Hamiltonian Holography: A different approach to the AdS/CFT correspondence

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This is the bachelor thesis of Alexander Tabler *atampler@physics.auth.gr* carried out in the Institute of Theoretical Physics, Physics Department, Aristotle University of Thessaloniki, under supervision of Associate Professor A. Petkou.

Disclaimer: At the time of writing this thesis I have very limited knowledge of many important components of the AdS/CFT duality, namely string theory or other theories of quantum gravity, renormalization, as well as strictly conformal field theories. As a result of this, some aspects of what I have written here are only descriptively understood. However the AdS/CFT provides us with a concrete dictionary to applications, for which only specific elements of these components are needed, making this thesis (with lots of background reading and my constant nagging of my supervisor) possible.

Abstract

To be written in the end

Contents

Chapter 1

Introduction

The gauge/gravity duality is one of the most important results of string theory so far. It refers to the correspondence (duality or equivalence) of seemingly different physical theories: **String theory** on the one hand, which in some limits (low energy or string tension) becomes a quantum theory of gravity or even a theory of classical supergravity (general relativity in more than 4 spacetime dimensions), and **quantum field theory** on the other hand, which is a theory with no gravity whatsoever. An explicit example of the conjectured duality is the AdS/CFT Correspondence, also known as the Maldacena conjecture proposed by Juan Maldacena in 1997 [1]. The latter states that Type IIB String Theory compactified in specific 10-dimensional backgrounds involving Anti-de Sitter spacetimes (AdS), in particular in $AdS_5 \times S^5$, is *exactly* equivalent to a Supersymmetric Yang-Mills (SYM) gauge theory which is a Conformal Field Theory (CFT) in 4 dimensions. This includes a map between observables, correlators, operators of the field theory and dynamics of the fields in string theory. In addition there is as we will see both a strong/weak coupling duality, meaning that weak coupling in one theory is related to strong coupling in its dual theory, as well as an IR/UV duality, which means that high energy processes on one theory correpond to low energy processes in the other theory. This is part of what makes this duality so useful: strong coupling regimes are difficult to handle in perturbation theory and one can use the dual weakly coupled theory to read off information about the strongly coupled one. This particular conjecture has been thoroughly tested [2] in the limits where computations can be made (classical supergravity), but it is believed to be generally true (all values of the coupling constants). Moreover since its proposal other realizations of gauge/gravity duality have been discovered, which in turn leads us to believe that these are just particular cases of something deeper and more general.

1.1 Evidence for the gauge/gravity duality

The duality is still a conjecture in all but a few of its versions (that have been evaluated explicitly), but there are deep ideas of physics that point towards it. Here we state the most important hints for a general gauge/gravity duality (note that the AdS/CFT duality is a specific case of this and involves more evidence about its credibility):

Holography (XXX ramallo, boer) Firstly the so called *holographic principle* proposed by the study of thermodynamics of black holes in string theory states that the degrees of freedom of a theory of quantum gravity on some manifold actually live on the boundary of the manifold. This is thought to be a general property of theories of quantum gravity.

To make this more precise let us consider a quantum field theory in d dimensional spacetime ($d-1$ spacial part (R_{d-1}) + time). The number of degrees of freedom are measured by the entropy S (actually by $\sim e^S$) which in this theory is an extensive quantity, meaning it is proportional to the (spacial) volume of the system:

$$S_{QFT} \sim Vol\,(R_{d-1}) \tag{1.1}$$

In a theory of quantum gravity in $b+1$ spacetime dimensions (b spacial part (R_b) + time) on the other hand entropy, thus the number of degrees of freedom, is not extensive since it is bounded by the entropy of a black hole fitting on the R_b. The Hawking-Bekenstein formula tells us that the entropy of a black hole is proportional to the *area* of its event horizon

$$S_{BH} = \frac{1}{4G_N} A_{EH} \tag{1.2}$$

where G_N is Newton's constant (in appropriate dimensions if needed). The area in this context is proportional to the area of a boundary manifold $\partial R_b \sim R_{b-1}$ of the spacial part R_b. Consistency in matching the number of degrees of freedom is reached only if $b = d$

$$S_{BH} \sim S_{QFT} \sim Area\,(\partial R_d) \sim Vol\,(R_{d-1}) \tag{1.3}$$

or consequently if the gravity theory lives in more dimensions than the field theory. This is a situation similar to a *hologram* which is a d-dimensional system which encodes $d+1$-dimensional information. In analogy to this, the study of AdS/CFT and gauge/gravity duality in general is called *holography* although it is thought to be a specific case of the holographic principle.

Large N Limit (XXX boer, zaffaroni) Secondly, a strong hint is the idea that the large N limit of a Young-Mills gauge theory (e.g. large number of colours in a gauge theory with an $SU(N)$ gauge group) is actually equivalent to string theory (XXX 16 of boer). In this limit one can perturbatively expand the gauge theory partition function (ASK) in terms of $1/N$ and get

$$Z = \sum_{g \geq 0} N^{2-2g} f_g(\lambda) \tag{1.4}$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling which is fixed, g_{YM}^2 is the Yang-Mills coupling (gauge theory) g is the genus (number of "holes") of the corresponding Riemann surface of the graph expansion and f_g is a Polyakov path integral. In string theory, the *loop expansion* in the string coupling g_s is

$$Z = \sum_{g \geq 0} g_s^{2g-2} Z_g \tag{1.5}$$

which is astonishingly similar if we take the string coupling g_s to equal $1/N$.

Other Hints(XXX boer rammalo) Another hint towards the gauge/gravity duality is provided by the fact that gravity in three dimensions can be described to some extend by a topological field theory called *Chern-Simons* theory. This theory is described on a three dimensional manifold (2+1) with a boundary and can be reduced to studying the 1+1 field theory on the boundary. Lastly a strongly coupled condensed matter system can be studied in a way where new weakly coupled degrees of freedom emerge, which astonishingly live in one extra dimension and their corresponding theory is gravity.

1.2 Results of AdS/CFT

The results of the gauge/gravity duality, specifically of the AdS/CFT correspondence reach far beyond string theory. Applications have been made in very different domains: strong coupling of QCD and electroweak theories (calculation of the quark-antiquark potential), black hole physics and quantum gravity (black hole thermodynamics), relativistic hydrodynamics and condensed matter physics (transport coefficients and viscocities), quantum mechanics (quantum hall effect) (ASK).

1.3 Outline

The AdS/CFT correspondence provides us with a concrete dictionary to read off information about the dual field theory, specifically one can compute correlation functions of the field theory in the strong coupling regime which is nearly impossible with perturbative methods. The outline of this work is as following: We will begin by describing general features of conformal field theories (CFTs) and of Anti-de Sitter spacetimes and why we use it. Then we will introduce the AdS/CFT correspondence and give a full picture of the duality and the various limits that simplify it, as well as its particular, simple "recipe" (that uses classical gravity) to compute 2-point functions. After reviewing the different methods to carry this out we introduce the Hamiltonian version of the recipe and discuss various results. Finally we study the related problem, the inverted quantum harmonic oscillator (XXX check again)

Chapter 2

The AdS/CFT Correspondence

2.1 Conformal field theories

Symmetry principles are of paramount importance in physical theories - Lorentz (Poincare) symmetries are the backbone of the formulation of classical and quantum field theories: our axioms require that these theories are invariant under the action of corresponding groups. It is therefore natural to look for generalizations of these symmetries since crucial physical information is involved. A very straight-forward case would be to investigate theories that enjoy scale invariance i.e. are invariant under scale transformations of spacetime coordinates. Theories that are scale-invariant are thought also to be *conformally invariant* (XXX zaffaroni) and quantum field theories that are conformally invariant are called Conformal Field Theories ($CFTs$). We will state the most important features of conformal transformations, of the conformal group and algebra, and of conformal field theories.

2.1.1 Conformal transformations

Conformal transformations are generalized coordinate transformations

$$x^{\mu} \rightarrow x'^{\mu} = x'^{\mu}\left(x^{\nu}\right) \tag{2.1}$$

such that they rescale the line element of a manifold but preserve the angles between lines on the manifold. This means that the transformed metric tensor satisfies[1]

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega^{2}\left(x\right) g_{\mu\nu} \Longrightarrow ds'^{2} \rightarrow \Omega^{2}\left(x\right) ds^{2} \tag{2.2}$$

[1] In fact this type of transformation works differently depending on whether the metric $g_{\mu\nu}$ is fixed or fluctuating (dynamical). If $g_{\mu\nu}$ is dynamical as in General Relativity the transformation is a *differomorphism* and the corresponding symmetry a gauge symmetry. On the other hand, if the metric $g_{\mu\nu}$ is simply a fixed background metric the transformation corresponds to a global physical symmetry (XXX Tong). We will study the case of fixed *flat* background $\eta_{\mu\nu}$.

A CFT is then *invariant* under this type of transformation. This means that there is no "preferred" length or energy scale in the theory but we will discuss this and other effects of conformal invariance later. Examples of conformal transformations would of course include all Poincare transformations (translations, rotations & boosts) which have $\Omega^2(x) = 1$, and also dilatations

$$x^\mu \to x'^\mu = \lambda x^\mu \quad , \quad \Omega = \lambda$$

To study all kinds of conformal transformations we need to look into the infinitesimal transformations of the form 2.2. We take

$$\begin{aligned} x'^\mu &= x^\mu + v^\mu(x) \ , \quad v^\mu \ll 1 \\ \Omega(x) &= 1 + \frac{\omega(x)}{2} \\ g_{\mu\nu} &= \eta_{\mu\nu} \end{aligned} \tag{2.3}$$

and inserting into 2.2 while neglecting orders higher than $\sim v^\mu$ and $\sim \omega$ we have

$$\begin{aligned} dx'^\mu &= dx^\mu + \partial_\alpha v^\mu dx^\alpha \\ &\Rightarrow \ [...] \ \Rightarrow \\ \partial_\mu v_\nu + \partial_\nu v_\mu &= \omega(x)\,\eta_{\mu\nu} \end{aligned} \tag{2.4}$$

we can solve $\omega(x)$ in terms of v^μ by taking the trace

$$\omega(x) = \frac{2}{d}\partial^\mu v_\mu$$

where d is the number of dimensions. The defining equation for infinitesimal conformal transformations is then

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{d}\left(\partial^\alpha v_\alpha\right)\eta_{\mu\nu} = 0 \tag{2.5}$$

The solutions v^μ are also called the *conformal Killing vectors.* This equation has an infinite number of solutions in two dimensions[2] (XXX tong zaff) which means that two-dimensional conformal transformations are an infinite-dimensional group. In more than two dimensions however there is a finite number of solutions, and one can show (XXX springer) that the transformation in $v^\mu(x)$ is at most quadratic in terms of x^μ in $d \geq 3$. The general solution involves four kinds of transformations listed in the table:

[2] In fact, all holomorphic functions $v(x)$ are solutions to (2.1.5) and generate conformal transformations.

Transformation	Infinitesimal	Finite	Generator
Translation	$x'^{\mu} = x^{\mu} + a^{\mu}(\text{constant})$	$x'^{\mu} = x^{\mu} + a^{\mu}(\text{constant})$	$P^{\mu} = -i\partial^{\mu}$
Lorentz	$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu}$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$	$x'^{\mu} = M^{\mu}{}_{\nu}x^{\nu}$,$M = e^{\omega}$	$J^{\mu\nu} = i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)$
Dilatation	$x'^{\mu} = \lambda x^{\mu}$	$x'^{\mu} = \lambda x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
Special Conformal	$x'^{\mu} = x^2 a^{\mu} - 2\left(a\cdot x\right)x^{\mu}$	$x'^{\mu} = \frac{x^{\mu} - x^2 a^{\mu}}{1 - 2a\cdot x + a^2 x^2}$	$K^{\mu} = -i\left(2x^{\mu}\left(x\cdot\partial\right) - x^2\partial^{\mu}\right)$

We recognise of course the first two transformations as the transformations of the Poincare group. Dilatations correspond to the scale invariance we discussed about and the only new thing is the special conformal transformation generated by K_{μ}. We also note that a finite special conformal transformation can map a finite point $x^{\mu} = \frac{a^{\mu}}{a^2}$ to infinity: $1 - 2a\cdot\frac{a}{a^2} + a^2\left(\frac{a}{a^2}\right)^2 = 0$. This means that we should define the conformal transformations on a compactification of flat space that includes points at infinity.

2.1.2 Conformal group

The generators satisfy the following commutation relations (xxx zaff, aharony)

$$\begin{aligned}
\left[J_{\mu\nu},\, J_{\rho\sigma}\right] &= i\eta_{\mu\rho}J_{\nu\sigma} + \text{permutations} \\
\left[J_{\mu\nu},\, P_{\rho}\right] &= i\left(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}\right) \\
\left[J_{\mu\nu},\, K_{\rho}\right] &= i\left(\eta_{\mu\rho}K_{\nu} - \eta_{\nu\rho}K_{\mu}\right) \\
\left[J_{\mu\nu},\, D\right] &= 0 \\
\left[D,\, P_{\mu}\right] &= iP_{\mu} \\
\left[D,\, K_{\mu}\right] &= -K_{\mu} \\
\left[K_{\mu}, P_{\nu}\right] &= -2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D
\end{aligned} \tag{2.6}$$

The first line is the algebra of the Lorentz group $SO\left(1, d-1\right)$. The next three state that D is a scalar, and P_{μ} & K_{μ}vectors. The next two lines indicate that P_{μ} and K_{μ} are ladder operators to D (this will be important later), and the last equation means that P and K close on a Lorentz transformation and a dilatation. These relations close the algebra of the conformal group. Counting the number of generators we have $d + \frac{d(d-1)}{2}$ corresponding to the Poincare transformations (translations and Lorentz transformations $\sim SO\left(1, d-1\right)$) plus $1 + d$ for dilatations and special conformal transformations. All in all these are

$$\frac{\left(d+1\right)\left(d+2\right)}{2} = \dim\left(SO\left(2, d\right)\right) \tag{2.7}$$

One can check that conformal transformations are indeed isomorphic to the $SO\left(2, d\right)$ group by assembling

$$J_{MN}=\begin{pmatrix} J_{\mu\nu} & \frac{K_\mu-P_\mu}{2} & -\frac{K_\mu+P_\mu}{2} \\ -\frac{K_\mu-P_\mu}{2} & 0 & D \\ \frac{K_\mu+P_\mu}{2} & -D & 0 \end{pmatrix} \qquad M,\,N=1,\ldots d+2 \tag{2.8}$$

we can check that J_{MN} is a Lorentz rotation in $d+2$ dimensional space with signature $(2,d)$ $(\eta_{MN}=diag\,(-1,1,\ldots,1,-1))$.

$$[J_{MN},J_{RS}]=i\eta_{MR}J_{NS}-i\eta_{NR}J_{MS}+i\eta_{NS}J_{MR}-i\eta_{MS}J_{NR} \tag{2.9}$$

There is also a discrete symmetry that serves as a conformal transformation, the so-called *inversion*

$$x^\mu \to x'^\mu=\frac{x^\mu}{x^2} \tag{2.10}$$

and this completes the full conformal group which is now isomorphic to $O\,(2,d)$. With this transformation, the special conformal transformation can be rewritten as an inversion followed by a translation and again an inversion (XXX springer)

$$\frac{x'^\mu}{x'^2}=\frac{x^\mu}{x^2}-a^\mu \tag{2.11}$$

Constraints on the Energy-Momentum Tensor As we said, theories that are scale invariant are thought to enjoy full conformal invariance. to see this let us look into the energy-momentum tensor: From Noether's theorem we know that invariance of a theory under some transformation requires the conservation of the corresponding current

$$\partial_\mu J^\mu=0 \tag{2.12}$$

defined by

$$J^\mu=T^\mu{}_\nu\delta x^\nu$$

Invariance under translations where $\delta x^\mu=a^\mu$ requires the conservation of the energy-momentum tensor[3]

$$\partial^\mu T_{\mu\nu}=0 \tag{2.13}$$

Similarly, invariance under scale transformations requires that the energy-momentum tensor is *traceless*[4]

$$T^\mu{}_\mu=0 \tag{2.14}$$

[3]Actually $\nabla^\mu T_{\mu\nu}=0$ in curved spacetime.

