ of a system is determined by a unitary operator U . An operator is unitary if and only if:

$$U\,U^{\dagger} = I$$

where I is the identity operator , and $U^{\dagger}$ the hermitian conjugate of U .

Suppose at $t = 0$ the system has the pure state $|\Psi\rangle_0$. At later time t the new ket $|\Psi\rangle_t$ is given by the evolution operator U_t :

$$|\Psi\rangle_t \,=\, U_t\,|\Psi\rangle_0$$

Lets see how the density matrix transforms at later times . By the definition of the density matrix we have :

$$\rho_t \equiv |\Psi\rangle_t\,\langle\Psi|_t = U_t\,|\Psi\rangle_0\,\langle\Psi|_0\,U_t^{\dagger} = U_t\,\rho_0\,U_t^{\dagger}$$

We can now calculate ρ_t^2 :

$$\rho_t^2 = \left(U_t\,\rho_0\,U_t^{\dagger}\right)\left(U_t\,\rho_0\,U_t^{\dagger}\right) \;\Rightarrow$$

$$\rho_t^2 = U_t\,\rho_0\,\left(U_t^{\dagger}U_t\right)\,\rho_0\,U_t^{\dagger} \;\Rightarrow$$

$$\rho_t^2 = U_t\,\rho_0\,I\,\rho_0\,U_t^{\dagger} \;\Rightarrow$$

$$\rho_t^2 = U_t\,\rho_0\,I\,\rho_0\,U_t^{\dagger} \;\Rightarrow$$

$$\rho_t^2 = U_t\,\rho_0^2\,U_t^{\dagger} \;\Rightarrow \quad \left(since\ \rho_0^2 = \rho_0\right)$$

$$\rho_t^2 = U_t\,\rho_0\,U_t^{\dagger} \;\Rightarrow$$

$$\rho_t^2 = \rho_t$$

We see that the density operator satisfies the purity condition (3.3.1) for every time t . This very important conclusion states that if we start with a pure state of a quantum system then the state remains pure as long as the evolution operator is unitary.

3.7 No Cloning Theorem

A very important theorem in Quantum Information is the no - cloning theorem. It was stated by Wootters and Zurek and Dieks in 1982 [3] . The theorem states that we cannot create identical copies of an unknown quantum state We must assume the cloning is done with a unitary operator, hence the name " no- cloning " .

3.7.1 Proof of the Theorem

Imagine we have the system A in state $|\Psi\rangle_A$ which we want to copy and another system B in the initial state $|e\rangle_B$. Assuming the systems are independent, the composite system will be in the state $|\Psi\rangle_A \otimes |e\rangle_B$. Let us suppose that there exists a unitary operator U [8] that acts as copier :

$$U\,|\phi\rangle_A\,|e\rangle_B = |\phi\rangle_A\,|\phi\rangle_B \tag{3.7.1}$$

for all possible states $|\phi\rangle_A$.

Now , we know that every unitary operator preserves the inner products between states. So , let's take the inner product of the two states $|\phi\rangle_A\,|e\rangle_B$ and $|\psi\rangle_A\,|e\rangle_B$:

$$\langle e|_B\,\langle\phi|_A\,|\psi\rangle_A\,|e\rangle_B = \langle\phi|_A\,\psi\rangle_A$$

But if we use equation (3.7.1) we get :

$$\langle e|_B\,\langle\phi|_A\,|\psi\rangle_A\,|e\rangle_B = \langle e|_B\,\langle\phi|_A\,U^\dagger U\,|\psi\rangle_A\,|e\rangle_B =$$

$$= \langle\phi|_B\,\langle\phi|_A\,|\psi\rangle_A\,|\psi\rangle_B = (\langle\phi|_A\,\psi\rangle_A)^2$$

So we conclude that :

$$\langle\phi|_A\,\psi\rangle_A = (\langle\phi|_A\,\psi\rangle_A)^2 \quad \Rightarrow$$

$$\langle\phi|_A\,\psi\rangle_A = 1 \Rightarrow |\phi\rangle_A = |\psi\rangle_A \qquad or \qquad \langle\phi|_A\,\psi\rangle_A = 0 \Rightarrow \{|\phi\rangle_A\,,\,|\psi\rangle_A \quad orthogonal\}$$

But since the above conditions do not hold for two arbitrary states we deduce that the existence of the operator U is impossible. Hence we have proved the no cloning theorem.

If quantum cloning was possible we could use it to send messages faster than the speed of light. We could see how this would be possible by considering the following scenario. Suppose we create

[8] Do not confuse U here with the evolution operator we mentioned before

two particles a and b with entangled spins on the z axis . Then the state of the whole system would be :

$$|\Psi_{ab}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_{z,\,a}\rangle\,|\downarrow_{z,\,b}\rangle + |\downarrow_{z,\,a}\rangle\,|\uparrow_{z,\,b}\rangle\right)$$

We then sent particle a to observer A and particle B to observer B. Observers A and B are very far from each other. Lets also assume that they have decided beforehand that observer B will ether make a measurement of the spin in the z axis or he will not perform any measurement. Observer A could know whether B decided to measure the particle or not by making many clones of the state of particle a . Then he would measure the spin of all the cloned particles (which share the same state with particle a) in the z axis. If he found all the particles to have spin up or spin down he would deduce that B made a measurement of particle b and collapsed the wave-function to ether spin down or spin up .[9] If on the other hand observer A found that approximately half of the particles have spin up and the other half spin down, he would deduce that B did not perform a measurement. Hence if cloning was really allowed we could send information faster than light, violating causality.[10]

[9] If particle a has spin up then particle b has spin down , and if a has spin down particle b has spin up

[10] Credit to Giorgos Katsianis

Chapter 4

Information Loss in Black Hole Evaporation

In Chapter 2 we saw that Black Holes radiate particles via Hawking Radiation. In Chapter 3 we introduced all the necessary tools (entropy, density matrix e.t.c) from Quantum Information Theory. We are ready at this point to show what exactly we mean by information loss in Black Hole evaporation.

4.1 The Entropy Problem

In 1972 Bekenstein [4] published a paper in which he presented arguments that Black Holes must be associated with an entropy proportional to the surface area of the horizon. The main reason for assigning an entropy to a black hole is so that the second Law of Thermodynamics in not violated.

Suppose we take a box containing some gas and throw it in the black hole. The gas in the box has some entropy , so after it has fallen into the black hole the total entropy of the universe has decreased if the black hole does not have any entropy. Notice however, that as the box falls into the black hole , the mass of the black hole increases by a small amount and so does the surface of the horizon [1]. If we want to save the 2_{nd} law of Thermodynamics we should assign an entropy S_{BH} to the Black Hole so that :

$$\frac{dS_{total}}{dt} = \frac{dS_{matter}}{dt} + \frac{dS_{BH}}{dt} \geq 0$$

and the 2_{nd} Law still holds. Lets see now a simplified argument which will help us understand why the entropy should be proportional to the area of the black hole :

[1] $R = \frac{2GM}{c^2}$

The entropy of a system is a measure of the maximum amount of information that can be contained in the system :

$$S_{BH} \sim max\ number\ of\ bits\ of\ information \tag{4.1.1}$$

Next, suppose we have a black hole of mass M and we want to send one bit of information in. So, we choose a massless particle and throw it in the black hole. But since we want just one bit of information, the wave length λ of the particle should be in order of the Schwarzschild radius of the Black hole :

$$\lambda_p \sim R_s$$

If the particles's wavelength were of order less than the Schwarzschild radius[2] then we could say more than just whether the particle is inside the Black hole or not, but remember we want to sent the least amount of information we can send in the black hole[3] . Since the wavelength is of order of R_S then the energy of the particle will be :

$$E_p \sim \frac{\hbar c}{\lambda_p} \sim \frac{\hbar c}{R_s}$$

So the energy of the Black Hole increases by E_p :

$$dE_{BH} \sim E_p \sim \frac{\hbar c}{R_s} \Rightarrow$$

$$c^2 dM_{BH} \sim \frac{\hbar c}{R_s} \Rightarrow$$

$$dM_{BH} \sim \frac{\hbar c}{c^2 R_s} \Rightarrow$$

$$dM_{BH} \sim \frac{\hbar}{c R_s} \tag{4.1.2}$$

Now we can use the usual equation for the Schwarzschild radius :

$$R_s = \frac{2GM_{BH}}{c^2} \Rightarrow$$

$$dM_{BH} = \frac{c^2}{2G} dR_s \tag{4.1.3}$$

[2]If the wavelength were larger than R_S then we would not know for sure whether the particle is inside the black hole or not