[4]*The energy-momentum tensor is defined as* $T_{\mu\nu}=\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$, *where* S *is the action of the theory. We can consider the case of scale transformations i.e.* $\delta g^{\mu\nu}=\delta\lambda g^{\mu\nu}$. *But if the theory is scale-invariant we have* $\frac{\delta S}{\delta\lambda}=0\Rightarrow\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta\lambda}=T_{\mu\nu}g^{\mu\nu}=T^\mu{}_\mu=0$. *(XXX dixon)*

If a theory is Poincare and scale invariant then we can see that the conformal currents are automatically conserved using the defining equation (2.1.5) (XXX Zaf):

$$\partial_\mu J^\mu = \partial_\mu \left(T^\mu{}_\nu v^\nu\right) = \left(\partial_\mu T^\mu{}_\nu\right) v^\nu + T^{\mu\nu}\partial_{(\mu}v_{\nu)} = 0 + T^{\mu\nu}\frac{1}{d}\left(\partial^\rho v_\rho\right)\eta_{\mu\nu} = 0 \qquad (2.15)$$

2.1.3 Conformal quantum field theories

There are some things to note in the transition to quantum theories. Firstly, even if a classical theory is conformally invariant, in the corresponding quantum field theory the invariance is lifted by the introduction of the renormalization scale (XXX Zaf) and the energy momentum tensor is not traceless anymore. The same is also true if the theory is defined on curved background (the Weyl anomaly, XXX tong). Conformal invariance can be obtained under strict conditions (XXX Zaff:

1. **Fixed Points of the renormalization group**. The renormalization group introduces an equation which describes the evolution of coupling constants w.r.t. the energy scale, which breaks scale invariance. However at points of these couplings where the so-called β-function

$$\beta = \mu\frac{\partial g}{\partial \mu} \qquad (2.16)$$

   is zero, the coupling constants g are *fixed* w.r.t. the energy and the theory is scale invariant.

2. **Finite theories**. In theories with no divergences whatsoever the β-function is vanishing everywhere and the coupling constants are like a line (or a manifold if there are more than one) of fixed points and conformal invariance is preserved. An example is the theory discussed in the Maldacena conjecture, $\mathcal{N} = 4$ Super-Yang-Mills gauge theory.

2.1.4 Effects of conformal invariance

Absence of mass and S-matrix Conformal invariance is a very restrictive symmetry. Firstly, scale invariance means there is *no preferred length scale*. This means that there can be no special effects on some Compton wavelength which means that there can be no massive field in a CFT. Also, a typical Hamiltonian operator in the form of $P^\mu P_\mu$ or P^0 does not commute with all the operators of the algebra e.g. D, which is reflected on the fact that any specific energy or mass of some state can be rescaled via some conformal transformation anywhere from 0 to ∞. More specifically, $P^\mu P_\mu$ corresponds to the Casimir of the Poincare group so it is a good quantum number for Poincare-invariant theories, but it is not a Casimir of the full conformal group. The S-matrix formulation thus becomes irrelevant.

Primary Operators and Labels Since usual observables are irrelevant, our interest focuses on operators that are "well behaved" under scale transformations. We are interested in the eigen-operators of the dilatation transformation with eigenvalue $-i\Delta$, called the scaling dimension of the operator. This means that under dilatations

$$\phi(x) \to \lambda^{\Delta}\phi(\lambda x) \tag{2.17}$$

Also the commutation relations (2.1.6) signify that P^{μ} is a raising operator for eigenvectors of D and K^{μ} is a lowering operator for eigenvectors of D. The objects of interest are operators that are annihilated by K^{μ}at some point, and called *primary operators* and the operators obtained by application of the raising operator are called descendants. The action of the conformal transformation operators on these fields is

$$\begin{aligned}
\left[P^{\mu}, \Phi(x)\right] &= i\partial_{\mu}\Phi(x) \\
\left[J^{\mu\nu}, \Phi(x)\right] &= \left[i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu} + \Sigma^{\mu\nu}\right)\right]\Phi(x) \\
\left[D, \Phi(x)\right] &= i\left(-\Delta + x^{\mu}\partial_{\mu}\right)\Phi(x) \\
\left[K^{\mu}, \Phi(x)\right] &= \left[i\left(x^2\partial^{\mu} - 2x^{\mu}x^{\nu}\partial_{\nu} + 2x^{\mu}\Delta\right) - 2x_{\nu}\Sigma^{\mu\nu}\right]\Phi(x)
\end{aligned}$$

where $\Sigma^{\mu\nu}$ are finite-dimensional representations of the Lorentz group. In fact a primary operator is classified by its scaling dimension Δ as well as any other Lorentz quantum numbers, and the set of primary operators plus their quantum numbers $(\mathcal{O}_i, \Delta_i, j_i)$ determines the spectrum of the CFT and it substitutes the usual labeling of states/operators.[5]

Correlation functions Conformal invariance solely determines the form of n-point functions.

- 1-point functions are identically zero
- 2-point functions are of the form

$$\langle O_i(x)\, O_j(y)\rangle = \frac{A\delta_{ij}}{|x-y|^{2\Delta_i}} \tag{2.18}$$

This is the result we will actually use.

[5] The same classification as in operators is true for *states* in a CFT. In fact there is a so-called state to operator map that links the states of the theory to local operators (XXX Tong Dixon)

- 3-point functions are of the form

$$\langle O_i(x_i)\, O_j(x_j)\, O_k(x_k)\rangle = \frac{A\lambda_{ijk}}{|x_i - x_j|^{\Delta_i+\Delta_j-\Delta_k}\, |x_j - x_k|^{\Delta_j+\Delta_k-\Delta_i}\, |x_k - x_i|^{\Delta_k+\Delta_i-\Delta_j}} \tag{2.19}$$

2.1.5 Remarks

Some remarks to close off

1. Unitarity of a CFT actually imposes a bound on the conformal dimension of operators of various spins. For example, for scalar fields the bound is

$$\Delta \geq \frac{(d-2)}{2} \tag{2.20}$$

2. There are more ways to classify states and operators. These involve quantum numbers of subgroups of $SO(2,d)$, e.g. of its maximal compact subgroup $SO(2) \times SO(d)$. (XXX zaff, aha)

3. In the presence of supersymmetry the conformal group becomes the larger super conformal group. (XXX zaf)

2.2 Anti-de Sitter Space

2.2.1 What is anti-de Sitter space?

Anti-de Sitter space in d-dimensions is a *maximally symmetric* Lorentzian manifold with constant *negative* curvature. It emerges as the solution of the Einstein equations produced by the action with cosmological constant taking the appropriate signs for negative curvature

$$S = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} \left(\mathcal{R} - \Lambda \right) \tag{2.21}$$

which is the standard gravitational component of the Einstein-Hilbert action, the variation of which gives the Einstein field equations, keeping in mind that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$ (XXX GR action)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} \Lambda g_{\mu\nu} \tag{2.22}$$

We see a direct consequence of the equation of motion by taking the trace

$$R = \frac{d}{d-2} \Lambda \tag{2.23}$$

and substituting again we have

$$R_{\mu\nu} = \frac{\Lambda}{d-2} g_{\mu\nu} \tag{2.24}$$

If we further require that

$$\begin{aligned} \mathcal{R}_{\mu\nu\sigma\rho} &= \frac{R}{d\left(d-1\right)} \left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma} \right) \Rightarrow \\ \mathcal{R}_{\mu\nu\sigma\rho} &= \frac{\Lambda}{\left(d-2\right)\left(d-1\right)} \left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma} \right) \end{aligned} \tag{2.25}$$

the space becomes maximally symmetric (maximal number of Killing vectors) (XXX zaf). The solution with these constraints is classified according to its signature and the sign of the curvature as follows

1. When it has *Euclidean* signature and
   (a) *positive* curvature, it is an S^d sphere with $SO\left(d+1\right)$ isometry
   (b) *negative* curvature, it is an H^d hyperboloid with $SO\left(1,d\right)$ isometry
2. When it has *Minkowskian* signature and
   (a) *positive* curvature, it is called a de Sitter space (dS_d)
   (b) *negative* curvature, it is called an ***Anti-de Sitter*** space (AdS_d)

2.2.2 Coordinates of AdS_{d+1}

Anti-de Sitter space can be realised through many different coordinate patches, though not all cover the whole manifold.

Embedded AdS_{d+1} can be realised as the set of solutions[6] of

$$x_0^2 + x_{d+1}^2 - x_1^2 - \ldots - x_d^2 = R^2 \tag{2.26}$$

with

$$R^2 = \frac{(d-2)(d-1)}{\Lambda} \tag{2.27}$$

being the *radius*, embedded in $(d+2)$-dimensional $\mathbb{R}^{2,d}$ space with line element

$$ds^2 = -dx_0^2 - dx_{d+1}^2 + dx_1^2 + \ldots + dx_d^2 \tag{2.28}$$

The radius of AdS_{d+1} will be related to parameters in string theory in the context of the AdS/CFT correspondence: $\left(\frac{R}{l_s}\right)^4 = \lambda$ (XXX, Douglas CHECK). In this definition it is quite clear that AdS_{d+1} has $SO(2,d)$ isometry group. The $SO(2,d)$ isometry of AdS_{d+1} is in equivalence with the $SO(d+1)$ isometry of S^d (XXX petersen, zaf).

Global Another set of coordinates is given by (XXX zah aha) if we substitute in 2.26

$$\begin{aligned} x_0 &= R\cosh\rho\cos\tau \\ x_{d+1} &= R\cosh\rho\sin\tau \\ x_i &= R\sinh\rho\hat{x}_i\,, \quad \sum_{i=1}^{d}\hat{x}_i^2 = 1 \end{aligned} \tag{2.29}$$

which gives a metric

$$ds^2 = R^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2\right) \tag{2.30}$$

where $d\Omega_{d-1}^2$ is the line element of the sphere S^{d-1} . If we take $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$ the entire manifold is covered once, hence these coordinates are called *global*. In this version of the coordinates ∂_τ is a timelike killing vector non-vanishing on the whole manifold, so we refer to τ as global time. Initially it is taken to be periodic which leads to closed time-like curves. We eliminate them by simply ignoring the periodicity of τ and obtain the universal cover of AdS_{d+1}: Near $\rho = 0$ the line element becomes $ds^2 \simeq R^2\left(d\tau^2 + d\rho^2 + \rho^2\, d\Omega_{d-1}^2\right)$

[6]In fact, all the four different kinds of solutions with maximal symmetry can be realised as solutions of quadratic equations with appropriate signatures embedded in one higher dimension.

which is topologically similar to $S^1 \times \mathbb{R}^d$. The S^1 signifies the closed time-like curves in τ and must be "unwrapped" to $-\infty \leq \tau \leq +\infty$.. Additionally, we note that the isometry group $SO(2,d)$ of AdS_{d+1} has as we have seen a maximal compact subgroup $SO(2) \times SO(d)$ where the $SO(2)$ generates translation in the τ coordinate, while the $SO(d)$ rotates the x_i's (XXX Erdmenger).

Stereographic Another set of coordinates is defined by stereographic projection in the embedding space (XXX di vec, petersen). We introduce a transformation in the $\mathbb{R}^{2,d}$ space

$$x_0 = r\frac{1+y^2}{1-y^2} \tag{2.31}$$

$$x_\mu = r\frac{2y_\mu}{1-y^2} \tag{2.32}$$

where $\mu = 1, 2, \ldots, d+1$ and $y^2 = y_1^2 + \ldots + y_d^2 + y_{d+1}^2$. The metric $ds^2 = -dx_0^2 - dx_{d+1}^2 + dx_1^2 + \ldots + dx_d^2$ becomes

$$ds^2 = dr^2 - \frac{4r^2}{\left(1-y^2\right)^2}dy^2 \tag{2.33}$$

The defining equation of AdS_{d+1} is then $r = R$ and the metric is simply

$$ds^2 = \frac{4R^2}{\left(1-y^2\right)^2}dy^2 \tag{2.34}$$

Poincare The last set of coordinates (the one which will be used) is defined by introducing a Lorentz d-vector $\tilde{x}^\mu = \left(-t, \vec{\tilde{x}}\right)$ along with the $(d+1)$-th coordinate $u > 0$ and taking

$$\begin{aligned} x_0 &= \frac{1}{2u}\left(1 + u^2\left(R^2 + \vec{\tilde{x}}^2 - t^2\right)\right) \\ x_{d+1} &= R\,u\,t \\ x_d &= \frac{1}{2u}\left(1 - u^2\left(R^2 - \vec{\tilde{x}}^2 + t^2\right)\right) \\ x_i &= R\,u\,\tilde{x}_i \qquad i = 1, 2, \ldots d-1 \end{aligned} \tag{2.35}$$

which when substituted gives the very pretty form

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2\left(dx^\mu dx_\mu\right)\right) \quad u \geq 0 \tag{2.36}$$

Other useful forms are obtained by taking $u = 1/z$

$$ds^2 = R^2\left(\frac{dz^2 + dx^\mu dx_\mu}{z^2}\right) \quad z \geq 0 \tag{2.37}$$

or $u = e^r$

$$ds^2 = R^2\left(dr^2 + e^{2r}\left(dx^\mu dx_\mu\right)\right) \quad -\infty < r < +\infty \tag{2.38}$$

These sets of coordinates actually cover only *half* of the whole space. In the latter, the whole space is covered by sending $r \to -r$

$$ds'^2 = R^2\left(dr^2 + e^{-2r}\left(dx^\mu dx_\mu\right)\right) \quad -\infty < r < +\infty \tag{2.39}$$

We see in these forms of the metric why these coordinates are called Poincare: they contain slices isomorphic to d-dimensional Minkowski space, multiplied by an appropriate *warp* factor that rescales lengths in the slices. In particular the AdS_{d+1} isometry $x \to \lambda x,\ \ z \to \lambda z$ is directly related to dilatations in the slices.

Important loci of AdS_{d+1} include (XXX Zaff, aha):

- The **boundary** of AdS, which is the plane $u = \infty$ or equivalently $z = 0$ or $r = \infty$ (or $r = -\infty$ in the other patch). The metric blows up in this plane[7] but this is just a coordinate singularity and the metric can be conformally rescaled (e.g. $\tilde{ds}^2 = ds^2/u^2$) and boundary becomes $\mathbb{R}^{1,d-1}$ (XXX Erdmenger). This is actually the main point of this section: **The boundary of AdS_{d+1} is actually d-dimensional Minkowski space**[8]. The isometry group $SO\left(2,d\right)$ acts on the boundary as the conformal group acting on Minkowski space.

- The **horizon** of AdS, which is the plane $u = 0$ or equivalently $z = \infty$ or $e^r = 0$. One can see that the Killing vector ∂_t has zero norm at $u = 0$ and corresponds to a Killing horizon.

These will play a significant role in the "recipe" of the correspondence.

2.2.3 Euclidean AdS_{d+1}

Similarly to what we usually do in quantum field theory, one can consider a Euclidean continuation of the metric and send via a Wick rotation $x_{d+1} \to -ix_{d+1}$ which is equivalent to sending $\tau \to -i\tau$ and $t \to -it$ (even though the Poincare coordinates cover only half of the space). The line elements become

$$\begin{aligned} ds_E^2 &= R^2\left(\cosh^2\rho\, d\tau_E^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2\right) \\ &= R^2\left(\frac{dz^2 + dt^2 + d\vec{x}^2}{z^2}\right) \end{aligned} \tag{2.40}$$

[7] As we will see, in many cases we have to introduce a cut-off $z = \epsilon,\ \epsilon \ll 1$ in order to handle infinities. In this process the metric remains finite.

[8] To be more precise, the boundary of conformally compactified (points at infinity added) AdS_{d+1} is identical to the conformal compactification of d-dimensional Minkowski spacetime. This is also linked to the fact that the conformal group was properly defined on a compactified version of M^d.

The boundary plane $\mathbb{R}^{1,d-1}$ is replaced by $\mathbb{R}^d$, but the horizon now shrinks to a point (XXX zaf aha). One can then send the "point at infinity" which was the horizon to the boundary via compactification and thus the boundary becomes a compactified $\mathbb{R}^d$ which is equivalent to an S^d sphere. In fact the whole space is diffeomorphic to a $(d+1)$-dimensional ball in the embedding space $\mathbb{R}^{d+1}$ with metric

$$x_0^2 + \cdots x_d^2 \leq R^2, \quad ds^2 = \frac{dx^2}{R^2 - |x|^2} \tag{2.41}$$

In reference to the classification of maximally symmetric solutions we did before, the Euclidean version of AdS_{d+1} is actually a hyperboloid H^{d+1}.

2.2.4 Remarks

Some remarks to close off

1. In the global coordinates, one can transform the metric to become a compactified version of *half* of the *Einstein Static Universe* $\left(S^d \times R^1\right)$. In general, spacetimes that can be conformally compactified to the have a boundary of an Einstein Static Universe are called *asymptotically Anti-de Sitter spacetimes.*(XXX aha)

2. Massive particles propagating in Anti-de Sitter background can never reach the boundary (infinite geodesic distance). Massless particles on the other hand can reach the boundary and back in finite time. To see this, transform the Poincare metric (2.2.17) via $z = \rho\cos\theta$, $x^\mu = \rho\sin\theta$ where $\rho = e^\tau$ and $\cos\theta = \frac{1-r^2}{1+r^2}$. The metric becomes $ds^2 = -R^2\left(\frac{1+r^2}{1-r^2}\right)^2 d\tau^2 + \frac{4R^2}{(1-r^2)^2}\left(dr^2 + r^2 d\Omega_{d-1}^2\right)$ and a massless geodesic becomes simply $\frac{dr}{d\tau} = \frac{1+r^2}{2}$, which by integration gives a finite time $T = \frac{\pi}{2}$ (XXX aha, di vecc).

3. In general an Anti-de Sitter background permits tachyonic fields $\left(m^2 < 0\right)$. However energy conservation imposes the Breitenlohner-Freedman bound

$$m^2R^2 \geq -\frac{d^2}{4} \tag{2.42}$$

2.3 The AdS/CFT Correspondence

We studied this far Conformal Field Theories, which are Quantum Field Theories with conformal invariance, in fixed Minkowski spacetime background and the topology of Anti-de Sitter spacetimes. In this section we will study the *correspondence* of these CFTs with gravitational theories in AdS backgrounds.

2.3.1 Statement and evidence

The full AdS/CFT correspondence or Maldacena conjecture as we said before states the equivalence or duality between (i) Type IIB supersting theory on $AdS_5 \times S^5$ where AdS_5 and S^5 have the same radius R and $(ii)\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory in 4 spacetime dimensions with an $SU(N)$ gauge group, which is an example of a CFT (XXX dhoker). Equivalence means that a precise map between states/fields on the string theory side and local gauge invariant operators on the SYM theory side must be established, as well as a correspondence for correlators. The precise map was not formulated in the original paper (XXX mald paper) and was given in later papers (XXX asfootnote gubser, Witten).

Similarly to the discussion about the general gauge/gravity duality given in the Introduction (XXX cite intro?), we start by pointing out that the duality is still a correspondence. Various limits can be more easily tested as stated further on, but in its full form it remains unproven. However there is a lot of evidence supporting a relation between Type IIB string theory on a background involving AdS_{d+1} and a CFT in d-dimensions. Firstly, the same general evidence of gauge/gravity duality also applies here: Holography, in which the CFT_d gives a holographic description of physics in the AdS_{d+1}(XXX aha) and the large N limit ('t Hooft limit) of gauge theories. Furthermore, as we have explicitly seen in the previous sections, *the isometry group of AdS_{d+1} is the same as the symmetry group of CFT_d $(SO(2,d))$*. This is extended even in the presence of supersymmetry (XXX ramallo). Lastly, as we saw the boundary of AdS is *exactly* (compactified) Minkowski space in one less dimension, and the isometry group of AdS acts on the boundary as the conformal group in Minkowski space, thus making a relation between the two theories probable.

2.3.2 Formulation of the correspondence

Fields in the AdS side are called *bulk* fields $\phi(z, x^\mu)$, while operators $\mathcal{O}$ of the CFT are called *boundary* operators since they "live" on the AdS boundary. For each operator of conformal dimension Δ in the field theory side there is a corresponding bulk field on the gravity side, where the Δ is related to the mass of the field as we will see explicitly later. The **fundamental statement** of the correspondence states that *the boundary values of the* *bulk fields $\phi_0(x^\mu)$ are identified with the* ***sources*** *that couple to their dual operators*[9] *and the partition function of string theory on $AdS_5 \times S^5$ coincides with the partition function of $\mathcal{N}=4$ SYM on the boundary of AdS_5.* By making this identification, the string theory partition function is obtained by performing the path integral with the restriction that the boundary value is $\phi_0(x^\mu)$ (XXX di vechia, skenderis, aha) and it is now a functional of the boundary values, and is equal to the partition function of the field theory

$$Z_{String}[\phi_0] \equiv \int_{\phi\to\phi_0} \mathcal{D}\phi\, e^{-S[\phi]} = \left\langle \exp\left(-\int \mathcal{O}\phi_0 d^d x\right)\right\rangle_{CFT} \equiv e^{W[\phi_0]} \tag{2.43}$$

Where $W[\phi_0]$ is the generating functional of connected n-point functions, and the expectation value is over the CFT path integral. Then n-point functions are calculated by taking functional derivatives w.r.t. the boundary field ϕ_0 as shown in the Appendix A. As we will see there is more to the discussion about the partition function and the generating functional being functionals of the boundary values of the fields, as the term “boundary value” is not always well defined because the fields have divergent and/or vanishing modes on the boundary. We will present the full recipe to an approximation of this relation in the next chapter. We see the peculiar thing about the correspondence: even in its general case, by doing purely gravitational calculations on the AdS side we obtain correlation functions for the dual field theory.

Parameter Identification The correspondence requires identification of various parameters on each theory’s side. Parameters on the string theory side include the dimensionless string coupling constant g_s relevant to string splitting and joining, a dimensionful string length l_s relevant to world-sheet fluctuations and the radius R of AdS_5 and S^5. On the SYM theory side parameters are the rank N of the $SU(N)$ gauge group and a dimensionless Yang-Mills coupling constant g_{YM}. The mapping between parameters reads(XXX boer, ramallo, dhoker)

$$g_{YM}^2 = 4\pi g_s \tag{2.44}$$

$$R_{AdS}^4 = R_S^4 = 4\pi g_s N l_s^4 \tag{2.45}$$

which is also rewritten as

$$\left(\frac{R}{l_s}\right)^4 = 4\pi\lambda \tag{2.46}$$

where $\lambda = g_{YM}^2 N$ is the ’t Hooft coupling. From these identification one can make an important observation: Perturbative Yang-Mills is reliable when $\lambda \ll 1$ or equivalently when $R \ll l_s$. On the other hand, the *classical limit* for string theory is reliable when the

[9] See Appendix A

curvature radius is large compared to the string length: $R \gg l_s$ (more on the various limits bellow). It is clear that simultaneous perturbative methods for both theories are impossible with these identifications: *strong coupling in on side corresponds to weak coupling on the other side*. This is of course part of what makes the correspondence useful. If the correspondence holds indeed then one can study the weak coupling in string theory to learn about strongly coupled field theory, and use the weakly coupled field theory to learn about strongly coupled string theory (XXX aha).

2.3.3 Forms and limits of the correspondence

2.3.3.1 Strong Version

The correspondence is thought to hold for *all* values of the parameters N and $g_s = g_{YM}^2$. This is the *strong form* of the correspondence and it is difficult to prove this full version, since there is no good definition of non-perturbative string theory, which even at the classical limit (tree level) is not completely solvable (XXX Boer). Furthermore, we don't know the spectrum of operators on the SYM side at strong coupling, where perturbative methods cannot be used (XXX dhoker).

2.3.3.2 The 't Hooft Limit - Weaker Version

In the 't Hooft limit we have a somewhat weaker form of the AdS/CFT correspondence. The 't Hooft limit is achieved by taking $N \to \infty$ (as we saw in the Introduction) while keeping $\lambda = g_{YM}^2 N$ fixed. This is a well-defined, systematic expansion (in terms of $1/N$) in the SYM side (XXX zaf). In the string theory side, the string coupling can be written as $g_s = \lambda/N$ so the 't Hooft limit corresponds to the weak coupling in string theory (classical string theory - tree approximation) (XXX dhoker, Di vecc).

2.3.3.3 The Large λ Limit - Weakest Version

As we saw in the parameter identification, an even simpler form of the AdS/CFT correspondence is obtained if we assume both $N \to \infty$ and $\lambda \to \infty$ (which corresponds to taking both $g_s \to 0$ and $l_s \to 0$). In this "saddle-point" approximation, string theory becomes effective classical IIB supergravity. This limit of the correspondence has actually been well tested (XXX boer) and this is of course the limit that we will assume and use in detail (from now on any reference to the correspondence implies this limit).

2.3.4 Kaluza-Klein reduction on S^5

The full $AdS_5 \times S^5$ metric is

$$ds^2 = \frac{R^2}{u^2} du^2 + \frac{u^2}{R^2} \left(dx^\mu dx_\mu \right) + R^2 d\Omega_5^2 \tag{2.47}$$

which with the parameter identification we saw will be[10]

$$ds^2 = \frac{l_s^2 \sqrt{4\pi g_s N}}{u^2} du^2 + \frac{u^2}{l_s^2 \sqrt{4\pi g_s N}} \left(dx^\mu dx_\mu \right) + l_s^2 \sqrt{4\pi g_s N} d\Omega_5^2 \tag{2.48}$$

Taking $u = \frac{R^2}{z}$ we have the familiar form

$$ds^2 = R^2 \frac{dz^2}{z^2} + \frac{R^2}{z^2} \left(dx^\mu dx_\mu \right) + R^2 d\Omega_5^2 \tag{2.49}$$

In the following steps, we will have to solve the equation of motion in this background. If we consider the E.o.M. of a massless scalar it is simply the 10-dimensional analogue of a Laplace equation

$$\Box \psi = 0 \tag{2.50}$$

and as the metric is a product of AdS_5 and S^5 the Laplacian can by decomposed (XXX ramallo)

$$\Box = \Box_{AdS_5} + \Box_{S^5} \tag{2.51}$$

We can expand the solution ψ in Kaluza-Klein towers of "spherical harmonics" of $SO(6)$ (the isometry group of S^5)

$$\psi(x, \Omega) = \sum_l \phi_l(x) Y_l(\Omega) \tag{2.52}$$

and the $\Box_{S^5}$ part is just the Casimir operator in $SO(6)$, and the spherical harmonics are eigenfunctions

$$\Box_{S^5} Y_l(\Omega) = -\frac{m_l^2}{R^2} Y_l(\Omega) \tag{2.53}$$

with the eigenvalues given by

$$R^2 m_l^2 = l(l+4), \qquad l = 0, 1, 2, \ldots \tag{2.54}$$

Now the reduced equation of motion has become a Klein-Gordon equation in AdS_5 back-

[10] This metric comes from the solution of N stacked D3-branes, which in full is $ds^2 = \left(1 + \frac{R^4}{u^4}\right)^{-1/2} \left(dx^\mu dx_\mu \right) + \left(1 + \frac{R^4}{u^4}\right)^{1/2} \left(du^2 + u^2 d\Omega_5^2 \right)$, where $R^4 = l_s^4 4\pi g_s N$. In the $u \ll R$ limit, this reduces to the metric above.(XXX aha, Dhoker)

ground with a mass m_l^2 restricted by the spectrum given above

$$\Box_{AdS_5}\phi_l = m_l^2\phi_l \tag{2.55}$$

The relation between mass and the spherical harmonic quantum numbers $R^2 m_l^2 = l\,(l+4)$ is similar (and related) to the relation between mass m of the bulk fields and conformal dimension Δ of the dual operators (XXX ramallo, dhoker). In fact as we will see the relation for scalars in d-dimensions is[11]

$$R^2 m^2 = \Delta\,(\Delta - d) \tag{2.56}$$

Similar relations hold for higher spin fields (XXX aha)

- scalars: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2}\right)$
- spinors: $\Delta = \frac{1}{2}\left(d + 2\,|m|\right)$
- vectors: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2)^2 + 4m^2}\right)$
- p-forms: $\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2p)^2 + 4m^2}\right)$
- first-order $\frac{d}{2}$-forms (even d): $\Delta = \frac{1}{2}\left(d + 2\,|m|\right)$
- spin-3/2: $\Delta = \frac{1}{2}\left(d + 2\,|m|\right)$
- massless spin-2: $\Delta = d$

2.3.5 The weak AdS/CFT dictionary[12] and holographic renormalization

The fundamental statement of the correspondence in the *weak form* we saw before ($N \to \infty$ and $\lambda \to \infty$), which is a "saddle-point" approximation, comes to the very simple form (XXX skenderis)

$$S_{on-shell}\,[\phi_0] = -W_{CFT}\,[\phi_0] \tag{2.57}$$

where the l.h.s. is the on-shell supergravity action, and the r.h.s. is the connected graph generating function of the CFT. Correlation function are computed as seen in the Ap-

[11] With this definition it seems seems that $\Delta = l + d$. In (XXX dhoker, erdmenger) the KK decomposition is directly w.r.t. Δ: $\psi\,(x,\Omega) = \sum_\Delta \phi_\Delta\,(x)\,Y_\Delta\,(\Omega)$.

[12] For calculation of two-point functions

pendix (XXX check, cite)

$$\begin{aligned}
\langle \mathcal{O}(x) \rangle &= \left. \frac{\delta S_{on-shell}[\phi_0]}{\delta \phi_0(x)} \right|_{\phi_0=0} \\
\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle &= - \left. \frac{\delta^2 S_{on-shell}[\phi_0]}{\delta \phi_0(x_1)\, \delta \phi_0(x_2)} \right|_{\phi_0=0} \\
\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle &= (-1)^{n-1} \left. \frac{\delta^n S_{on-shell}[\phi_0]}{\delta \phi_0(x_1) \ldots \delta \phi_0(x_n)} \right|_{\phi_0=0}
\end{aligned}$$

and so on. But this is not a well defined expression: the l.h.s. diverges due to the infinite volume of spacetime, and the r.h.s. diverges because the evaluation of the on-shell action on divergent boundary fields is divergent and needs to be renormalized to $S^{ren}_{on-shell}[\phi_0]$ and the boundary value of the fields is not simply $\phi|_{z=0}$.

The precise dictionary to holographically renormalize is presented in (XXX skenderis) and consists of:

Equation of Motion We find the bulk equation of motion in the AdS_{d+1} background.

Solution We find the (asymptotic if needed) solution of the equation of motion *with prescribed but arbitrary* Dirichlet boundary conditions. The solution (in the massive scalar case) contains, as we will see, two linearly independent solutions that behave as $\phi \sim \phi_1 z^{d-\Delta}$ and $\phi \sim \phi_2 z^{\Delta}$ near $z = 0$ (where Δ is the scaling dimension of the corresponding operator in the field theory side) and one is generally normalizable while the other is not (XXX zaff).

Regularization Since the leading term of the field $\phi \sim \phi_1 z^{d-\Delta}$ is either divergent or vanishing (only when $d = \Delta$ it approaches a constant finite value) at the boundary, we require that in leading term

$$\phi(z, x) \to z^{d-\Delta} \phi_0(x) \tag{2.58}$$

and we also require that the solution be *regular* at the horizon $z = \infty$. This fixes the solution by lifting the independence of the two solutions, and the $\phi_0(x)$ in (2.3.16) (XXX check) is identified with the source of the dual operator (XXX zaf). The normalizable mode ϕ_2 is related to the vacuum expectation value of the dual operator (1-pt function) (XXX erdmenger, zaf, ramallo). Furthermore, for the regularization of the on-shell action, we compute its value on the solutions of the equations of motion (often this will just give boundary terms), but the *holographic coordinate is restricted to a cut-off* $z \geq \epsilon$ and the boundary terms are evaluated at $z = \epsilon$ and we will take $\epsilon \to 0$ in the end. $S_{on-shell}[\phi_0]$ then is denoted as $S_{reg}[\phi_0, \epsilon]$.