[3]Of course if one wanted to be more precise he should prove why this is the least amount of information

Combining equations (4.1.2) and (4.1.3) :

$$\frac{c^2}{2G}dR_s \sim \frac{\hbar}{cR_s} \Rightarrow$$

$$R_s dR_s \sim \frac{2G\hbar}{c^3} \Rightarrow$$

$$d\left(R_s^2\right) \sim \frac{2G\hbar}{c^3}$$

And since the area of the horizon is proportional to the square of the radius $A \sim R_s^2$:

$$dA \sim \frac{2G\hbar}{c^3} \tag{4.1.4}$$

We see therefore that if we add one bit of information into the Black Hole the surface of the the horizon changes by a constant of number. We deduce therefore that the most amount of information a black hole can contain (integrating eq (4.1.4)) is proportional to the surface area :

$$Max\ number\ of\ bits\ of\ information \sim \frac{c^3}{2G\hbar}A$$

But from equation (4.1.1) we get that :

$$S_{BH} \sim \frac{c^3}{2G\hbar}A$$

In fact we can find the exact relation between entropy and the surface area of the horizon without the proportionality constant by using the Hawking temperature we derived in the Chapter 2. The Hawking Temperature is given by the formula [4] :

$$T_H = \frac{\hbar c^3}{8\pi GM}$$

and

$$A = 4\pi R_s^2 = 4\pi\left(\frac{2GM_{BH}}{c^2}\right)^2 = \frac{16\pi M_{BH}^2 G^2}{c^4} \Rightarrow$$

$$M_{BH}^2 = \frac{c^4}{16\pi G^2}A$$

Now , from usual thermodynamics we have :

$$T_H dS_{BH} = dE \Rightarrow \frac{\hbar c^3}{8\pi GM_{BH}}\, dS_{BH} = c^2\, dM_{BH} \Rightarrow$$

[4] With all the constants c , G , $\hbar$

$$dS_{BH} = \frac{8\pi G}{\hbar c} M_{BH} dM_{BH} \Rightarrow dS_{BH} = \frac{4\pi G}{\hbar c} d\left(M_{BH}^2\right) \Rightarrow$$

$$dS_{BH} = \frac{4\pi G}{\hbar c} \frac{c^4}{16\pi G^2} dA \Rightarrow dS_{BH} = \frac{c^3}{4G\hbar} dA \Rightarrow$$

$$S_{BH} = \frac{c^3}{4G\hbar} A$$

The fact that black Holes have entropy may be necessary if we want to save 2_{nd} Law of Thermodynamics, but this poses another more serious problem. We know that the entropy of a macroscopic system is given by :

$$S = LogN$$

where N is the number of microstates the system has for given macroscopic parameters (e.g. Temperature) . So the number of microstates a black hole has is :

$$N = e^{S_{BH}}$$

For a solar Black Hole $N \sim 10^{10^{77}}$, which is an incredibly big number. It is not at all clear where is such a big number of states is encoded. In General Relativity Black holes have no - hair, meaning they quickly settle to the same equilibrium geometries that are described at most by three parameters. So Black Holes seem to be the simplest macroscopic objects found in nature. But on the other hand the entropy that a Black Hole has implies it must have a big number of microstates. Where these deformations of the geometry exist still remains an unsolved problem.

4.2 Information Loss

We have seen how Black Holes emit particles through Hawking Radiation. What we have not yet done is to calculate the state of the radiation. There is a simple picture which describes the way a Black Hole evaporates that will help us find the the state of the Radiation. In the vacuum pairs, of particles and antiparticles are continuously being created and annihilated. For example pairs of electrons and positrons are created and destroyed all the time even in vacuum. If there is a strong electric field present it is possible to “pull apart” these pairs and make them real, in the sense that we can detect two opposite charge currents in the laboratory . This phenomenon is called the Schwinger effect [5].

The same effect could in principle happen near the Black Hole horizon, where instead of the electric field we have the gravitational field of the Black Hole. If by chance one particle is created just inside the horizon and its antiparticle just outside the horizon then one inside would fall into the singularity and the one outside could escape to infinity, where it would be detected by an observer as Hawking Radiation. The following analysis in this chapter will show that the state of the radiation is mixed even though we started with a pure state. This is normal because what we did was to consider only the particles that go to infinity and not the "inside particles" that they are entangled with. If we consider both the "inside" and the "outside" system then the state will be pure , which is consistent with unitary evolution. The problem arises when the Black Hole evaporates completely there is no "inside" system any more and we will only be left with the Hawking Radiation. So it seems that there is a fundamental violation of unitarity in Quantum Mechanics where pure states only evolve to pure states. This is what we mean by information Loss in Black Hole evaporation.

4.3 Vacuum State in Different Basis

In chapter 2 we saw that we can decompose the field operator Φ in many different basis. We can choose any coordinate time t and decompose the field into its positive and negative frequency modes with respect to this choice of t . Even in flat space-time accelerated observers detect particles even though for inertial observers we have the vacuum or no particle state [5]. In fact this is the reason why (for the Black Hole geometry) the incoming vacuum state at past null infinity appears as Hawking radiation for an observer at future null infinity.

Let the field Φ be expanded in some basis $\left\{f_{n(x)}\right\}$:

$$\Phi = \sum_n \left\{ f_{n(x)} \mathbf{a_n} + f^\star_{n(x)} \mathbf{a_n^\dagger} \right\}$$

In this basis, the vacuum state is defined as :

$$\mathbf{a_n} \left|0_a\right\rangle = 0 \qquad \forall\, n$$

Since in curved space-time there is no unique choice of the time coordinate t , we can choose a different time $\widetilde{t}$ which correspond to a different basis $\left\{h_{n(x)}\right\}$:

$$\Phi = \sum_n \left\{ h_{n(x)} \mathbf{b_n} + h^\star_{n(x)} \mathbf{b_n^\dagger} \right\}$$

Now in this basis the vacuum would be defined as :

$$\mathbf{b_n} \left|0_b\right\rangle = 0 \qquad \forall\, n$$

We wish to find the relation between the vacuum states $\left|0_a\right\rangle$, $\left|0_b\right\rangle$. This is relatively easy to do by

[5] See Chapter 2 : Unruh Effect

defining the usual inner product :

$$(f, g) = \imath \int d\Sigma^{\mu} \left(f \partial_{\mu} g^{\star} - g^{\star} \partial_{\mu} f \right)$$

From the definition of the inner product we have for the orthonormal basis $\left\{ f_{n(x)} \right\}$:

$$(f_m,\, f_n) = \delta_{mn}, \quad (f_m,\, f_n^{\star}) = 0, \quad (f_m^{\star},\, f_n^{\star}) = -\delta_{mn}$$

From the two different expansions we have :

$$\sum_n \left\{ f_{n(x)} \mathbf{a_n} + f_{n(x)}^{\star} \mathbf{a_n^{\dagger}} \right\} = \sum_n \left\{ h_{n(x)} \mathbf{b_n} + h_{n(x)}^{\star} \mathbf{b_n^{\dagger}} \right\} \Rightarrow$$

$$\mathbf{a_m} = \sum_n \left\{ \alpha_{mn} \mathbf{b_n} + \beta_{mn} \mathbf{b_n^{\dagger}} \right\}$$

with $\alpha_{mn} = \left(h_{n(x)}, f_{m(x)} \right)$, $\beta_{mn} = \left(h_{n(x)}^{\star}, f_{m(x)} \right)$. From the definition of the vacuum $|0_a\rangle$ we have :

$$\mathbf{a_m} |0_a\rangle = 0 \;\Rightarrow$$

$$\sum_n \left\{ \alpha_{mn} \mathbf{b_n} + \beta_{mn} \mathbf{b_n^{\dagger}} \right\} |0_a\rangle = 0 \tag{4.3.1}$$

The above equation has the solution :

$$|0_a\rangle = C e^{-\frac{1}{2} \sum_{m,n} \mathbf{b_m^{\dagger}} \gamma_{mn} \mathbf{b_n^{\dagger}}} |0_b\rangle$$

where γ is the symmetric matrix :

$$\gamma = \frac{1}{2} \left(a^{-1} \beta + \left(a^{-1} \beta \right)^T \right)$$