Counterterms The counterterms of the on-shell action are defined as

$$S_{ct}\left[\phi_0\left(x,\epsilon\right),\epsilon\right] = -\text{divergent terms of } S_{reg}\left[\phi_0,\epsilon\right] \tag{2.59}$$

Often these are simply ignored in the process because their effect is just "contact terms" in the correlator (XXX zaff)

Renormalization To obtain the renormalized action used to calculate the n-point functions, we first define the *subtracted action*

$$S_{sub}\left[\phi_0\left(x,\epsilon\right),\epsilon\right] = S_{reg}\left[\phi_0,\epsilon\right] + S_{ct}\left[\phi_0\left(x,\epsilon\right),\epsilon\right] \tag{2.60}$$

which has a finite limit at $\epsilon \to 0$. If we take this limit *after the variations* we obtain the *renormalized* action used to calculate the n-point functions

$$S_{ren}\left[\phi_0\left(x\right)\right] = \lim_{\epsilon\to 0} S_{sub}\left[\phi_0\left(x,\epsilon\right),\epsilon\right] \tag{2.61}$$

2.3.6 Interpretation of the $(d+1)$th dimension

One thing that has to be discussed is the interpretation of the extra dimension in the Anti-de Sitter spacetime. It is argued in (XXX boer, zaff, energy scale) that the extra dimension of the AdS is related to the energy scale in the CFT. One way to illustrate this is if we consider a dilatation in the field theory side that leaves the theory invariant $x \to \lambda x$. This corresponds to the $SO\left(2,d\right)$ isometry $x^\mu \to \lambda x^\mu,\ z \to \lambda z$ of the $ds^2 = \frac{dz^2+dx^2}{z^2}$ of the AdS_{d+1} and we can see that the holographic coordinate z is related to the energy scale μ, in particular we roughly take

$$\frac{1}{z} = \mu \tag{2.62}$$

This identification reveals another important aspect of the correspondence: high energy processes on the gravity side are associated with the horizon $z = \infty$ and low energy processes are associated with the boundary $z = 0$, while in the field theory side the boundary corresponds to high energy processes as we see from 2.62. This is referred to as the UV/IR duality (XXX energy scale)

More evidence of the holographic coordinate being the energy scale of the field theory can be seen in "deformed" versions of the correspondence where one can solve explicitly for the supergravity solutions, which are *dual* to the renormalization group flows of field theory (XXX boer). In our case, we can see the similarity between standard renormalization in QFT and holographic renormalization, since we regulate the action with a cut-off $z = \epsilon$, and use counterterms to make it finite. Finally, as stated in (XXX boer) it has been shown that general $(d+1)$-dimensional coordinate transformations in the AdS_{d+1} side imply renormalization group equations in the field theory, making the RG equations similar to Poincare transformations, hence strengthening the argument that the energy scale is an AdS coordinate.

2.3.7 Remarks

Some remarks to close off

1. Often the dual operator of the CFT to some field in AdS can be found using symmetry, since they both have the same $SO\left(2,d\right)$ quantum numbers. This is also related to a generic property of the correspondence: global symmetries in the CFT correspond to isometries in the gravity side. This statement is also true for the S^5 product: the $SO\left(6\right)$ isometry of the S^5 sphere is related to the R-symmetry of supersymmetry. In fact this holds for any compact manifold M in $AdS_d \times M$, so the isometries of M become global symmetries in the field theory side.

2. Since the proposal of the prototype Maldacena conjecture many different forms of gauge/gravity correspondence have been found and tested. Examples include string theory compactified on products involving AdS_d with many values for d, e.g. $AdS_3 \times S^3 \times T^4$ which is dual to a 2-dimensional CFT (XXX boer). Also, strictly Anti-de Sitter space is not truly necessary, just "a manifold with the topological structure of AdS, plus a conformal boundary" (XXX zaff). Furthermore, examples where the conformal invariance is lifted and examples with less supersymmetry have also been found, although we are still far from a version of the correspondence where the field theory is a non-supersymmetric, non-scale invariant, $N = 3_{color}$ field theory, like QCD (XXX nastase, aha, moar?).

Chapter 3

Hamiltonian Dynamics in AdS/CFT

3.1 CFT 2-point Function of a Massive Scalar on fixed AdS Background

We have seen up to this point the general formulation of the correspondence and now we want to explicitly use the dictionary for the calculation of a two-point CFT function from a massive scalar in fixed AdS background, which is the simplest case. These calculations have been repeatedly carried out in (XXX peterson, ramallo, zaff, aha, dhoker) but we follow the method used in the Appendix of (XXX freedman).

3.1.1 Action and equation of motion

We work in Anti-de-Sitter spacetime of $d+1$ dimensions with metric

$$ds^2 = \frac{dz^2 + (dx)^2}{z^2} \tag{3.1}$$

The boundary of AdS_{d+1} in this picture is $\mathbb{R}^d$ space $z=0$ plus 1 point at $z=\infty$ (so it is S^d if compactified). The action of a (generally massive) scalar field is written as[1]

$$S[\phi] = \frac{1}{2}\int d^dx\, dz\, \sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] \tag{3.2}$$

The equation of motion by variation of the above action reads

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0 \tag{3.3}$$

[1]The μ, ν indices here run from $1,\ldots,d+1$, not to be confused with terms like $dx^\mu dx_\mu$, $\eta_{\mu\nu}$ or x^μ which imply the Minkowski slices where the indices run in one less dimension $\mu = 1,\ldots,d$

and since $\sqrt{g} = z^{-d-1}$ and $g^{\mu\nu} = z^2 \widetilde{\eta}_{\mu\nu} (= z^2 \delta_{\mu\nu}$ if we change to Euclidean $AdS)^2$ we can substitute and get the **equation of motion**

$$z^{d+1} \partial_z \left(z^{1-d} \partial_z \phi \right) + z^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \phi - m^2 \phi = 0 \tag{3.4}$$

Comment This is were the dictionaries in the bibliography differ: Most applications follow the prescription formulated by Witten in (XXX witten maybe float), by solving the Dirichlet problem with a bulk-to-boundary propagator $K(z, x'^\mu - x^\mu)$, and higher order point function with bulk-to-bulk propagators. We will follow the prescription formulated in (XXX gubser gleb, maybe foot, MORE freedman): solution, reguralization and all the rest by solving exactly the equations of motion via a Fourier transform.

We will regulate the metric as discussed in the previous chapter, and put the boundary of AdS at the point $z_0 = \epsilon$ with $\epsilon \ll 1$ and we will carefully take the limit $\epsilon \to 0$ in the end.

3.1.2 The conformal dimension Δ

If we investigate modes independent of "Minkowski" coordinates x^μ we find

$$z^{d+1} \partial_z \left(z^{1-d} \partial_z \phi \right) - m^2 \phi = 0 \tag{3.5}$$

and substituting power-like solutions $\phi \sim z^\Delta$ we have two x^μ-independent solutions

$$\Delta (\Delta - d) = m^2 \tag{3.6}$$

with solutions

$$\Delta_\pm = \frac{1}{2} \left(-d \pm \sqrt{d^2 + 4m^2} \right) \tag{3.7}$$

We henceforth denote as Δ the largest of the two solutions of 3.7 (note that if Δ is a solution, then so is $d - \Delta$, which is the other solution) and the general solution is then

$$\phi(z) = \phi_0 z^{d-\Delta} + \phi_1 z^\Delta \tag{3.8}$$

with ϕ_0 and ϕ_1 constants. Note that the first solution is the leading term near the boundary and is generally not normalizable there (XXX zaff & moar check) . This tells us that fields

[2] $\widetilde{\eta}_{\mu\nu} = \operatorname{diag}\{1, -1, 1, \ldots, 1\}$ so that the Minkowski slice has the "usual" metric $\eta_{\mu\nu} = \operatorname{diag}\{-1, 1, \ldots, 1\}$. This is a slight abuse of notation as in the first the indices run in $d+1$ dimensions, while in the latter in d dimensions.

with $\Delta \neq d$ will either vanish or diverge at the boundary, and fields with $\Delta = d$ (massless case) will aproach a constant value. We want to evaluate the fields in the bulk in terms of their values at the boundary, which will be identified as the source for the dual conformal operator (which has conformal dimension Δ). So for massive fields we indentify the source of the operator with the non-vanishing or non-divergent part of the boundary field, i.e. since

$$\phi \to \phi_0 \epsilon^{d-\Delta} \text{ as } z \to \epsilon \tag{3.9}$$

But 3.7 is exactly the relation between the Kaluza-Klein reduction and the induced mass from 10 dimensions we saw earlier. As stated, the Δ is related (actually it is equal) to the conformal dimension of the dual boundary operator that couples to the bulk field ϕ. One way to stress this is to study the invariance of the "modification-term" of the field theory action Appendix A

$$S[\phi] + i \int d^d x \, \mathcal{O}(x) \, \phi_0(x) \tag{3.10}$$

and how its terms need to transform under a dilatation. We saw before that a dilatation $x \to \lambda x$ in the CFT corresponds to the $SO(2,d)$ isometry $x^\mu \to \lambda x^\mu$, $z \to \lambda z$ of the $ds^2 = \frac{dz^2 + dx^2}{z^2}$ metric of the AdS_{d+1}. Under this dilatation the definition of the conformal dimension states that

$$\begin{aligned} \mathcal{O}(x) &\to \lambda^{\Delta} \mathcal{O}(\lambda x) \\ \mathcal{O}'(x') &= \lambda^{-\Delta} \mathcal{O}(x) \end{aligned} \tag{3.11}$$

and under the $x^\mu \to \lambda x^\mu$ dilatation in the CFT

$$d^d x' = \lambda^d \, d^d x \tag{3.12}$$

and in order for the modified term to be conformally invariant it is necessary that the source ϕ_0 transforms as

$$\phi_0'(x') = \lambda^{\Delta - d} \phi_0(x) \tag{3.13}$$

So that

$$\int d^d x' \, \mathcal{O}'(x') \, \phi_0'(x') = \int d^d x \, \mathcal{O}(x) \, \phi_0(x) \tag{3.14}$$

Back in the AdS side now, we started out by saying that the whole bulk field $\phi(z, x^\mu)$ is a scalar, so under AdS isometries like the one discussed above so we have

$$\phi'(z', x'^\mu) = \phi(z, x^\mu) \tag{3.15}$$

and if we impose this constraint on the leading term at the boundary we have

$$\phi_0' \lambda^{d-\Delta} \epsilon^{d-\Delta} = \phi_0 \epsilon^{d-\Delta} \tag{3.16}$$

which gives the needed relation

$$\phi_0' \left(x'\right) = \lambda^{\Delta-d} \phi_0 \left(x\right) \tag{3.17}$$

and we can see why Δ is the scaling dimension of the dual operator.

3.1.3 Explicit solution, on-shell action and the 2-point fuction

Returning to the search of the solution of the full equation 3.4 we note that adding the the x^μ dependence should not change the behaviour of the solution near $z = \epsilon$. This means that the asymptotic solution $\phi(z) = \phi_0 z^{d-\Delta} + \phi_1 z^\Delta$ should still hold for modes dependent of x^μ and ϕ_0, ϕ_1 are now functions of the coordinates $\phi_{0,1}(z, x^\mu)$ and that the x^μ coordinates are irrelevant to the z-evolution of the solutions and we can Fourier transform via

$$\phi(z, x^\mu) = \frac{1}{(2\pi)^{d/2}} \int d^d k \, e^{ik\cdot x} \phi(z, k^\mu) \tag{3.18}$$

where k^μ is the the wave-vector of the Minkowski slice. With this definition,

$$\partial_z \phi(z, x^\mu) = \frac{1}{(2\pi)^{d/2}} \int d^d k \, e^{ik\cdot x} \partial_z \phi(z, k^\mu) \tag{3.19}$$

and

$$\partial_\mu \phi(z, x^\mu) = ik_\mu \frac{1}{(2\pi)^{d/2}} \int d^d k \, e^{ik\cdot x} \phi(z, k^\mu) \tag{3.20}$$

so that

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi(z, x^\mu) = -\left(\eta^{\mu\nu} k_\mu k_\nu\right) \frac{1}{(2\pi)^{d/2}} \int d^d k \, e^{ik\cdot x} \phi(z, k^\mu) \tag{3.21}$$

Substituting we get the equation of motion in Fourier space

$$z^{d+1} \partial_z \left(z^{1-d} \partial_z \phi\right) - z^2 k^2 \phi - m^2 \phi = 0 \tag{3.22}$$

where now $\phi = \phi(z, k^\mu)$. It is then easy to see that the action 3.2 takes the form (XXX freedman)

$$\begin{aligned} S\left[\phi(z, k^\mu)\right] &= \frac{1}{2} \int dz \int d^d k \int d^d k' \, (2\pi)^d \, \delta\left(k^\mu + k'^\mu\right) z^{-d+1} \\ & \left[\partial_z \phi(z, k^\mu) \, \partial_z \phi\left(z, k'^\mu\right) + \left(k^2 + \frac{m^2}{z^2}\right) \phi(z, k^\mu) \, \phi\left(z, k'^\mu\right)\right] \end{aligned} \tag{3.23}$$

where the $(2\pi)^d \delta(k^\mu + k'^\mu)$ arises from the x^μ-integration of the $e^{i(k+k')\cdot x}$ coming from all the quadratic terms. The AdS/CFT dictionary tells us we have to evaluate the on-shell action, (on the solutions of the equation of motion). We integrate by parts and get the boundary term(XXX check this) which corresponds to the value of the generating functional in the saddle-point approximation

$$S^{on-shell}[\phi(z,k^\mu)] = \frac{1}{2}\int d^dk \int d^dk' \, (2\pi)^d \delta(k^\mu + k'^\mu) \lim_{z\to\epsilon} z^{-d+1}\left[\phi(z,k^\mu)\,\partial_z\phi(z,k'^\mu)\right] \tag{3.24}$$

and we remind that we will vary this action on the boundary with the prescribed but arbitrary value of the field at the boundary

$$\phi(\epsilon,k^\mu) = \phi_{bnd.}(k^\mu) \tag{3.25}$$

so that $S^{on-shell} = S^{on-shell}[\phi_{bnd.}(k^\mu)]$.