We will not derive this solution for the general case but only for the case of one mode . In the case of one mode equation (4.3.1) becomes :

$$\left(\alpha \mathbf{b} + \beta \mathbf{b}^{\dagger} \right) |0_a\rangle = 0 \quad \Rightarrow$$

$$\left(\mathbf{b} + \frac{\beta}{\alpha} \mathbf{b}^{\dagger} \right) |0_a\rangle = 0 \quad \Rightarrow$$

$$\left(\mathbf{b}+\gamma\mathbf{b}^\dagger\right)|0_a\rangle = 0 \qquad where \ \ \gamma = \frac{\beta}{\alpha} \tag{4.3.2}$$

Lets prove that equation (4.3.2) has a solution of the form :

$$|0_a\rangle = Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}|0_b\rangle$$

where C is a normalization constant and μ is a number that we have to determine with respect to γ . A very useful relation we will use is the following :

$$\mathbf{b}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n = \left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n\mathbf{b} + 2n\mathbf{b}^\dagger\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^{n-1} \qquad \forall\, n \geq 1 \tag{4.3.3}$$

Proof. From the commutator relation : $\left[\mathbf{b},\, \mathbf{b}^\dagger\right] = 1$ we have :

$$\mathbf{b}\mathbf{b}^\dagger - \mathbf{b}^\dagger\mathbf{b} = 1 \;\Rightarrow\; \mathbf{b}\mathbf{b}^\dagger = \mathbf{b}^\dagger\mathbf{b} + 1$$

Lets start with the right side of the formula we want to prove :

$$\mathbf{b}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n = \mathbf{b}\,\underbrace{\left(b^\dagger b^\dagger\ldots b^\dagger b^\dagger\right)}_{\mathbf{2n}} = \left(\mathbf{b}\mathbf{b}^\dagger\right)\underbrace{\left(b^\dagger\ldots b^\dagger\right)}_{\mathbf{2n-1}} =$$

$$= \left(1+\mathbf{b}^\dagger\mathbf{b}\right)\underbrace{\left(b^\dagger\ldots b^\dagger\right)}_{\mathbf{2n-1}} = \left(\mathbf{b}^\dagger\right)^{2n-1} + \mathbf{b}^\dagger\mathbf{b}\left(\mathbf{b}^\dagger\right)^{2n-1} =$$

$$= \left(\mathbf{b}^\dagger\right)^{2n-1} + \mathbf{b}^\dagger\left(\mathbf{1}+\mathbf{b}^\dagger\mathbf{b}\right)\left(\mathbf{b}^\dagger\right)^{2n-2} = \left(\mathbf{b}^\dagger\right)^{2n-1} + \left(\mathbf{b}^\dagger\right)^{2n-1} + \mathbf{b}^\dagger\mathbf{b}^\dagger\mathbf{b}\left(\mathbf{b}^\dagger\right)^{2n-2} =$$

$$= \ldots\ldots\ldots = 2n\left(\mathbf{b}^\dagger\right)^{2n-1} + \left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n\mathbf{b} \;=\; 2n\mathbf{b}^\dagger\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^{n-1} + \left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n\mathbf{b}$$

If we expand now the exponential in the power series we get :

$$Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger} = C\sum_{n=0}\frac{\mu^n}{n!}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n$$

we will now calculate $\mathbf{b}\left(Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}\right)|0_b\rangle$ and see what we get :

$$\mathbf{b}\left(Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}\right)|0_b\rangle = C\mathbf{b}\left(\sum_{n=0}\frac{\mu^n}{n!}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n\right)|0_b\rangle \;=$$

$$= C\left(\sum_{n=0}\frac{\mu^n}{n!}\mathbf{b}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n\right)|0_b\rangle = C\left(\mathbf{b}\,|0_b\rangle + \sum_{n=1}\frac{\mu^n}{n!}\mathbf{b}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n|0_b\rangle\right) =$$

$$= C\left(0 + \sum_{n=1} \frac{\mu^n}{n!}\left[\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n \mathbf{b} + 2n\mathbf{b}^\dagger\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^{n-1}\right]|0_b\rangle\right) =$$

$$= C\left(\sum_{n=1} \frac{2n\mu^n}{n!}\left[\mathbf{b}^\dagger\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^{n-1}\right]|0_b\rangle\right) =$$

$$= 2\mu\mathbf{b}^\dagger\left(\sum_{n=1} C\frac{\mu^{n-1}}{(n-1)!}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^{n-1}\right)|0_b\rangle =$$

$$= 2\mu\mathbf{b}^\dagger\left(C\sum_{n=0}\frac{\mu^n}{n!}\left(\mathbf{b}^\dagger\mathbf{b}^\dagger\right)^n\right)|0_b\rangle =$$

$$= 2\mu\mathbf{b}^\dagger\left(Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}\right)|0_b\rangle \quad \Rightarrow$$

$$\mathbf{b}\left(Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}\right)|0_b\rangle = 2\mu\mathbf{b}^\dagger\left(Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}\right)|0_b\rangle \tag{4.3.4}$$

Comparing the following equations ((4.3.2) *and* (4.3.4)) :

$$\left(\mathbf{b} + \gamma\mathbf{b}^\dagger\right)|0_a\rangle = 0 \quad , \quad \mathbf{b}\left(Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}\right)|0_b\rangle - 2\mu\mathbf{b}^\dagger\left(Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}\right)|0_b\rangle = 0$$

we see that if $\mu = -\frac{1}{2}\gamma$ then : □

$$|0_a\rangle = Ce^{\mu\mathbf{b}^\dagger\mathbf{b}^\dagger}|0_b\rangle$$

is a solution to equation $\left(\mathbf{b} + \gamma\mathbf{b}^\dagger\right)|0_a\rangle = 0$, which is what we wanted to prove. If we expand $|0_a\rangle$ we have :

$$|0_a\rangle = C_0|0_b\rangle + C_2\mathbf{b}^\dagger\mathbf{b}^\dagger|0_b\rangle + C_4\mathbf{b}^\dagger\mathbf{b}^\dagger\mathbf{b}^\dagger\mathbf{b}^\dagger|0_b\rangle + \ldots\ldots$$

where $C_n = C\left(-\frac{\gamma}{2}\right)^n\frac{1}{n!}$

We see therefore that although in the $\left\{f_{n(x)}\right\}$ basis we have the vacuum state , in the $\left\{h_{n(x)}\right\}$ basis we have a part with zero particles b , a part with two particles b , and so on . This is equivalent to saying that different observers will in general have different vacuum states . The goal in the next sections is to describe an accurate visual description of particle creation near the horizon and finally show how the initial vacuum state becomes an entangled superposition of outgoing and ingoing particle states.

4.4 How Wavemodes move in Spacetime

In order to study the Hawking Radiation (as we have mentioned in previous chapter) one should solve the wave equation of the field Φ for the Schwarzschild metric. Although there isn't a complete analytical solution to the equation there are many approximate solutions. We will use the "eikonical approximation" which will provide us with a satisfying qualitative view of how the field evolves in each spacetime slice. We will consider only the harmonic $l = 0$ (s wave) in this treatment.

4.4.1 Wavemodes in Flat Spacetime

From the wave equation in flat Spacetime :

$$\Box\Phi = 0$$

we see that each wavemode travels at the speed of light as a massless particle. Lets depict a wavemode on two spacelike slices in flat spacetime :

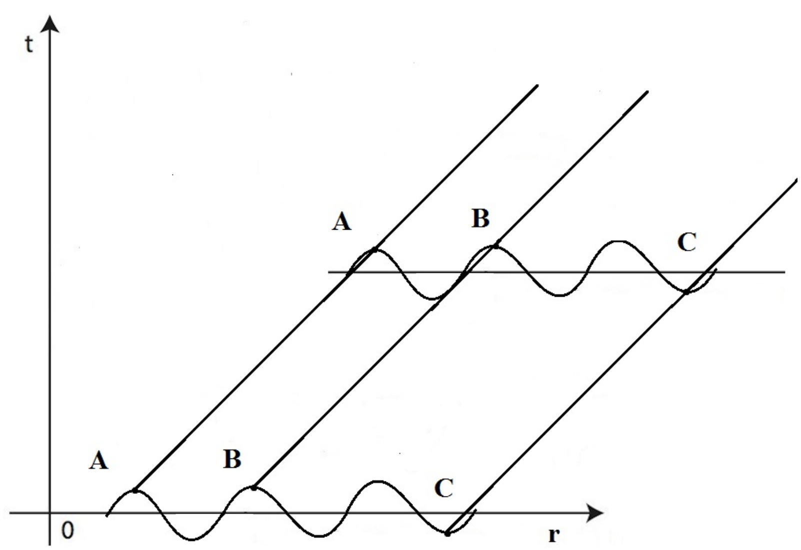

Figure 4.4.1: Evolution of one wavemode in two $t = constant$ spacelike slices for Flat Spacetime

At each point in the initial spacelike slice the wavemode has an amplitude and a phase. For every point in the slice we draw radial null geodesics which in flat space-time are straight lines at 45^0 angle , as seen in the above figure. For every geodesic we assign the same phase as the point in the initial spacelike slice. Each null geodesic has therefore the same phase. This process defines exactly the evolution of a wavemode from an initial slice to the future of the slice. The two $t = constant$ spacelike slices in the figure conserve the form of the initial wavemode since the distance between the points AB remains the same in each slice. Hence, the wavemode does not get deformed as the slice moves in time, which is characteristic of flat space-time. Lets clarify two questions that may have appeared in the readers mind.

First, why does each null geodesic have the same phase ? This is not at all obvious but we will use it without proof in our discussion although it can be proved for the geometrical approximation of waves [6]. Furthermore, the reader may have wondered what would happen if we used another spacelike slice that is not $t = constant$, but some other form. Surely, the wavemode would appear deformed in this choice of slice because the distance between AB will not remain the same. Does that suggest that we would have particle creation even in flat space-time? It can be proved that, although the wavemodes can get deformed even in flat spacetime by an unconventional but still valid choice of spacelike slices, the deformation can not get big enough for particle creation .

4.5 Slicing the Black Hole Geometry

Since we have seen the way a wavemode evolves at a later spacelike slice, we must now choose how to slice the Black Hole geometry, so that we cover both the outside and inside regions of the horizon. Lets consider the Schwarzschild metric :

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

There are three important regions we must take into account in constructing the spacelike slice :

- In the $r > \frac{2GM}{c^2}$ region, lets say $r = \frac{3GM}{c^2}$ we take the spacelike surface $t = constant$. (t Schwarzschild coordinate). This part of the spacelike slice is similar to the one in Minkowski Spacetime, since as r gets bigger and bigger the Schwarzschild metric approaches the Minkowski metric asymptotically. We call this part of the spacelike slice S_{out} .
- Inside the Horizon $r < \frac{2GM}{c^2}$, the light cones tip sides or as it is usually referred space and time interchange roles i.e the $t = constant$ slice becomes timelike and the $r = constant$ becomes spacelike. Because we want our slice to be spacelike we use a $r = constant$ slice . We take the slice to be at $r = \frac{GM}{c^2}$ which far from the horizon at $r = \frac{2GM}{c^2}$ and from the singularity $r = 0$. The reason we want it to be far from the singularity is because we want to avoid regions with very big curvature, so that quantum gravity corrections do not become very big. We name this part of the spacelike slice as S_{in} .

[6] Remember we are looking only for a qualitative picture of how wavemodes get deformed ,which in turn leads to particle creation

- The only part that remains is the part from $r = \frac{GM}{c^2}$ to $r = \frac{3GM}{c^2}$ that crosses the horizon at $r = \frac{2GM}{c^2}$. We connect this part with a spacelike surface called S_{con} . The Schwarzschild coordinates blow up in the horizon so in order to depict the connector we will use another time coordinate τ that for large r we let $\tau \to t$. The fact that such a spacelike connector exists, although not explicitly proved here , can be understood if we draw it in another coordinate system like a Penrose diagram and then transform it to Schwarzschild coordinates.

The Spacelike slice in the $r - \tau$ plane is shown below :

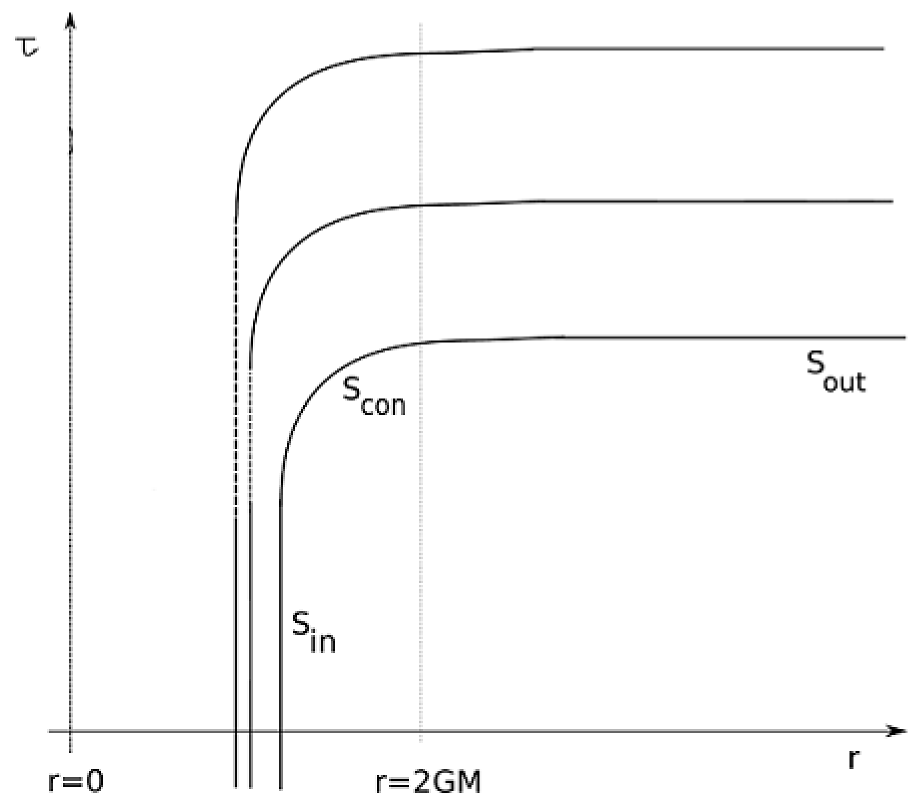

Figure 4.5.1: Spacelike slice depicted in the $r - \tau$ plane

Finally we have to determine how to evolve our slice at later "times" . Generally this is arbitrary as long as the slice remains spacelike and does not cross the singularity $r = 0$. These two conditions can be easily fulfilled by advancing the S_{out} part of the slice to a later time $t + \Delta$ and the S_{in} part to a smaller r so that it never reaches the singularity $r = 0$. The evolution of each slice can be also depicted in the Penrose Diagram below :

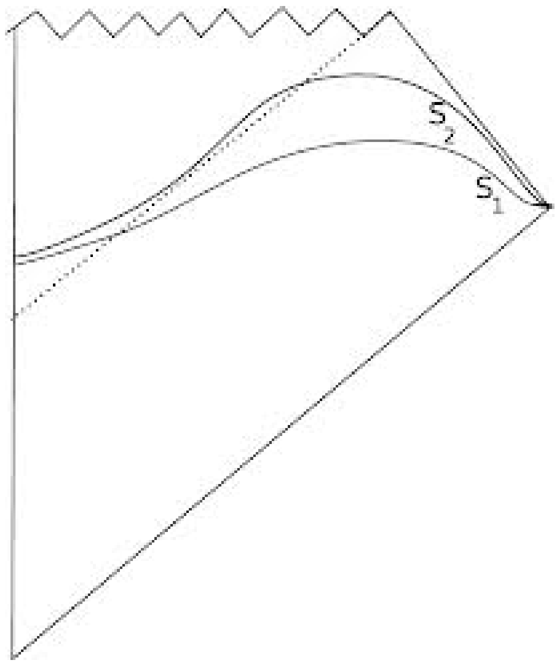

Figure 4.5.2: The evolution of a spacelike slice in Penrose Diagram

We see that the outer part S_{out} approaches the future null infinity without the S_{in} part getting close to the singularity , but at the same time the whole slice remains spacelike everywhere.