Following (XXX freedman) as always, we look for the solution to the equation of motion and we set

$$z\,k = \xi \tag{3.26}$$

and transform again

$$\phi = \xi^{\frac{d}{2}} G(\xi) \tag{3.27}$$

which when substituted in 3.22 gives [...]

$$\xi^2\partial_\xi^2 G(\xi) + \xi\partial_\xi G(\xi) - \left(\xi^2 + \left[m^2 + \frac{d^2}{4}\right]\right) G(\xi) = 0 \tag{3.28}$$

which is the *modified Bessel equation* for index $\nu = \sqrt{m^2 + \frac{d^2}{4}}$ which is written

$$\nu = \Delta - \frac{d}{2} \tag{3.29}$$

The general solution is

$$G_\nu(\xi) = c_1\mathcal{I}_\nu(\xi) + c_2\mathcal{K}_\nu(\xi) \tag{3.30}$$

where $\mathcal{I}_\nu(\xi)$ is the modified bessel function fo first kind and $\mathcal{K}_\nu(\xi)$ of the second kind. In the previous notation

$$\phi(z,k^\mu) = (kz)^{\frac{d}{2}}\, G(kz) \tag{3.31}$$

In order to impose the boundary condition 3.25 as well as the *regularity on the horizon* we need the part of the 3.31 that satisfies

$$\lim_{z\to\epsilon}\Phi^{\epsilon}\left(z,k^{\mu}\right)=1\quad,\quad\lim_{z\to\infty}\Phi^{\epsilon}\left(z,k^{\mu}\right)=0 \tag{3.32}$$

and then the solution which we will substitute in the on-shell action can be written as

$$\phi\left(z,k^{\mu}\right)=\Phi^{\epsilon}\left(z,k^{\mu}\right)\phi_{bnd.}\left(k^{\mu}\right)\epsilon^{d-\Delta} \tag{3.33}$$

One such solution is when we choose $c_1 = 0$ in 3.30 (since the $\mathcal{I}_{\nu}(\xi)$ is exponentially increasing)

$$\phi\left(z,k^{\mu}\right)=\frac{z^{d/2}}{\epsilon^{d/2}}\frac{\mathcal{K}_{\nu}(kz)}{\mathcal{K}_{\nu}(k\epsilon)}\phi_{bnd.}\left(k^{\mu}\right)\epsilon^{d-\Delta} \tag{3.34}$$

which vanishes at $z=\infty$ and $\mathcal{K}_{\nu}(\xi)$ has the needed $\epsilon^{d-\Delta}$ behaviour as $z\to\epsilon$ (XXX dhoker, erdmenger). With this solution, the term in the on-shell action is

$$\begin{aligned}\lim_{z\to\epsilon}z^{-d+1}\left[\phi\left(z,k^{\mu}\right)\partial_z\phi\left(z,k'^{\mu}\right)\right]&=\epsilon^{2(d-\Delta)}\phi_{bnd.}\left(k^{\mu}\right)\,\phi_{bnd.}\left(k'^{\mu}\right)\lim_{z\to\epsilon}\frac{\partial_z\left(z^{d/2}\mathcal{K}_{\nu}(kz)\right)}{\epsilon^{d/2}\mathcal{K}_{\nu}(k\epsilon)}\\&=\epsilon^{2(d-\Delta)}\phi_{bnd.}\left(k^{\mu}\right)\,\phi_{bnd.}\left(k'^{\mu}\right)\frac{d}{d\epsilon}\ln\left(\epsilon^{d/2}\mathcal{K}_{\nu}(k\epsilon)\right)\end{aligned} \tag{3.35}$$

Substituting in the on shell action 3.24 we get an expression like

$$S^{on-shell}\left[\phi_{bnd.}\right]=\frac{1}{2}\epsilon^{d-2\Delta+1}\int d^dk\int d^dk'\;(2\pi)^d\,\delta\left(k^{\mu}+k'^{\mu}\right)\phi_{bnd.}\left(k^{\mu}\right)\,\phi_{bnd.}\left(k'^{\mu}\right)\frac{d}{d\epsilon}\ln\left(\epsilon^{d/2}\mathcal{K}_{\nu}(k\epsilon)\right) \tag{3.36}$$

and it is easy to perform the variation now (XXX erdmenger)

$$\begin{aligned}\left\langle\mathcal{O}_{\Delta}(k)\,\mathcal{O}_{\Delta}\left(k'\right)\right\rangle&=-\frac{\delta^2 S^{on-shell}\left[\phi_{bnd.}\right]}{\delta\phi_{bnd}\left(k^{\mu}\right)\delta\phi_{bnd}\left(k'^{\mu}\right)}\\&=-\left(2\pi\right)^d\delta\left(k^{\mu}+k'^{\mu}\right)\epsilon^{d-2\Delta}\frac{d}{d\epsilon}\ln\left(\epsilon^{d/2}\mathcal{K}_{\nu}(k\epsilon)\right)\end{aligned} \tag{3.37}$$

and the for the final result we use the asymptotic forms of the modified Bessel function (XXX dhoker)

$$\mathcal{K}_{\nu}(x)=x^{-\nu}\left(a_0+a_1x^2+\ldots\right)+x^{\nu}\ln(x)\left(b_0+b_1x^2+\ldots\right) \tag{3.38}$$

with a_i, b_i depending on ν. The result for 3.37 is

$$
\begin{aligned}
\left\langle \mathcal{O}_\Delta(k)\,\mathcal{O}_\Delta\left(k'\right)\right\rangle \;=\;& (2\pi)^d\,\delta\left(k^\mu + k'^\mu\right)\epsilon^{d-2\Delta} \\
& \left[-\frac{d}{2} + \nu\left(1 + c_2 k^2\epsilon^2 + \ldots\right)\right. \\
& \left. -\frac{2\nu b_0}{a_0} k^{2\nu}\epsilon^{2\nu}\ln(k\epsilon)\left(1 + d_2 k^2\epsilon^2\right)\right]
\end{aligned}
\tag{3.39}
$$

where c_i, d_i are related to a_i, b_i. The usual "momentum conservation" term $(2\pi)^d\,\delta\left(k^\mu + k'^\mu\right)$ is dropped and the power-like terms of the expression yield delta-function "contact" terms when we Fourier transform back to position space and are not of interest. The only non-trivial term is the $\frac{2\nu b_0}{a_0}k^{2\nu}\epsilon^{2\nu}\ln(k\epsilon)$ and the ratio $\frac{2\nu b_0}{a_0}$ is given in (XXX dhoker)

$$
\frac{2\nu b_0}{a_0} = \frac{(-1)^{\nu-1}}{2^{2\nu-2}\Gamma(\nu)^2} \tag{3.40}
$$

and of course $\epsilon^{d-2\Delta} = \epsilon^{-2\nu}$ and the physical result in Fourier momentum space is

$$
\left\langle \mathcal{O}_\Delta(k)\,\mathcal{O}_\Delta(-k)\right\rangle = -\frac{2\nu b_0}{a_0}k^{2\nu}\ln(k\epsilon) \tag{3.41}
$$

and the the Fourier transformation to position gives the **final result** (XXX erdmenger, dhoker, freedman)

$$
\left\langle \mathcal{O}_\Delta(x)\,\mathcal{O}_\Delta(y)\right\rangle = \frac{(2\Delta - d)}{\pi^{d/2}}\frac{\Gamma(\Delta)}{\Gamma\left(\Delta - \frac{d}{2}\right)}\frac{1}{|x-y|^{2\Delta}} \tag{3.42}
$$

3.2 The Hamiltonian Approach

3.2.1 Preliminaries

This application referres to a novel way to the calculation of the CFT two-point function from the Hamiltonian dynamics in the AdS bulk side: we will consider the system described by the classical gravity action in AdS to be a Hamiltonian system, solve the equations of motion, and apply the AdS/CFT dictionary. This means that we have to select a "preferred" coordinate in the AdS side to become the "time" coordinate of our Hamiltonian formalism. In this process we will see that canonical transformations of the phasespace variables play an important role.

We will study the case for AdS_{3+1} ($d = 3$ in all formulas for generic d) for simplicity. We will firstly use a rescaled version of the AdS coordinate patch 2.39

$$ds^2 = \left(dr^2 + e^{-2r/R}\left(-dt^2 + d\vec{x}^2\right)\right) \tag{3.43}$$

with $\vec{x} = \left(x^1, x^2\right)$ (and we also note $x^i = (-t, \vec{x})\,,\ i = 0, 1, 2$). As we saw in the second chapter, the horizon in this patch is the plane $r = -\infty$ and the horizon is a single point at $r = +\infty$.

3.2.2 Euclidean AdS_4/CFT_3 in Hamiltonian formalism

The action for a massive scalar with no interaction is in any patch

$$S\left[\phi\right] = -\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{3.44}$$

with $\phi = \phi\left(r, t, \vec{x}\right)$ and if we substitute for the patch mentioned

$$\sqrt{-g} = e^{-3r/R} \quad \text{and} \quad g^{\mu\nu} = \mathrm{diag}\left\{1, -e^{-2r/R}, e^{-2r/R}, e^{-2r/R}\right\} \tag{3.45}$$

the action in this patch is

$$S\left[\phi\right] = -\frac{1}{2}\int d^4x\, e^{-3r/R}\left[\left(\partial_r\phi\right)^2 - e^{2r/R}\left[\left(\partial_t\phi\right)^2 - \left(\partial_{x_1}\phi\right)^2 - \left(\partial_{x_2}\phi\right)^2\right] + m^2\phi^2\right] \tag{3.46}$$

As in the previous calculation of the 2-point function, the "Minkowski" coordinates x^i are irrelevant to the holographic evolution of data along the r coordinate and we can Fourier transform. We will firstly transform only the "space" coordinates via

$$\phi\left(r, t, \vec{x}\right) = \int \frac{d^2p}{\left(2\pi\right)^2} e^{-i\vec{p}\cdot\vec{x}}\phi\left(r, t, \vec{p}\right) \tag{3.47}$$

and as usual, all the quadratic terms in the fields and their derivatives yield $\delta\left(\vec{p}+\vec{p}'\right)$ delta functions ("momentum conservation" terms) after integration of the $e^{i(\vec{p}+\vec{p}')\cdot\vec{x}}$ in the transformed $\vec{x}$ coordinates, and we are left with Fourier modes of the form

$$\phi\left(r,t,\vec{p}\right)\phi\left(r,t,-\vec{p}\right) \tag{3.48}$$

We may take the field to be real, in which case we can just write

$$\phi\left(r,t,-\vec{p}\right)=\phi^{*}\left(r,t,\vec{p}\right) \tag{3.49}$$

and the transformed action has the form (absorbing the outer $(-)$ factor)

$$\begin{aligned} S\left[\phi\right]&=\frac{1}{2}\int\frac{d^{2}p}{(2\pi)^{2}}\int dr\int dt\, e^{-3r/R}\left[e^{2r/R}\partial_{t}\phi^{*}\partial_{t}\phi-\partial_{r}\phi^{*}\partial_{r}\phi-\left[e^{2r/R}\vec{p}^{2}+m^{2}\right]\phi^{*}\phi\right]\\ &=\frac{1}{2}\int\frac{d^{2}p}{(2\pi)^{2}}\int dr\int dt\, e^{-r/R}\left[\partial_{t}\phi^{*}\partial_{t}\phi-e^{-2r/R}\partial_{r}\phi^{*}\partial_{r}\phi-\left[\vec{p}^{2}+e^{-2r/R}m^{2}\right]\phi^{*}\phi\right] \end{aligned} \tag{3.50}$$

and from the boundary point-of-view we see that the action $S=\int\mathcal{L}d^{4}x$ has a Lagrangian density of the form

$$\mathcal{L}=T-V \tag{3.51}$$

This will of course change under a Wick rotation to Euclidean AdS (time) to the logarithm of a *Boltzmann factor* e^{-E}

$$iS=i\int\left[T-V\right]d^{4}x\rightarrow-\int\left[T+V\right]d^{4}x \tag{3.52}$$

As we stated, in order to study the Hamiltonian dynamics of the AdS action, we need to choose a "preferred" coordinate which we will identify with the "time" of canonical formalism. It is clear that in the study of holography this has to be the "holographic" coordinate r (in this patch): "time" will run from the horizon $r=+\infty$ to the bondary $r=-\infty$. We change variables

$$z=Re^{r/R} \tag{3.53}$$

to the more useful coordinate patch 2.37

$$ds^{2}=R^{2}\left(\frac{dz^{2}+\left(-dt^{2}+d\vec{x}^{2}\right)}{z^{2}}\right) \tag{3.54}$$

where now as we saw $z \in [0, +\infty)$. Taking in to account that in this patch

$$\sqrt{-g} = \left(\frac{R}{z}\right)^4 \quad \text{and} \quad g^{\mu\nu} = \left(\frac{z}{R}\right)^2 \text{diag}\left\{1, -1, 1, 1\right\} \tag{3.55}$$

the action takes the form

$$\begin{aligned} S[\phi] &= -\frac{1}{2}\int d^4x \left(\frac{R}{z}\right)^4 \left[\left(\frac{z}{R}\right)^2 (\partial_r \phi)^2 - \left(\frac{z}{R}\right)^2 \left[(\partial_t \phi)^2 - (\partial_{x_1}\phi)^2 - (\partial_{x_2}\phi)^2\right] + m^2\phi^2\right] \\ &= -\frac{1}{2}\int d^4x \left[\left(\frac{R}{z}\right)^2 (\partial_r \phi)^2 - \left(\frac{R}{z}\right)^2 \left[(\partial_t \phi)^2 - (\partial_{x_1}\phi)^2 - (\partial_{x_2}\phi)^2\right] + \left(\frac{R}{z}\right)^4 m^2\phi^2\right] \end{aligned} \tag{3.56}$$

where now

$$\phi = \phi\left(z, t, \vec{x}\right) \tag{3.57}$$

We can now Fourier transform in all the "Minkowski" coordinates via

$$\phi\left(z, t, \vec{x}\right) = \int \frac{d^2p\, d\omega}{(2\pi)^3} e^{-i(\vec{p}\cdot\vec{x} + \omega\cdot t)} \phi\left(z, \omega, \vec{p}\right) \tag{3.58}$$

and as before the quadratic terms will give delta function conservation terms $\delta\left(\vec{p} + \vec{p}'\right)$ and $\delta\left(\omega + \omega'\right)$ and again we can assume that the fields are real and write the terms as

$$\phi\left(r, -\omega, -\vec{p}\right) = \phi^*\left(r, \omega, \vec{p}\right) \tag{3.59}$$

The action with integration limits becomes[3]

$$S[\phi] = \frac{1}{2}\int \frac{d^2p\, d\omega}{(2\pi)^3} \int_0^{+\infty} dz \left[\left(\frac{R}{z}\right)^2 \partial_z\phi^*\partial_z\phi - \left(\frac{R}{z}\right)^2 \left[\omega^2 - \vec{p}^2\right] \phi^*\phi + \left(\frac{R}{z}\right)^4 m^2\phi^*\phi\right] \tag{3.60}$$

so since $S = \int \mathcal{L}\, d^dx$ the Lagranian density is half the expression in square brackets

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{R}{z}\right)^2 \partial_z\phi^*\partial_z\phi - \left(\frac{R}{z}\right)^2 \left[\omega^2 - \vec{p}^2\right] \phi^*\phi + \left(\frac{R}{z}\right)^4 m^2\phi^*\phi\right] \tag{3.61}$$

We introduce the **canonical conjugate momentum** to the canonical coordinate ϕ as usual

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial\left(\partial_z \phi\right)} \tag{3.62}$$

[3]The limits of integration absorb the $(-)$ factor of the action when reversed.

we get the expressions

$$\Pi_\phi = \frac{1}{2}\left(\frac{R}{z}\right)^2 \partial_z \phi^* \quad \Rightarrow \quad \partial_z \phi^* = 2\left(\frac{z}{R}\right)^2 \Pi_\phi \tag{3.63}$$

$$\Pi_{\phi^*} = \frac{1}{2}\left(\frac{R}{z}\right)^2 \partial_z \phi \quad \Rightarrow \quad \partial_z \phi = 2\left(\frac{z}{R}\right)^2 \Pi_{\phi^*} \tag{3.64}$$

and we construct the Hamiltonian density $\mathcal{H}$

$$\mathcal{H} \quad = \quad \Pi_\phi \partial_z \phi + \Pi_{\phi^*} \partial_z \phi^* - \mathcal{L} \tag{3.65}$$

and substituting all the $\partial_z \phi$ expressions from 3.63, 3.64 we get the expression

$$\mathcal{H} = 2\left(\frac{z}{R}\right)^2 \Pi_{\phi^*}\Pi_\phi + \frac{1}{2}\left(\frac{z}{R}\right)^2 \left[\omega^2 - \vec{p}^2\right] \phi^*\phi - \frac{1}{2}\left(\frac{z}{R}\right)^4 m^2 \phi^*\phi \tag{3.66}$$

and we can write the action via

$$\mathcal{L} = \Pi_\phi \partial_z \phi + \Pi_{\phi^*} \partial_z \phi^* - H \tag{3.67}$$

as

$$\begin{aligned} S\left[\phi\right] = \int \frac{d^2p\, d\omega}{(2\pi)^3} \int_0^{+\infty} dz \\ \left[\Pi_\phi \partial_z \phi + \Pi_{\phi^*} \partial_z \phi^* - \frac{1}{2}\left(4\left(\frac{z}{R}\right)^2 \Pi_{\phi^*}\Pi_\phi + \left(\frac{z}{R}\right)^2 \left[\omega^2 - \vec{p}^2\right] \phi^*\phi - \left(\frac{z}{R}\right)^4 m^2 \phi^*\phi\right)\right] \end{aligned} \tag{3.68}$$

We see that in this set of canonical variables $\{(\phi^*, \phi), (\Pi_{\phi^*}, \Pi_\phi)\}$ there is coupling of our canonical variables with "time", which is somewhat difficult to handle. We thus impose the rescaling of canonical coordinates

$$\phi = \frac{z}{R} f \quad , \qquad \phi^* = \frac{z}{R} f^* \tag{3.69}$$

which in turn means

$$\partial_z \phi = \frac{f}{R} + \frac{z}{R}\partial_z f \quad , \qquad \partial_z \phi^* = \frac{f^*}{R} + \frac{z}{R}\partial_z f^* \tag{3.70}$$

Substituting in 3.61 we get after some calculation and manipulation

$$\mathcal{L} = \frac{1}{2}\left[\partial_z f^* \partial_z f - \left[\omega^2 - \vec{p}^2\right] f^* f + \left(\frac{R}{z}\right)^2 \left(m^2 + \frac{2}{R^2}\right) f^* f\right] + \partial_z \left(\frac{f^* f}{2z}\right) \tag{3.71}$$

We see that 3.71 has a total "time" derivative term that in Hamiltonian dynamics leads to canonical transformations and can generally be neglected. However the term is divergent because of the integration limits (the divergence comes form the boundary at $z = 0$) and one has to discuss what to do with it: in QFT it is standard that these kind of divergences are dropped and one works with the remaining physically sensible terms. As we will see, total derivative terms that are divergent are related to holographic renormalization.

Canonical conjugate momenta are again defined as

$$p_f = \frac{\partial \mathcal{L}_{dropped}}{\partial\left(\partial_z f\right)} \tag{3.72}$$

where we have now dropped the total derivative. This gives

$$p_f = \frac{1}{2}\partial_z f^* \quad \Rightarrow \quad \partial_z f^* = 2p_f \tag{3.73}$$

$$p_{f^*} = \frac{1}{2}\partial_z f \quad \Rightarrow \quad \partial_z f = 2p_{f^*} \tag{3.74}$$

The induced canonical transformation from 3.69 that relates Π_ϕ to p_f is

$$\Pi_\phi = \frac{R}{z}\left(p_f + \frac{1}{2z}f^*\right) \quad , \quad \Pi_{\phi^*} = \frac{R}{z}\left(p_{f^*} + \frac{1}{2z}f\right) \tag{3.75}$$

The straight forward calculation of momenta conjugate to f, f^* given by 3.69 without having manipulated the Lagrangian to get a total derivative term would have given rescaled expressions of previous momenta $\tilde{p_f} = \frac{z}{R}\Pi_\phi$. However, the Lagrangian (and consequently the Hamiltonian) then involves coupled terms in the form of $\tilde{p_f}f$. Thus the boundary term generates the correct canonical transformations that decouples the data, and one can check explicitly that the Poisson brackets in the new set of variables $\{(f^*, f), (p_{f^*}, p_f)\}$ satisfy the appropriate relations. (XXX maybe appendix?)

We construct the new Hamiltonian again from 3.71

$$\mathcal{H} = p_f \partial_z f + p_{f^*}\partial_z f^* - \mathcal{L}_{dropped} \tag{3.76}$$

and substituting again all the $\partial_z \phi$ expressions from 3.73, 3.74 we get the expression (dropping the total derivative term)

$$\mathcal{H} = \frac{1}{2}\left[4p_{f^*}p_f + \left[\omega^2 - \vec{p}^2\right] f^* f - \frac{1}{z^2}\left(R^2 m^2 + 2\right) f^* f\right] \tag{3.77}$$

We write

$$a^2 = R^2 m^2 + 2 \tag{3.78}$$

so the final expression for our Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left[4p_{f^*}p_f + \left[\omega^2 - \vec{p}^2\right] f^* f - \frac{a^2}{z^2} f^* f\right] \tag{3.79}$$

and we immediately understand that the a^2 parameter is related to the conformal dimension Δ of the dual operator. The "2" above seems arbitrary, but it is just for our case where $d = 3$. In fact, in the general case AdS_{d+1} the rescaling of the field is

$$\phi = z^{\frac{d-1}{2}} f \tag{3.80}$$

which reduces to 3.69 for $d = 3$. If one works out the general Hamiltonian in similar spirit (we won't do it here) the expression reads (XXX check once more)

$$\mathcal{H} = \frac{1}{2}\left[4p_{f^*}p_f - k^2 f^* f - \frac{R^2m^2 + \frac{d^2-1}{4}}{z^2} f^* f\right] \tag{3.81}$$

where $k^2 = \eta_{\mu\nu}k^\mu k^\nu$ and now $a^2 = R^2m^2 + \frac{d^2-1}{4}$ which reduces to 3.66 for $d = 3$ as it should.

The **equations of motion** produced by the final system 3.79 are

$$\dot{f} = 2p_{f^*} \quad , \quad \dot{p}_f = -\frac{1}{2}\left[\omega^2 - \vec{p}^2\right] f^* + \frac{a^2}{2z^2} f^* \tag{3.82}$$

$$\dot{f}^* = 2p_f \quad , \quad \dot{p}_{f^*} = -\frac{1}{2}\left[\omega^2 - \vec{p}^2\right] f + \frac{a^2}{2z^2} f \tag{3.83}$$

which lead to the decoupled equations

$$\ddot{f} + \left[\omega^2 - \vec{p}^2\right] f - \frac{a^2}{z^2} f = 0 \tag{3.84}$$

$$\ddot{f}^* + \left[\omega^2 - \vec{p}^2\right] f^* - \frac{a^2}{z^2} f^* = 0 \tag{3.85}$$

and we see that we only need to study one mode since both $f = f(z, \omega, p)$ and $f^* = f(z, -\omega, -p)$ (for real fields) behave the same way.

Let's now discuss the general form of 3.84, 3.85. We see that for Lorentzian signature the behaviour of the fields is governed by the sign of $\left[\omega^2 - \vec{p}^2\right]$: Non-"tachyonic" fields in AdS have $\left[\omega^2 - \vec{p}^2\right] > 0$ and the fields are oscillations with time dependence. The case that is of interest to us is the one with Euclidean signature where we rotate

$$\omega \to i\omega \tag{3.86}$$

and

$$\left[\omega^2 - \vec{p}^2\right] \to - \left[\omega^2 + \vec{p}^2\right] = -\bar{\omega}^2 \tag{3.87}$$

or the Lorentzian case with tachyonic fields in AdS which admit $\left[\omega^2 - \vec{p}^2\right] < 0$. For simplicity we will hence only focus on the one with Euclidean signature. In these cases the equations 3.84,3.85 are time-dependent *inverted oscillators*. The phase space is similar to the usual saddle-point of a typical inverted oscillator (repulsive force) but it is altered by the time-dependent term. As we will see, the application of the AdS/CFT dictionary is deeply related to the stability analysis of these fields' phase space, and holography is now the study of a classical hamiltonian system.

We will present some trajectories in the (f, p_f)-plane (phase space) for the Euclidean system in the corresponding Appendix B, where we can roughly see the evolution of the system for a given set of initial (f, p_f) data.

We can immediately distinguish interesting cases in this formalism: the "conformal scalar" with $a^2 = 0$ which should simplify things greatly, and of course the massless scalar which in this context is $a^2 = 2$.

3.2.3 Conformal scalar $a^2 = 0$

In the case where $m^2 = -\frac{2}{R^2}$ things are simplified: the time dependent term vanishes. The equations of motion reduce to a simple inverted harmonic oscillator

$$\ddot{f}_{\bar{\omega}} - \bar{\omega}^2 f_{\bar{\omega}} = 0 \tag{3.88}$$

where

$$\bar{\omega} = (\omega, \vec{p}) \ , \ \ \bar{\omega}^2 = \omega^2 + \vec{p}^2 \tag{3.89}$$

and identically for the conjugate field f^*. The general solution is as usual[4]

$$f_{\bar{\omega}}(t) = A(\bar{\omega})\, e^{-\bar{\omega}t} + B(\bar{\omega})\, e^{+\bar{\omega}t} \tag{3.90}$$

$$f^*_{\bar{\omega}}(t) = A^*(\bar{\omega})\, e^{-\bar{\omega}t} + B^*(\bar{\omega})\, e^{+\bar{\omega}t} \tag{3.91}$$

and by 3.82, 3.83

$$p_{f_{\bar{\omega}}}(t) = \frac{\bar{\omega}}{2}\left(B^*(\bar{\omega})\, e^{+\bar{\omega}t} - A^*(\bar{\omega})\, e^{-\bar{\omega}t}\right) \tag{3.92}$$

$$p_{f^*_{\bar{\omega}}}(t) = \frac{\bar{\omega}}{2}\left(B(\bar{\omega})\, e^{+\bar{\omega}t} - A(\bar{\omega})\, e^{-\bar{\omega}t}\right) \tag{3.93}$$

Let's now remember what we have to do to applpy the AdS/CFT dictionary: firstly,

[4]We will hence denote $z \to t$ for aesthetic reasons.

we need to impose appropriate boundary conditions, namely one that forms the value of the field on the boundary $t = 0$ which we will identify with the source of the dual operator, and a second that keeps the solution *regular* on the horizon $t = +\infty$. This means that we have to choose

$$B(\bar{\omega}) = 0 \tag{3.94}$$

which in the context of initial boundary data is equivalent to

$$\begin{aligned} p_{f_{\bar{\omega}}}(0) &= -\frac{\bar{\omega}}{2} f^*_{\bar{\omega}}(0) && (3.95) \\ p_{f^*_{\bar{\omega}}}(0) &= -\frac{\bar{\omega}}{2} f_{\bar{\omega}}(0) && (3.96) \end{aligned}$$

and the *parametric* equations now become

$$\begin{aligned} f_{\bar{\omega}}(t) = A(\bar{\omega})\, e^{-\bar{\omega}t} \quad &, \quad f^*_{\bar{\omega}}(t) = A^*(\bar{\omega})\, e^{-\bar{\omega}t} && (3.97) \\ p_{f_{\bar{\omega}}}(t) = -\frac{\bar{\omega}}{2} A^*(\bar{\omega})\, e^{-\bar{\omega}t} \quad &, \quad p_{f^*_{\bar{\omega}}}(t) = -\frac{\bar{\omega}}{2} A(\bar{\omega})\, e^{-\bar{\omega}t} && (3.98) \end{aligned}$$

Taking into account that the for the initial fields ϕ we had $\phi^*(\omega, \vec{p}) = \phi(-\omega, -\vec{p})$ we can write (XXX check, ask!)

$$\begin{aligned} f_{\bar{\omega}}(t) = A(\bar{\omega})\, e^{-\bar{\omega}t} \quad &, \quad f^*_{\bar{\omega}}(t) = A(-\bar{\omega})\, e^{-\bar{\omega}t} && (3.99) \\ p_{f_{\bar{\omega}}}(t) = -\frac{\bar{\omega}}{2} A(-\bar{\omega})\, e^{-\bar{\omega}t} \quad &, \quad p_{f^*_{\bar{\omega}}}(t) = -\frac{\bar{\omega}}{2} A(\bar{\omega})\, e^{-\bar{\omega}t} && (3.100) \end{aligned}$$

Notice how through the boundary conditions we selected a particular trajectory of the phase space, namely the stable manifold. Also it is now clear that the boundary value of the field is

$$f_{\bar{\omega}}(0) = A(\bar{\omega}) \tag{3.101}$$

which in this case is well-defined (non-divergent, non-vanishing) and we identify it with the source of the dual operator

$$A(\bar{\omega}) \equiv J(\bar{\omega}) \tag{3.102}$$

Secondly, we want to evaluate the variation of the on-shell action w.r.t. the sources $J(\bar{\omega})$. The variation of the action is of the form

$$\begin{aligned} \delta S = \int \left\{ \delta p \cdot \dot{q} + p \cdot \delta \dot{q} - \frac{\partial H}{\partial p} \cdot \delta p - \frac{\partial H}{\partial q} \cdot \delta q \right\} dt \\ = \int \left\{ \left[\dot{q} - \frac{\partial H}{\partial p} \right] \cdot \delta p + \left[-\dot{p} - \frac{\partial H}{\partial q} \right] \cdot \delta q \right\} dt + [p \cdot \delta q]_{t_i}^{t_f} \end{aligned}$$

If we evaluate this on the equations of motion the first term vanishes and we are left with the boundary term. The on-shell variation in our case is then

$$\delta S_{on-shell}=\int\frac{d^2p}{(2\pi)^2}\frac{d\omega}{2\pi}\left[p_{f_{\bar{\omega}}}\delta f_{\bar{\omega}}+p_{f^*_{\bar{\omega}}}\delta f^*_{\bar{\omega}}\right]_0^{\infty} \tag{3.103}$$

and since the fields are regular on the horizon $t=\infty$ the variation is

$$\delta S_{on-shell}=\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\frac{d\omega}{2\pi}\bar{\omega}\left[A(\bar{\omega})\,\delta A(-\bar{\omega})+A(-\bar{\omega})\,\delta A(\bar{\omega})\right] \tag{3.104}$$

and the second order is

$$\delta^2 S_{on-shell}=\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\frac{d\omega}{2\pi}\bar{\omega}\left[2\delta A(\bar{\omega})\,\delta A(-\bar{\omega})+\overbrace{A(\bar{\omega})\,\delta^2 A(-\bar{\omega})+A(-\bar{\omega})\,\delta^2 A(\bar{\omega})}^{\text{will give 0?}}\right] \tag{3.105}$$

The AdS/CFT recipe reads

$$S_{on-shell}[A]\equiv -W[J] \tag{3.106}$$

in our saddle-point (classical gravity) approximation and thus

$$\langle\mathcal{O}(-\bar{\omega})\,\mathcal{O}(\bar{\omega})\rangle=-\frac{\delta^2 S_{on-shell}}{\delta A(\bar{\omega})\,\delta A(-\bar{\omega})} \tag{3.107}$$

Substituting we get the value for the 2-point function

$$\langle\mathcal{O}(-\bar{\omega})\,\mathcal{O}(\bar{\omega})\rangle=-\bar{\omega} \tag{3.108}$$

Using the formula for Fourier transformations in d-dimensions

$$f(x)=\int\frac{d^dp}{(2\pi)^d}e^{i\vec{p}\cdot\vec{x}}\,|\vec{p}|^n=2^n\pi^{-d/2}\frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)}\frac{1}{x^{n+d}} \tag{3.109}$$

we transform back to position space for $n=1$ and $d=3$ and get the final answer

$$\langle\mathcal{O}(x)\,\mathcal{O}(0)\rangle=\frac{1}{\pi^2}\frac{1}{x^4} \tag{3.110}$$

Let's do some consistency checks: We started out for $a^2=0$ or $m^2=-\frac{2}{R^2}$. Substituting in 3.7 we get

$$\Delta=2 \tag{3.111}$$

which is consistent with the $x^{-2\Delta}$ form of 3.110. In particular 3.42 reduces exactly to 3.110 for $d=3$ and $\Delta=2$, so we have the correct two-point function.

One thing to note is the importance of the initial boundary data 3.95, 3.96: essentially these are required to produce the $\bar{\omega}^1$ power term e need to have the final result 3.110. As we will see, this will be a general feature of the initial boundary data in this approach.