4.6 Evolution of Wavemodes in Black Hole Slicing - Particle Creation

We have so far described the way a wavemode evolves in each spacelike slice [7] , and also the exact form of the slices in Black Hole Geometry and how they evolve at later times [8]. We will now combine the two, and see how the initial "vacuum" mode gets distorted near the horizon, which in turn leads to particle creation. In the following figure we depict the evolution of the initial wavemode in two later slices for the Black Hole Geometry :

[7] See section : How Wavemodes move in Spacetime
[8] See section : Slicing the Black Hole Geometry

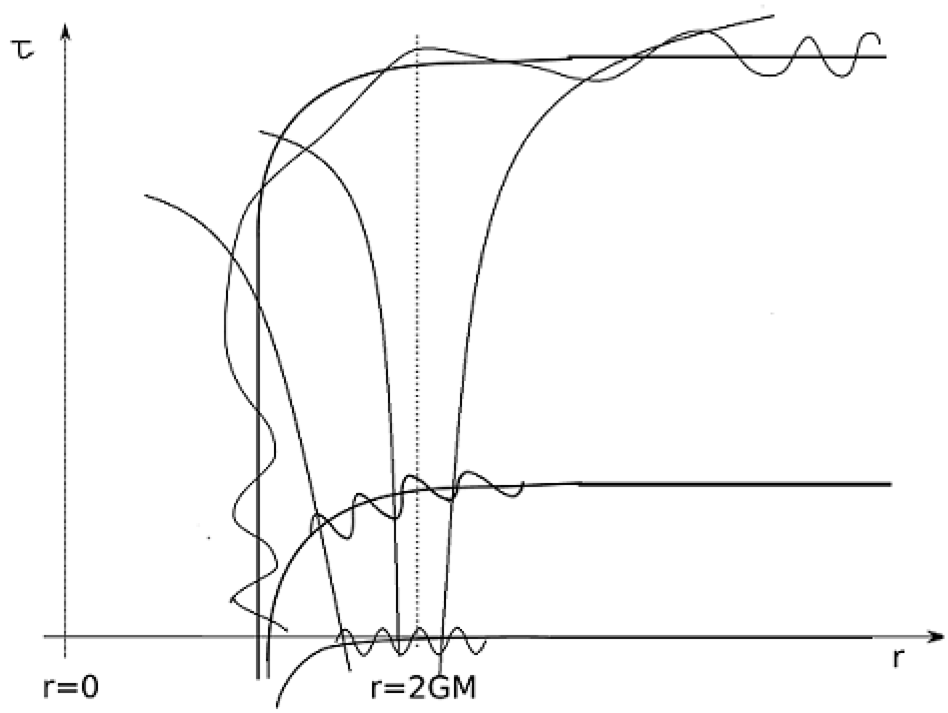

Figure 4.6.1: Stretching of Wavemodes in Black Hole Slicing

The lines of constant phase are also drown in the diagram , which as we have discussed before are radial null geodesics. So we can draw the wavemode in the next slice as seen above. We notice that the wavemode gets deformed. This is because null geodesics inside and outside the horizon "peel off" as time goes on. The ones just inside the horizon evolve towards smaller r , whereas the ones just outside eventually approach $r = \infty$.

The evolution of the modes in the $r - \tau$ diagram can provide us with some information, at least qualitatively, about the stretching of the modes. Although these observations can be proved mathematically we will not present the proofs here [6]. By looking at the diagram we observe that:

- Only modes very close to the horizon get stretched enough so that we have particle creation in the S_{out} , S_{in} areas of the spacelike slice.
- Modes with longer wavelengths get distorted first. Practically the higher the frequency of the wavemode the longer the mode must evolve so that there is enough stretching for particle creation.

4.7 State of the System

We have derived previously that the vacuum state $|0_a\rangle$ can be written as :

$$|0_a\rangle = Ce^{-\frac{1}{2}\sum_{m,n} \mathbf{b}^\dagger_\mathbf{m} \gamma_{mn} \mathbf{b}^\dagger_\mathbf{n}} |0_b\rangle$$

In our case we can break the creation operators $\mathbf{b}^\dagger_\mathbf{k}$ into those on S_{out} which we call $\mathbf{b}^\dagger_\mathbf{k}$, and those on S_{in} which we call $\mathbf{c}^\dagger_\mathbf{k}$. The state turns out to be (without proof) :

$$|0_a\rangle = Ce^{\sum_\mathbf{k} \gamma \mathbf{b}^\dagger_\mathbf{k} \mathbf{c}^\dagger_\mathbf{k}} |0_b\rangle |0_c\rangle$$

If we consider the state of the system for two wavemodes :

$$|\Psi_{k_1}\rangle = Ce^{\gamma \mathbf{b}^\dagger_{\mathbf{k_1}} \mathbf{c}^\dagger_{\mathbf{k_1}}} |0_{k_1}\rangle |0_{k_1}\rangle \qquad and \qquad |\Psi_{k_2}\rangle = Ce^{\gamma \mathbf{b}^\dagger_{\mathbf{k_2}} \mathbf{c}^\dagger_{\mathbf{k_2}}} |0_{k_2}\rangle |0_{k_2}\rangle$$

The operator $\mathbf{b}^\dagger_{\mathbf{k_1}}$ creates a quantum of momentum k_1 in the region S_{out} , and the operator $\mathbf{c}^\dagger_{\mathbf{k_1}}$ creates a quantum of momentum k_1 in the region S_{in}. Similarly for the operators $\mathbf{b}^\dagger_{\mathbf{k_2}}$, $\mathbf{c}^\dagger_{\mathbf{k_2}}$.

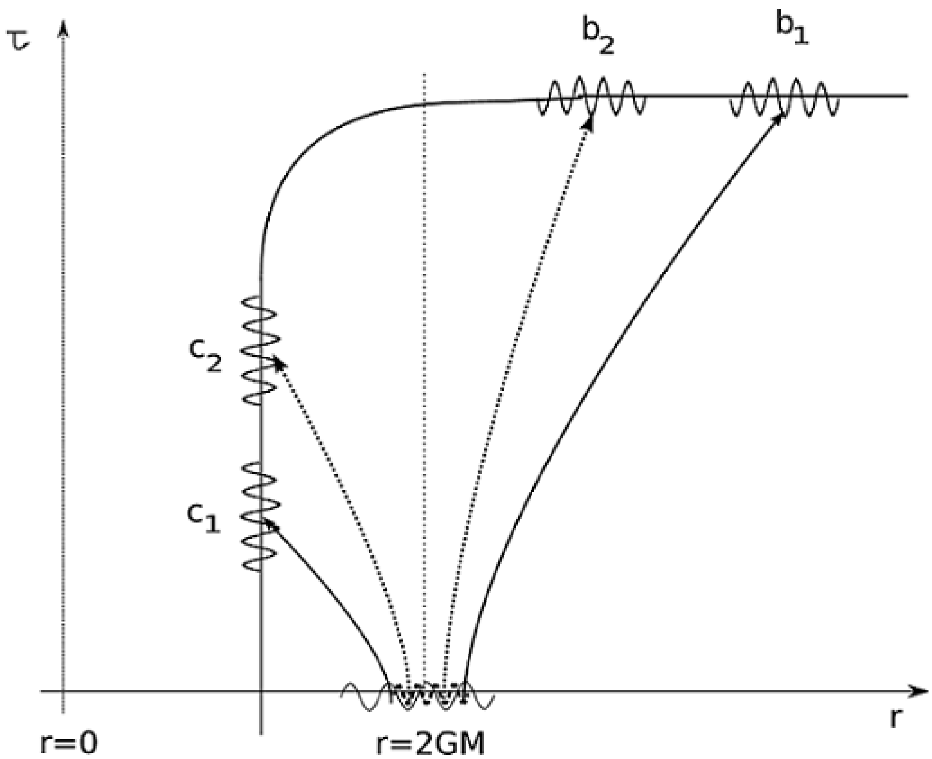

Figure 4.7.1: Pairs of Particles created from two wavemodes near the Black Hole Horizon

In the above figure we assume that $\lambda_2 < \lambda_1$. Since the two wavemodes have different wavelengths, they take different times to get the same distortion and create the pairs b, c shown in the figure. So the particles do not overlap in the same spacelike slice (the corresponding operators commute) and the complete state can be written as the tensor product :

$$|\Psi\rangle = |\Psi_{k_1}\rangle \otimes |\Psi_{k_2}\rangle \otimes |\Psi_{k_3}\rangle \otimes \dots \qquad \forall\, k \subseteq \mathbb{R}$$

If we expand the exponential of the $|\Psi_{k_1}\rangle$ we get [9] :

$$|\Psi_{k_1}\rangle = C\left(|0_{b,k_1}\rangle \otimes |0_{c,k_1}\rangle + \gamma\, \mathbf{b}^{\dagger}_{\mathbf{k_1}}\, |0_{b,k_1}\rangle \otimes\, \mathbf{c}^{\dagger}_{\mathbf{k_1}}\, |0_{c,k_1}\rangle + \frac{\gamma^2}{2!}\, \mathbf{b}^{\dagger}_{\mathbf{k_1}} \mathbf{b}^{\dagger}_{\mathbf{k_1}}\, |0_{b,k_1}\rangle \,\otimes\, \mathbf{c}^{\dagger}_{\mathbf{k_1}} \mathbf{c}^{\dagger}_{\mathbf{k_1}}\, |0_{c,k_1}\rangle + \dots\right) \Rightarrow$$

$$|\Psi_{k_1}\rangle = C\left(|0_{b,k_1}\rangle \otimes |0_{c,k_1}\rangle + \gamma\, |1_{b,k_1}\rangle \otimes\, |1_{c,k_1}\rangle + \gamma^2\, |2_{b,k_1}\rangle \,\otimes\, |2_{c,k_1}\rangle + \dots\right)$$

[9]We will use the relation: $\left(\mathbf{b}^{\dagger}_{\mathbf{k_1}}\right)^n \left|0_{b,k_1}\right\rangle = n! \left|n_{b,k_1}\right\rangle$

We notice from the above expression that the outgoing b quanta and the ingoing c quanta are always created in pairs.

4.8 Mixed State of the Hawking Radiation

We will now proceed to find out whether the outgoing quanta b, are in a mixed or in a pure state. We saw in the previous section that the state of the whole system is:

$$|\Psi\rangle = |\Psi_{k_1}\rangle \otimes |\Psi_{k_2}\rangle \otimes |\Psi_{k_3}\rangle \otimes \dots \qquad \forall\, k \subseteq \mathbb{R}$$

We will first just consider the state of one wavemode $|\Psi_{k_1}\rangle$ and then with the help of the properties of Von Neumann entropy [10] we will be able to generalize for the whole state $|\Psi\rangle$. The state of one wavemode $|\Psi_{k_1}\rangle$, which we will call in short $|\Psi_1\rangle$ is :

$$|\Psi_1\rangle = C\left(|0_{b_1}\rangle \otimes |0_{c_1}\rangle + \gamma\ |1_{b_1}\rangle \otimes\ |1_{c_1}\rangle + \gamma^2\ |2_{b_1}\rangle\ \otimes\ |2_{c_1}\rangle + \dots\right)$$

The normalization C is given by the formula [11] :

$$C^2 \sum_{n=0}^{\infty} \gamma^{2n} = 1 \quad \Rightarrow \quad C^2 = \frac{1}{\sum_{n=0}^{\infty} \gamma^{2n}}$$

The quanta c_1 lie in the region inside the horizon and the b_1 just outside the horizon, which escape as Hawking Radiation. From the state $|\Psi_1\rangle$ we see that the state of the b quanta are entangled with the state of the c quanta. Since we are interested in the state of the Hawking Radiation (e.g the b quanta) we have the familiar case[12] of finding the reduced density matrix of a system which is entangled. It probably has become obvious at this point that the state of the b quanta will turn out to be mixed. But in order to prove this we must first find the reduced density matrix for the b quanta.

4.8.1 Reduced Density Matrix- Entropy of the b Quanta

The density matrix for the state $|\Psi_1\rangle$ is :

$$\rho_1 = |\Psi_1\rangle \langle\Psi_1| \quad \Rightarrow$$

[10] See chapter 3: Entropy of the Product of two states
[11] It is important to mention that γ turns out to be less than 1, so that every series converses
[12] See Chapter 2 : Mixed States from Pure States- Reduced Density Matrix

$$\rho_1 = C^2 \left(|0_{b_1}\rangle \otimes |0_{c_1}\rangle + \gamma \; |1_{b_1}\rangle \otimes \; |1_{c_1}\rangle + \ldots \right) \left(\langle 0_{b_1}| \otimes \langle 0_{c_1}| + \gamma \; \langle 1_{b_1}| \otimes \langle 1_{c_1}| + \ldots \right)$$

From the above density matrix we can easily find the reduced density matrix for the b quanta, by tracing over the basis of the c quanta $\{ \, |0_{c_1}\rangle , |1_{c_1}\rangle , |2_{c_1}\rangle , \ldots \} = \{ |n_{c_1}\rangle , \quad with \, n = 0,1,2,3,\ldots \}$:

$$\rho_b = Tr_c\left(\rho_1\right) = \sum_n \langle n_{c_1}| \, \rho_1 \, |n_{c_1}\rangle \; \Rightarrow$$

$$\rho_{b_1} = C^2 \left(|0_{b_1}\rangle \langle 0_{b_1}| + \gamma^2 \, |1_{b_1}\rangle \langle 1_{b_1}| + \gamma^4 \, |2_{b_1}\rangle \langle 2_{b_1}| + \ldots \right) \; \Rightarrow$$

$$\rho_{b_1} = C^2 \sum_{n=0}^{\infty} \left(\gamma^{2n} \, |n_{b_1}\rangle \langle n_{b_1}| \right), \qquad with \; C^2 = \frac{1}{\sum\limits_{n=0}^{\infty} \gamma^{2n}}$$

We can also write the reduced density matrix in matrix form, in the basis : $\{ \, |0_{b_1}\rangle , |1_{b_1}\rangle , |2_{b_1}\rangle , \ldots \} = \{ |j_{b_1}\rangle , \quad with \; j = 0,1,2,3,\ldots \}$:

$$\rho_{b_1} = C^2 \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \gamma^2 & 0 & \vdots & 0 \\ 0 & 0 & \gamma^4 & 0 & \vdots \\ \vdots & \cdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \gamma^{2n} \end{bmatrix}$$

We see that the matrix ρ_{b_1} is diagonal , and has eigenvalues $\left\{ \, C^2, C^2\gamma^2, C^2\gamma^4, \ldots\ldots, C^2\gamma^{2n} \, \right\}$.

Now, in order to determine whether this is a pure or a mixed state we will calculate the entropy $S_{\rho_{b_1}}$:

$$S_{\rho_{b_1}} = -\sum_i \lambda_i log\left(\lambda_i\right) = -\sum_{n=0}^{\infty} C^2 \gamma^{2n} log\left(C^2\gamma^{2n}\right) =$$

$$= -\left(C^2 log\left(C^2\right) + C^2\gamma^2 log\left(C^2\gamma^2\right) + C^2\gamma^4 log\left(C^2\gamma^4\right) + \ldots\ldots \right)$$

Although the whole series above may seem difficult to calculate exactly [13] we can easily prove that $S_{\rho_{b_1}}$ is greater than zero. In order to prove that we only have to notice that:

$$C^2\gamma^{2n} = \frac{\gamma^{2n}}{\sum_{n=0}^{\infty}\gamma^{2n}} < 1 \qquad \forall\, n \subseteq \mathbb{N}$$

I we take the logarithm of the above inequality we find :

$$log\left(C^2\gamma^{2n}\right) < 0 \;\Rightarrow$$

$$-C^2\gamma^{2n} log\left(C^2\gamma^{2n}\right) > 0 \;\Rightarrow$$

$$\sum_{n=0}^{\infty}\left(-C^2\gamma^{2n} log\left(C^2\gamma^{2n}\right)\right) > 0 \;\Rightarrow$$

$$S_{\rho_{b_1}} > 0$$

So far we have proved what we anticipated , that the system of b quanta at least for one wavemode, is mixed . What remains to be done now, is just to prove that state of b quanta for all wavemodes is mixed.