3.2.4 Massless scalar $a^2 = 2$

In this case $m^2 = 0$ and the equations of motion involve a time dependent singular term

$$\ddot{f}_{\bar{\omega}} - \left(\bar{\omega}^2 + \frac{2}{t^2}\right) f_{\bar{\omega}} = 0 \tag{3.112}$$

For a general solution we transform

$$f_{\bar{\omega}}(t) = \sqrt{t} g_{\bar{\omega}}(t) \tag{3.113}$$

and the general solution is found in terms of modified Bessel functions[5]

$$f_{\bar{\omega}}(t) = \sqrt{t}\left[A(\bar{\omega}) \mathcal{I}_{3/2}(\bar{\omega}t) + B(\bar{\omega}) \mathcal{K}_{3/2}(\bar{\omega}t)\right] \tag{3.114}$$

and using the explicit formulas

$$\begin{aligned} \mathcal{I}_{3/2}(x) &= \sqrt{\frac{1}{2\pi x}}\left(e^x\left(1 - \frac{1}{x}\right) + e^{-x}\left(1 + \frac{1}{x}\right)\right) \qquad (3.115) \\ \mathcal{K}_{3/2}(x) &= \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + \frac{1}{x}\right) \qquad (3.116) \end{aligned}$$

we can write

$$f_{\bar{\omega}}(t) = C_1(\bar{\omega}) e^{\bar{\omega}t}\left(1 - \frac{1}{\bar{\omega}t}\right) + \left(C_1(\bar{\omega}) + C_2(\bar{\omega})\right) e^{-\bar{\omega}t}\left(1 + \frac{1}{\bar{\omega}t}\right) \tag{3.117}$$

where the $C_1(\bar{\omega})$, $C_2(\bar{\omega})$ are simply related to $A(\bar{\omega})$, $B(\bar{\omega})$ and the conjugate momentum is consequently

$$p_{f_{\bar{\omega}}}(t) = \frac{\bar{\omega}}{2}\left[C_1^*(\bar{\omega})\, e^{\bar{\omega}t}\left(1 - \frac{1}{\bar{\omega}t} + \frac{1}{(\bar{\omega}t)^2}\right) - \left(C_1^*(\bar{\omega}) + C_2^*(\bar{\omega})\right)\, e^{-\bar{\omega}t}\left(1 + \frac{1}{\bar{\omega}t} + \frac{1}{(\bar{\omega}t)^2}\right)\right] \tag{3.118}$$

We see that this is maybe the simplest non-trivial case: The form of the solutions is exponentials multiplied by polynomials of $(\omega t)^{-1}$ (and for half-integer Bessel index the polynomials have finite terms) and these polynomials make the solutions divergent on the

[5] This is not surprising: We initially transformed $\phi \sim t f$ and then $f = t^{1/2} g$ so altogether $\phi \sim t^{3/2} \times$ [*Bessel functions*] which is consistent with the $\phi = \xi^{d/2} G$ ansatz of 3.27 so we naturally come to the same result.

boundary $t = 0$. We have to properly select our initial boundary data which will be identified with the source of the dual operator. As we saw in the conventional 2-point function recipe, the source is generally identified with the *leading term* of the divergent solution $f_{\bar{\omega}}(t)$. In this case the divergences are poles of order one so the source is of the form

$$\begin{aligned} f_{\bar{\omega}}(t) &= \frac{1}{t} J(\bar{\omega}) + \mathcal{O}(1) \\ \text{to leading term} \quad J(\bar{\omega}) &= \frac{1}{\bar{\omega}} \left(\left(C_1(\bar{\omega}) + C_2(\bar{\omega}) \right) e^{-\bar{\omega} t} - C_1(\bar{\omega}) e^{\bar{\omega} t} \right) \end{aligned} \tag{3.119}$$

The terms that will contribute to the calculation of the two-point function come from the variation of the action, so these are of the form $p_{f_{\bar{\omega}}} \delta f_{\bar{\omega}}$ and we are only interested in their values on the boundary $t = 0$ and on the horizon $t = \infty$. In a general case, this term will give on the boundary divergent terms ($\sim t^{-n}$), vanishing terms ($\sim t^{+m}$) and finite terms, which in this case are produced by the terms proportional to t in the power series of $p_{f_{\bar{\omega}}}$ which encode the information about the two-point functions. The immediate impulse is to subtract all terms that are divergent in the variation δS via correct counterterms and keep the finite terms that give us physically sensible information in the $\frac{\delta^2 S}{\delta J \delta J}$ term. However, this **does not** lead to correct terms for the two-point function: in generalisation of what we saw before it is clear that we need a term proportional to some power of $\bar{\omega}$ other than one in Fourier space. So how do we find the correct counterterms?

Things are much simpler if we consider that the system is Hamiltonian: essentially what we did in the simple case of the conformal scalar was to define the coordinates and conjugate momenta so that their (leading) values on the boundary are *independent*, and we choose the initial data so that the boundary conditions are met on the boundary and the horizon and the correct power of the Fourier mode emerges. We see from 3.118 and 3.119 that the *leading* term of $p_{f^*_{\bar{\omega}}}(t)$ is (keep only the t^{-2} terms) is

$$\sim \frac{1}{2\bar{\omega}} \left(C_1(\bar{\omega})\, e^{\bar{\omega} t} - \left(C_1(\bar{\omega}) + C_2(\bar{\omega}) \right)\, e^{-\bar{\omega} t} \right) \frac{1}{t^2} \tag{3.120}$$

which is *exactly* $\sim J(\bar{\omega})$ on the boundary, thus clearly *not* independent of $f_{\bar{\omega}}$. In addition, $p_{f_{\bar{\omega}}}(t)$ does not have a term proportional to t to cancel out the divergence of $f_{\bar{\omega}}$ in the $p_{f_{\bar{\omega}}} \delta f_{\bar{\omega}}$ product. The correct way to handle these problems is to remember what we did in 3.71 when the canonical data was not *properly defined*: canonical transformations.

The *correct* boundary term to add to the Hamiltonian 3.79 is (XXX check again, maybe extend?!)

$$\frac{1}{2} \left(\frac{1}{t} + \bar{\omega}^2 t \right) f^*_{\bar{\omega}} f_{\bar{\omega}} \tag{3.121}$$

and the full *generating function* is

$$G_{gen}\left(f_{\bar{\omega}}^*, f_{\bar{\omega}}, P_{f_{\bar{\omega}}^*}, P_{f_{\bar{\omega}}}\right) = P_{f_{\bar{\omega}}} f_{\bar{\omega}} + P_{f_{\bar{\omega}}^*} f_{\bar{\omega}}^* - \frac{1}{2}\left(\frac{1}{t} + \bar{\omega}^2 t\right) f_{\bar{\omega}}^* f_{\bar{\omega}} \tag{3.122}$$

to which the induced canonical transformation is

$$\left\{f_{\bar{\omega}}^*, f_{\bar{\omega}}, p_{f_{\bar{\omega}}^*}, p_{f_{\bar{\omega}}}\right\} \to \left\{F_{\bar{\omega}}^*, F_{\bar{\omega}}, P_{f_{\bar{\omega}}^*}, P_{f_{\bar{\omega}}}\right\} \tag{3.123}$$

defined by

$$\left\{ \begin{array}{c} p_{f_{\bar{\omega}}} = \frac{\partial G}{\partial f_{\bar{\omega}}} \\ F_{\bar{\omega}} = \frac{\partial G}{\partial P_{f_{\bar{\omega}}}} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} P_{f_{\bar{\omega}}} = p_{f_{\bar{\omega}}} + \frac{1}{2}\left(\frac{1}{t} + \bar{\omega}^2 t\right) f_{\bar{\omega}}^* \\ F_{\bar{\omega}} = f_{\bar{\omega}} \end{array} \right\} + conjugates \tag{3.124}$$

Of course the boundary term is divergent on the boundary as well, but when we subtract the divergences in the process of renormalization we are left with the correct terms for the two-point function. Adding this divergent boundary term is equivalent to the holographic renormalization. (XXX check again). We write explicitly the solutions in the new momenta (coordinates stay the same)

$$\begin{aligned} f_{\bar{\omega}}(t) &= C_1(\bar{\omega}) e^{\bar{\omega}t}\left(1 - \frac{1}{\bar{\omega}t}\right) + \left(C_1(\bar{\omega}) + C_2(\bar{\omega})\right) e^{-\bar{\omega}t}\left(1 + \frac{1}{\bar{\omega}t}\right) && (3.125) \\ P_{f_{\bar{\omega}}}(t) &= \frac{\bar{\omega}^2}{2}\left[C_1^*(\bar{\omega})\, e^{\bar{\omega}t} + \left(C_1^*(\bar{\omega}) + C_2^*(\bar{\omega})\right) e^{-\bar{\omega}t}\right] t && (3.126) \end{aligned}$$

and similarly for the complex conjugate data.

The leading terms on the boundary $t = 0$ are from above

$$\begin{aligned} f_{\bar{\omega}} &= \frac{1}{t}\frac{C_2(\bar{\omega})}{\bar{\omega}} + \mathcal{O}(1) \Rightarrow \\ J(\bar{\omega}) &= \frac{C_2(\bar{\omega})}{\bar{\omega}} \end{aligned} \tag{3.127}$$

and in this picture the canonical conjugate momentum is in leading terms

$$P_{f_{\bar{\omega}}} = \frac{\bar{\omega}^2}{2}\left(2C_1^*(\bar{\omega}) + C_2^*(\bar{\omega})\right) t \tag{3.128}$$

which is independent of $J(\bar{\omega})$ as we need. We have not yet applied the *regularity condition* which does not change after the transformation: regularity at $t = \infty$ is equivalent to choosing

$$C_1(\bar{\omega}) = C_1^*(\bar{\omega}) = 0 \tag{3.129}$$

The condition lifts the independence of coordinates and momenta as in the conformal case and the momentum is now

$$P_{f_{\bar{\omega}}}\left(t\right)=\frac{\bar{\omega}^{3}}{2}J\left(-\bar{\omega}\right)t \tag{3.130}$$

We can now calculate the variation of the action

$$\delta S_{on-shell}=\frac{1}{2}\int\frac{d^{2}p}{\left(2\pi\right)^{2}}\frac{d\omega}{2\pi}\bar{\omega}^{3}\left[J\left(\bar{\omega}\right)\delta J\left(-\bar{\omega}\right)+J\left(-\bar{\omega}\right)\delta J\left(\bar{\omega}\right)\right] \tag{3.131}$$

and the second order is

$$\delta^{2}S_{on-shell}=\frac{1}{2}\int\frac{d^{2}p}{\left(2\pi\right)^{2}}\frac{d\omega}{2\pi}\bar{\omega}^{3}\left[2\delta J\left(\bar{\omega}\right)\delta J\left(-\bar{\omega}\right)+\overbrace{J\left(\bar{\omega}\right)\delta^{2}J\left(-\bar{\omega}\right)+J\left(-\bar{\omega}\right)\delta^{2}J\left(\bar{\omega}\right)}^{\text{will give 0?}}\right] \tag{3.132}$$

which again lead to the two-point function in Fourier space

$$\left\langle\mathcal{O}\left(-\bar{\omega}\right)\mathcal{O}\left(\bar{\omega}\right)\right\rangle=\bar{\omega}^{3} \tag{3.133}$$

Using again 3.109 we get the final result in coordinate space

$$\left\langle\mathcal{O}\left(x\right)\mathcal{O}\left(0\right)\right\rangle=\frac{12}{\pi^{2}}\frac{1}{x^{6}} \tag{3.134}$$

We do again the consistency checks: We have $m^2=0$ or $a^2=2$. Substituting again in 3.7 we get

$$\Delta=3 \tag{3.135}$$

which is consistent with the $x^{-2\Delta}$ form of 3.134 and once again 3.42 reduces exactly to 3.134 for $d=3$ and $\Delta=3$, so we have the correct normalization for the two-point function.

3.3 Conclusions

In the previous sections we identified of the so-called holographic coordinate with the "time" coordinate of a Hamiltonian system and we explicitly showed that the system in Euclidean AdS/CFT is *unstable*, in particular it is of ***time-dependent inverted oscillator*** type. In this analysis we came to two important results:

- Canonical transformations, which are implemented by adding and subtracting divergent (in general) boundary terms to the Lagrangian density, are equivalent to the Holographic Renormalization process in the sense that we add and subtract counterterms. However canonical transformations actively change the holographic evolution of the hamiltonian system and further study/interpretation of their physical meaning is needed.

- The boundary conditions on the boundary and the regularity condition on the horizon mean in the context of hamiltonian dynamics that we automatically select *one* trajectory from the system's phase space, namely one that satisfies the conditions:

  1. **$a^2 = 0$: Simple inverted harmonic oscillator**
     In this case as we said things are simple. In the $\left(f_{\bar{\omega}}, p_{f^*_{\bar{\omega}}}\right)$ plane it is easy to see from the parametric equations 3.90 and 3.93 that the phase space trajectories are
     $$p_{f^*_{\bar{\omega}}} = \pm\bar{\omega}\sqrt{f^2_{\bar{\omega}} - A(\bar{\omega})\, B(\bar{\omega})} \tag{3.136}$$
     which are sets of hyperbolas spanning the plane for different values of $A(\bar{\omega})$ and $B(\bar{\omega})$, and the asymptotics are the lines
     $$p_{f^*_{\bar{\omega}}} = \pm\bar{\omega} f_{\bar{\omega}} \tag{3.137}$$
     The boundary conditions for the Hamiltonian AdS/CFT are as we saw equivalent to choosing one of the asymptotics as the trajectory of holographic evolution. In particular we choose one that asymptotically goes to $f_{\bar{\omega}} = 0$ for $t \to \infty$, so we choose the *stable line*.

  2. **$a^2 = 2$: Time-dependent inverted harmonic oscillator**
     In this case we consider the phase space of the final (transformed) variables $\left(f_{\bar{\omega}}, P_{f^*_{\bar{\omega}}}\right)$. For the trajectories we need to find some function that satisfies
     $$S\left(f_{\bar{\omega}}, P_{f^*_{\bar{\omega}}}, C_1(\bar{\omega}), C_2(\bar{\omega}), t\right) = 0 \tag{3.138}$$
     In fact, an implicit function for the trajectories in phase space should be of the

form

$$S\left(f_{\bar{\omega}}, P_{f^*_{\bar{\omega}}}, C_1(\bar{\omega}), C_2(\bar{\omega})\right) = 0 \tag{3.139}$$

i.e. time-independent. But the system we want to depict, even the canonically transformed system, is not an autonomous system and thus the phase space changes as time evolves. What we can do, is rescale the variables and try to eliminate the exponential terms. We can write the parametric equations 3.125 and 3.126 as

$$\begin{aligned}
\bar{\omega} t f_{\bar{\omega}}(t) &= C_1(\bar{\omega}) e^{\bar{\omega} t}(\bar{\omega} t - 1) + (C_1(\bar{\omega}) + C_2(\bar{\omega})) e^{-\bar{\omega} t}(\bar{\omega} t + 1) && (3.140)\\
\frac{2}{\bar{\omega}^2 t} P^*_{f_{\bar{\omega}}}(t) &= \left(C_1(\bar{\omega})\, e^{\bar{\omega} t} + (C_1(\bar{\omega}) + C_2(\bar{\omega}))\, e^{-\bar{\omega} t}\right) t && (3.141)
\end{aligned}$$

and we can work out

$$\bar{\omega} t f_{\bar{\omega}}(t) - \frac{2(\bar{\omega} t - 1)}{\bar{\omega}^2 t} P^*_{f_{\bar{\omega}}}(t) = 2(C_1(\bar{\omega}) + C_2(\bar{\omega}))\, e^{-\bar{\omega} t} \tag{3.142}$$

and

$$\bar{\omega} t f_{\bar{\omega}}(t) - \frac{2(\bar{\omega} t + 1)}{\bar{\omega}^2 t} P^*_{f_{\bar{\omega}}}(t) = -2C_1(\bar{\omega})\, e^{\bar{\omega} t} \tag{3.143}$$

so multiplying we get the desired function

$$\left(\bar{\omega} t f_{\bar{\omega}} - \frac{2(\bar{\omega} t - 1)}{\bar{\omega}^2 t} P^*_{f_{\bar{\omega}}}\right) \cdot \left(\bar{\omega} t f_{\bar{\omega}} - \frac{2(\bar{\omega} t + 1)}{\bar{\omega}^2 t} P^*_{f_{\bar{\omega}}}\right) = -4C_1(\bar{\omega})(C_1(\bar{\omega}) + C_2(\bar{\omega})) \tag{3.144}$$

We see that the trajectories resemble hyperbolas in the rescaled variables with eccentricity depending on the initial data $(C_1(\bar{\omega}), C_2(\bar{\omega}))$. This is to be expected as the system is unstable. The unstable and stable manifolds that pass through the *hyperbolic* point $(0,0)$ are correspondingly the "lines"

$$\begin{aligned}
\bar{\omega} t f_{\bar{\omega}} - \frac{2(\bar{\omega} t - 1)}{\bar{\omega}^2 t} P^*_{f_{\bar{\omega}}} &= 0 && (3.145)\\
\bar{\omega} t f_{\bar{\omega}} - \frac{2(\bar{\omega} t + 1)}{\bar{\omega}^2 t} P^*_{f_{\bar{\omega}}} &= 0 && (3.146)
\end{aligned}$$

where the first one comes from initial data $C_1(\bar{\omega}) = -C_2(\bar{\omega})$ and the second one from inital data $C_1(\bar{\omega}) = 0$. We notice the second line $C_1(\bar{\omega}) = 0$ corresponds again to the boundary data we imposed for the two-point function in our Hamiltonian approach, as in the previous case.