4.8.2 Entropy of the Whole System

We have found the entropy of the b quanta for the wavemode k_1 but we are interested in the entropy of all the hawking radiation which consists of wavemodes of all k. The state of the whole system as we have mentioned before is the tensor product of the states of all wavemodes :

$$|\Psi\rangle = |\Psi_{k_1}\rangle \otimes |\Psi_{k_2}\rangle \otimes |\Psi_{k_3}\rangle \otimes \ldots \qquad \forall\, k \subseteq \mathbb{R}$$

We have explicitly discussed in chapter 3 [14] that the reduced reduced density matrix of the whole system for the b quanta is :

$$\rho_b = \rho_{b_1} \otimes \rho_{b_2} \otimes \rho_{b_3} \otimes \rho_{b_4} \otimes \ldots\ldots$$

Using the property of subadditivity of the Von Neumann entropy we have that :

$$S_{\rho_b} = S_{\rho_{b_1}} + S_{\rho_{b_2}} + S_{\rho_{b_3}} + \cdots > 0$$

[13] γ must be less than one so that the series converges

[14] See chapter 3 : Entropy of the Product of two States

since $S_{\rho_{b_1}}, S_{\rho_{b_2}}, S_{\rho_{b_1}}, \cdots > 0$

We have come to the conclusion that the state of all the b quanta that escape as Hawking Radiation , are in a mixed state since the entropy is larger than zero. This may at first glance not appear as a problem because the system of the b quanta is entangled with the system of c quanta. But as the b quanta are emitted to infinity the Black Hole slowly loses mass and eventually disappears, [15] so there are no c quanta that they are entangled with. All that remains is the Hawking Radiation which we just proved cannot be described by a pure state. It seems therefore that there is an inherent violation of unitarity in Black Hole evaporation, since a pure state evolves into a mixed state. Hawking [7] claimed that we must describe quantum mechanics through density matrices rather than pure states when we take gravity into account, since in any quantum gravity situation there is the possibility of virtual black holes popping in and out of existence from quantum vacuum fluctuations. This was a very radical and unsatisfying idea for someone who believed that unitarity was a fundamental postulate of Quantum Mechanics.

[15] There are theories however that suggest that Black Holes do not disappear completely, but exist as a “remnant”.

Chapter 5

The Black Hole Information Paradox

Paradoxes have always been the seeds for great paradigm shifts in the history of physics. There are generally two kinds of Paradoxes in physical theories.

The first kind of paradoxes arise when existing physical theories in certain extreme conditions give infinities that cannot possibly describe reality. I say possibly because one should not really predestine how nature comes out to be , but it is generally believed that infinities are strong indications of an incomplete physical theory. There are many examples of "Infinity" Paradoxes found in the History of physics. The infinite self energy of the electron is such an example, or the space- time singularities predicted by Einstein's General Theory of Relativity. Of course one of the most important ones was the ultraviolet catastrophe, which was a prediction of classical physics, that an ideal black body at thermal equilibrium will emit radiation with infinite power. Some of the infinity paradoxes like the ultraviolet catastrophe have lead to great revolutions in our understanding of nature (Quantum Mechanics), and some others remain unsolved even today.

The second kind of paradoxes arise when two or more principles of physics are inconsistent with each other. A deep belief among physicists is that there is an underlying consistent physical theory that describes nature. So any principles which are inconsistent with each other must be modified or replaced. An example of an "inconsistency" paradox is the conflict that appeared between the Principle of Relativity, the Galilean transformations of space and time and Maxwell's theory of Electromagnetism. Something had to be modified and in this case (as it is very well known) the Galilean transformations were replaced by Lorentz transformations. In the same way we can formulate the evaporation process of a Black Hole as a case where known principles of physics are inconsistent of each other. In this regard, we call this apparent conflict, as the Information Paradox.

5.1 Unitarity and the Equivalence Principle : Two conflicting Principles

In the previous chapter we saw that the Hawking Radiation of an evaporated Black Hole is in a mixed sate, which violates the Unitarity Principle of Quantum Mechanics. One could claim that this by itself is a paradoxical result since we derived the non unitary evolution of Black Hole evaporation by using Quantum field theory, which by itself is unitary. We will see that even if the Hawking Radiation was in a pure state there would still be a violation of unitarity. In this sense, there is really a paradox since both possible outcomes of the Black Hole evaporation (information loss or not) creates a conflict between known principles of physics. We will analyze the problem more extensively below :

Consider the formation and complete evaporation of a black hole by a spherical shell of collapsing mass M . The black hole radiates away slowly and finally evaporates completely. We assume also that the initial state of the collapsing mass is in the pure state $|\phi\rangle$. We present the Penrose diagram for the formation an evaporation of the Black Hole below [1] :

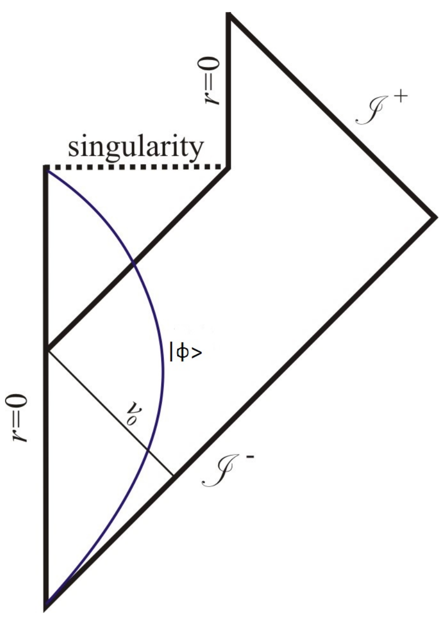

Figure 5.1.1: Penrose Diagram for Evaporating Black Hole

There are two postulates we will assume :

[1] For more about Penrose diagrams for evaporating black holes see: ($http://golem.ph.utexas.edu/~distler/blog/archives/001994.html$)

- The first postulate is the assumption that the formation and evaporation of a black hole is described by a unitary S matrix, much like ordinary scattering in Quantum Field Theory. We deduce therefore that the state of the Hawking Radiation after the black hole has evaporated completely is still in the state $|\phi\rangle$, if we use the Heisenberg Picture. The Hawking Radiation is depicted below :

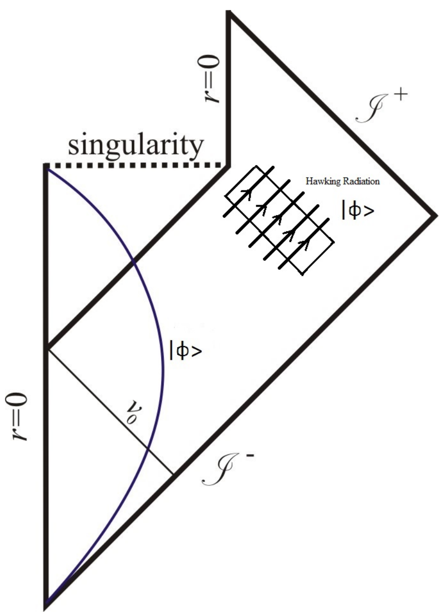

Figure 5.1.2: Penrose Diagram with Hawking Radiation

- The second postulate is the Equivalence Principle which is embedded in the General theory of Relativity. According to the equivalence principle, for scales significantly smaller than the radius of the Black Hole the horizon can be approximated by ordinary flat space. So the in-falling matter does not experience anything unusual as it crosses the horizon. This means then that the state $|\phi\rangle$ is intact at a region inside the horizon as shown below :

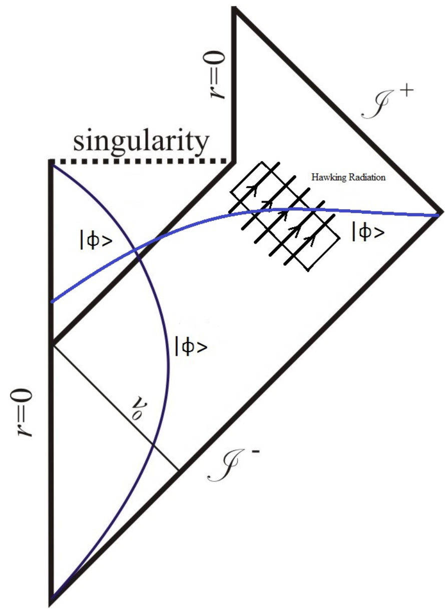

Figure 5.1.3: Spacelike Slice that contains both states $|\phi\rangle$

Here is the problem. If both postulates are true, then we can construct[2] a spacelike slice (blue line in the above figure) that contains both the sate of the in-falling matter and the state of the Hawking Radiation. The no cloning theorem[3] states that the process of unitary evolution cannot faithfully replicate quantum information in two separate sets of commuting degrees of freedom. If the infalling state is completely recorded inside the horizon (as the Equivalence Principle predicts) then it cannot be simultaneously found in the Hawking Radiation and vice verse.

As a conclusion we can sum up the Information Paradox in the following matrix:

$$\text{Black Hole Evaporation} \longrightarrow \begin{bmatrix} \text{Information IS conserved :} & \text{Violation of No-Cloning Theorem-Unitarity} \\ \\ \text{Information is NOT conserved :} & \text{Violation of Unitarity} \end{bmatrix}$$

[2]For an explicit construction of such slice in kruskal coordinates see section 2 of [8]
[3]See chapter 3: No Cloning Theorem

5.2 Black Hole Complementarity Principle

We have shown a serious conflict between two very well established principles of physics. From the perspective of the distant observer whatever gets thrown into the black hole gets thermalized near the horizon and eventually comes out through the hawking radiation, since the whole process can be described by an ordinary unitary S matrix. From the point of view of the in-falling observer however nothing special happens near the horizon and all the information is able to pass inside absolutely intact and in finite proper time. Both points of view suggest that there is a violation of the no- cloning theorem of Quantum Mechanics. How could both versions be true?

A close analogy to the above problem is the wave - particle duality problem that physicists encountered in the early 1900's . It was observed that particles appeared to have wave- like properties in some cases, and waves like the electromagnetic wave had particle like properties in some other cases (photoelectric phenomenon). This was very confusing since particle and waves seemed to be contradictory notions with each other. How could something be both ? Bohr's response to the problem was the very well known complementarity principle, which stated that neither waves nor particles are complete descriptions of reality, but that they are complementary to each other. Depending on what you want to measure in a specific experiment , different aspects will appear in different circumstances. Sometimes an experiment will unveil the wave characteristics while other experiments the particle characteristics, but never both at the same time. Since no experiment will produce contradictory results there is really no paradox.

In the same manner Leonard Susskind, Larus Thorlacius and John Uglum [9] proposed the Black Hole Complementarity Principle. In order for both stories of the distant and the in-falling observer to be true there should be a principle preventing these two observers communicating with each other. In other words cloning is fictitious since there is no way an observer can confirm that it happened. Every description of a single observer is consistent with unitarity and the equivalence principle. The problem appears when we try to describe reality in a global way , but if we restrict ourselves to one observer everything is consistent with the known principles of physics and no contradiction arises. There are many strategies one could imagine in order to show that the Black Hole Complementarity principle is violated, much like there were many experiments designed to disprove the uncertainty principle of Quantum Mechanics. We will consider only one such thought experiment and see that Black Hole complementarity is not violated.

5.2.1 Test of the B.H Complementarity Principle

We will now consider a certain strategy two observers can follow in order for one of them two observe the two clones of some quantum mechanical system in state $|\phi\rangle$. If such strategy is physically possible then it is obvious that the complementarity principle is violated. Imagine two observers

Alice (A) and Bob (B). Alice prepares a quantum system in state $|\phi\rangle$ and then falls into the Black Hole Horizon carrying the system with her. Both Alice and Bob have conspired to implement the following strategy : Bob will stay outside the black hole horizon as a Schwarzschild observer for Schwarzschild time Δt . The time that Bob stays outside the horizon should be enough so that he can retrieve the state $|\phi\rangle$ from the Hawking Radiation. After he has retrieved the state $|\phi\rangle$ he falls into the horizon. Alice on the other hand sends a signal from inside the horizon transmitting the state, so that Bob receives the signal just as he crosses the horizon and before hitting the singularity. In this way Bob will have both the state $|\phi\rangle$ from the Hawking Radiation and the state $|\phi\rangle$ that Alice send, verifying that cloning has occurred. We have now to determine whether such strategy is physically possible, because if it is, there would be a clear violation of unitarity and the complementarity principle. We must warn the reader that for detailed Gedanken experiments involving Black Hole Complementarity he should refer to [10]. We are more interested at the general idea of how B.H Complementarity works at this point.

It is convenient to use Kruskal coordinates since we wish to describe both the inside and the outside of the Black Hole Horizon. The metric in Schwarzschild coordinates is [4]:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

The transformation relations to Kruskal coordinates (U, V) is :

$$U = -e^{\frac{(r^\star - t)}{4M}} \qquad\qquad V = -e^{\frac{(r^\star + t)}{4M}}$$

where $r^\star = r + 2M\, log\left|\frac{r}{2M} - 1\right|$. The Schwarzschild line element then becomes[5] :

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} dUdV$$

The metric in Kruskal coordinates is presented below :

[4] $c = G = \hbar = 1$
[5] We take $d\Omega = 0$

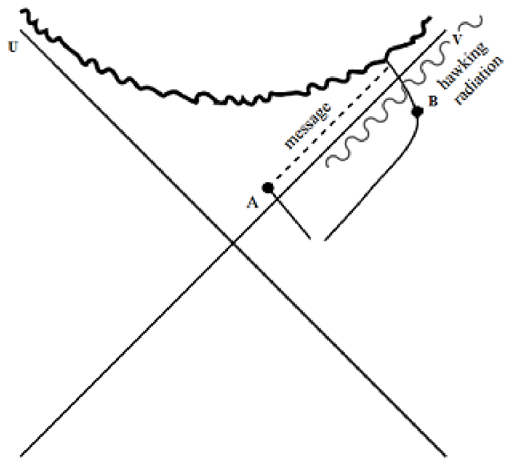

Figure 5.2.1: Alice (A) and Bob (B) in Kruskal coordinates

The singularity is represented by the hyperbola $UV = 1$ and the horizon by the line $U = 0$. From the diagram it is easy to see that the more time Bob waits outside the Black Hole horizon the smaller the value U becomes when he runs into the singularity. The smaller the U is , the less proper time Alice has in order to send a message that can reach Bob before hitting the singularity. This is expressed by the relation [6] :

$$\tau \sim Re^{\left(-\frac{\Delta t}{R}\right)} \tag{5.2.1}$$

where τ is the proper time Alice has before Bob is out of reach. Δt is the Schwarzschild time Bob waits outside the horizon before falling in. R is the Schwarzschild radius $(R = 2M)$.

In 1993 there was a paper published by Don Page [11] which showed that the Schwarzschwild time Δt necessary for Bob to reconstruct $|\phi\rangle$ out of the Hawking radiation is of order :

$$\Delta t \sim M^3 \tag{5.2.2}$$

From eq. (5.2.1) , (5.2.2) we see that :

$$\tau \sim Re^{\left(-\frac{M^3}{R}\right)}$$

[6] Almost all relations will be presented without formal proof

and since $R = 2M$:

$$\tau \sim Me^{\left(-M^2\right)}$$

From the uncertainty principle of energy and time we have that the energy of the signal Alice has to send is of order :

$$E_{signal} \cdot \tau \sim 1 \quad \Rightarrow E_{signal} \sim \frac{1}{\tau} \quad \Rightarrow E_{signal} \sim \frac{1}{M} e^{M^2}$$

Will will now show that the energy of the signal Alice has to send is bigger than the energy of the Black Hole :

$$\left.\begin{array}{r} E_{BH} = M \\ \\ E_{signal} \sim \frac{1}{M} e^{M^2} \end{array}\right| \frac{E_{signal}}{E_{BH}} = \frac{e^{M^2}}{M^2} > 1 \quad \forall M$$

So we see that if Alice wants to communicate to Bob , she must send such a high- energy pulse that would disturb the Black Hole geometry immensely. It is obvious then, that Bob cannot physically receive both clones before falling into the singularity. Black Hole Complementarity is therefore NOT violated.

5.3 Conclusion

For over 35 years the Paradox that Hawking discovered has not even come close to being resolved. It is very hard to find a mechanism in which information can escape from a black hole , and still be consistent with the semi-classical approximation that Hawking used. In fact if Quantum Gravity effects are needed to explain what is happening, then we could be very far from finding a complete solution. It is very confusing how or why quantum gravity effects can have such a big impact in the regions of energy and curvature that the calculations were made. Nevertheless, it seems that there are a lot more ideas and paradigm shifts that can emerge from the problem.

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