One can take it further and make the ansatz that *Euclidean holography is deeply related to unstable Hamiltonian systems and the boundary data for Euclidean AdS/CFT is equivalent to choosing the unique (in these cases) stable manifold in that unstable system.*

Chapter 4

The Inverted Oscillator and Coherent States

4.1 On Coherent States of Unstable Quantum Systems

4.2 Application: The Inverted Oscillator

4.3 Physical Content: Interpretation?

Appendix A

Functional Methods in QFT

In standard quantum field theory, we are interested in the *vacuum expectation value* of a (scalar) quantum field operator $\phi(x)$which can be calculated by the path integral

$$\langle 0| \hat{\phi}(x) |0\rangle = \int \mathcal{D}\phi e^{iS[\phi]}\phi(x) \tag{A.1}$$

The $\mathcal{D}\phi(x)$ denotes the (somewhat shady) product of field configurations

$$\mathcal{D}\phi(x) = \prod_x \int d\phi(x) \tag{A.2}$$

which can be sort of defined as (XXX aqft)

$$\prod_x \int d\phi(x) = \lim_{a\to 0} \prod_i \int d\phi(x_i) \tag{A.3}$$

where the i signifies that we have divided our spacetime in a lattice of size a and we take $a \to 0$. Doing this we should take a finite total volume, and take it to be infinite after the calculations.

More generally we are interested in the vacuum expectation value of strings of operators, in particular of *time-ordered* strings of operators, which we call the n-point function

$$G_n(x_1, x_2, \ldots x_n) = \langle 0| \mathcal{T}\left\{\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_n)\right\}|0\rangle \tag{A.4}$$

which in the previous notation is written as

$$G_n(x_1, x_2, \ldots x_n) = \int \mathcal{D}\phi e^{iS[\phi]}\phi(x_1)\phi(x_2)\ldots\phi(x_n) \tag{A.5}$$

One standard way to calculate this is to use the so-called *partition function* (XXX nastase)

$$Z\left[J\right]=\int\mathcal{D}\phi e^{iS[\phi]+i\int d^{d}x\,J(x)\phi(x)} \tag{A.6}$$

normalized to

$$Z\left[0\right]=\int\mathcal{D}\phi e^{iS[\phi]}=1 \tag{A.7}$$

where the we have "modified" the action by the $\int d^{d}x\,J\left(x\right)\phi\left(x\right)$ term, and the $J\left(x\right)$ is an arbitrary function called the *source* of the $\phi\left(x\right)$ quantum operator.

The n-point functions of the fields $\phi\left(x\right)$ are now calculated by taking the *functional derivative,* defined by

$$\frac{\delta J\left(y\right)}{\delta J\left(x\right)}=\delta^{d}\left(x-y\right) \tag{A.8}$$

plus Leibniz and chain-rules for differantiation (XXX aqft), of the partition function w.r.t. the sources $J\left(x\right)$ of the operators (also denoted as)

$$\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle^{(n)}=(-i)^{n}\frac{\delta}{\delta J\left(x_1\right)}\frac{\delta}{\delta J\left(x_2\right)}\cdots\frac{\delta}{\delta J\left(x_n\right)}Z\left[J\right]\Bigg|_{J=0} \tag{A.9}$$

and it is common to write (XXX aqft)

$$Z\left[J\right]=e^{iW[J]} \tag{A.10}$$

where $W\left[J\right]$ is called the *generating functional* for connected amplitudes. Since $Z\left[0\right]=1$ we have

$$\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle^{(n)}_{conn.}=(-i)^{n-1}\frac{\delta}{\delta J\left(x_1\right)}\frac{\delta}{\delta J\left(x_2\right)}\cdots\frac{\delta}{\delta J\left(x_n\right)}W\left[J\right]\Bigg|_{J=0} \tag{A.11}$$

It is also common and useful to perform a *Wick rotation* to Euclidean time

$$t\to-i\tau_E,\quad iS\to-S_E \tag{A.12}$$

so the path integral has a damped e^{-S_E} factor and is easier to perform. The expressions are substituted with

$$Z_E\left[J\right]=\int\mathcal{D}\phi e^{-S_E[\phi]+\int d^{d}x\,J(x)\phi(x)} \tag{A.13}$$

$$Z_E\left[J\right]=e^{-W[J]} \tag{A.14}$$

$$\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle^{(n)}_{E\;conn.}=(-i)^{n-1}\frac{\delta}{\delta J\left(x_1\right)}\frac{\delta}{\delta J\left(x_2\right)}\cdots\frac{\delta}{\delta J\left(x_n\right)}W\left[J\right]\Bigg|_{J=0} \tag{A.15}$$

Appendix B

Trajectories to the Euclidean AdS bulk field

In this Appendix we will present some plots for the system

$$\dot{f} = 2p_{f^*} \ , \ \dot{p}_{f^*} = \frac{1}{2}\left(\bar{\omega}^2 + \frac{a^2}{t^2}\right) f^* \tag{B.1}$$

for $d = 3$. The general solution to this system can be found in terms of modified Bessel functions

$$f\left(t\right) = \sqrt{t}\left[A\left(\bar{\omega}\right)\mathcal{I}_\nu\left(\bar{\omega}t\right) + B\left(\bar{\omega}\right)\mathcal{K}_\nu\left(\bar{\omega}t\right)\right] \tag{B.2}$$

where the Bessel index is

$$\nu = \sqrt{\frac{1}{4} + a^2} \tag{B.3}$$

and if we sustistute a^2 we have

$$\begin{aligned} \nu &= \sqrt{\frac{1}{4} + R^2m^2 + 2} \\ &= \sqrt{\frac{9}{4} + R^2m^2} \\ &= \sqrt{\frac{d^2}{4} + R^2m^2} \end{aligned} \tag{B.4}$$

which is consistent with the result from 3.29.

We notice a few things: Firstly these plots are not strictly a phase space for the system because the system is not autonomous, meaning explicitly time-dependent. One repercussion of this is that the trajectories may intersect in the (f, p_f) plane. Also the solutions are exponentialy increasing or decreasing making it very sensitive to initial data. We will just define some initial data f_0 and p_{f_0}at some point $t_0 \ll 1$ (because the solutions are also singular at $t = 0$ and $t = \infty$) and let the parametric solutions run for some finite time interval . This way we will have a rough picture about where the stable manifolds lie, since trajectories close to them should closely approach the equilibrium point $(0, 0)$. Secondly we will make different plots *only for half-integer values of* ν for simplicity. The cases we consider in the main chapter correspond to the first two plots $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$.

The plots were made using Mathematica:

```
Clear["Global`*"]

range=5; (*range of phasespace to investigate *)
spacing=.75; (* spacing in grid of initial data *)
grid=Table[{i,j},{i,-range,range,spacing},{j,-range,range,spacing}];
start=.5; (*starting point for trajectories in "time" since x in (0,
    infinity) and x=0 is singular *)
end=5;   (*ending point for trajectories in "time" , also a singular
    point *)
w=Sqrt[.5]; (* "frequency of the bessel functions *)
startpoint=start(* point for initial conditions *) ;

fn=Sqrt[x]*(c1*BesselI[n,w*x]+c2*BesselK[n,w*x]);
pn=1/2*D[fn,x];

lista=Table[i/2,{i,1,11,2}]

n=lista[[1]]; (* set value for n from list *)
sol=Solve[{(fn/.{x->startpoint})==fn0,(pn/.{x->startpoint})==pn0},{c1
    ,c2}]; (* solve  constants c1,  c2 in definition of position and
     momenum as functions of initial data fn0,pn0) *)
Print["Phasespace for Bessel index n=",n];
listofplots={};
For[i=-range,i<range+1,i=i+spacing,
        For[j=-range,j<range+1,j=j+spacing,
                dlist={fn/.{sol[[1,1]],sol[[1,2]]},pn/.{sol[[1,1]],
                    sol[[1,2]]}}/.{fn0->i,pn0->j}; (* {fn,pn} with
                    initial data pn0=i fn0=j, scanning the phasespace
                    *)
                AppendTo[listofplots,ParametricPlot[dlist,{x,start,
                    end},AxesLabel->{f,p},PlotStyle->Thickness
                    [0.0001],PlotRange->{{-range,range},{-range,range
                    }}]]
        ]
];
Show[listofplots,ListPlot[grid]]
```

These are:

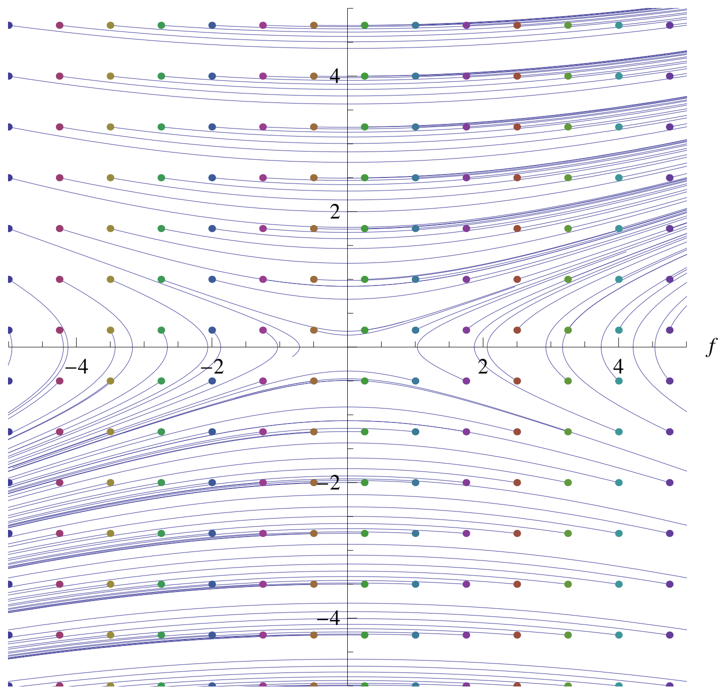

Figure B.0.1: Bessel index $\nu = \frac{1}{2}$

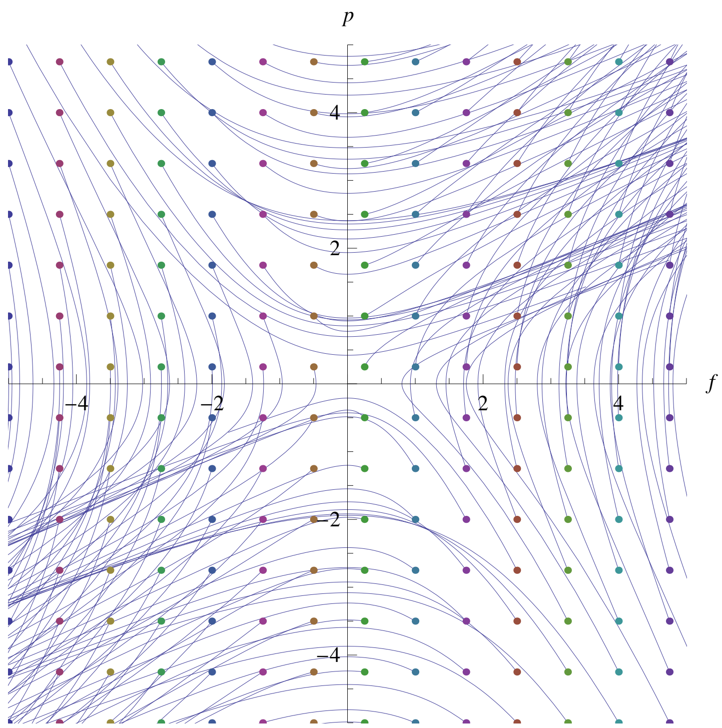

Figure B.0.2: Bessel index $\nu = \frac{3}{2}$

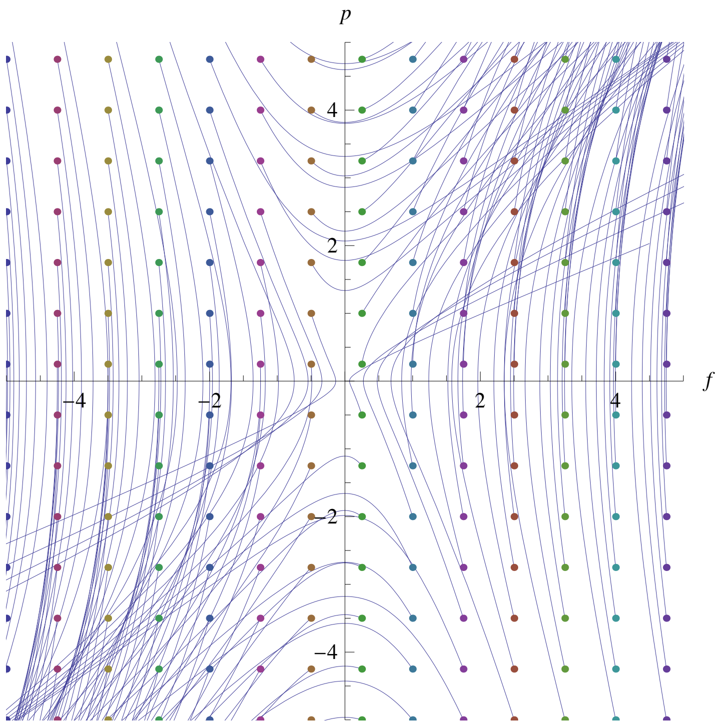

Figure B.0.3: Bessel index $\nu = \frac{5}{2}$

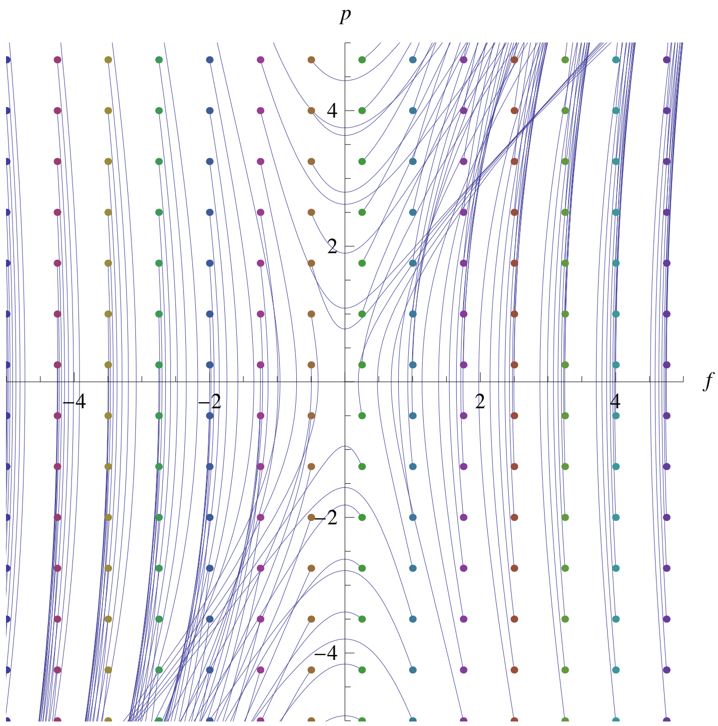

Figure B.0.4: Bessel index $\nu = \frac{7}{2}$

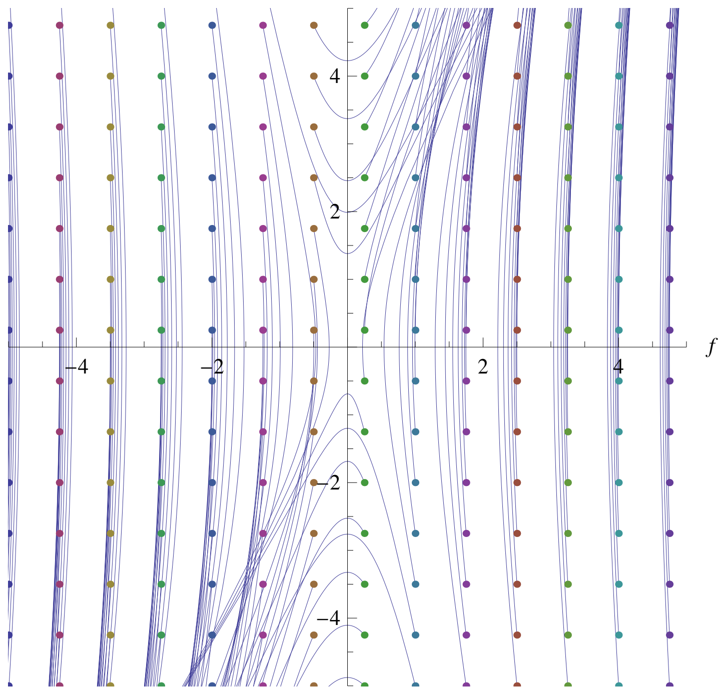

Figure B.0.5: Bessel index $\nu = \frac{9}{2}$

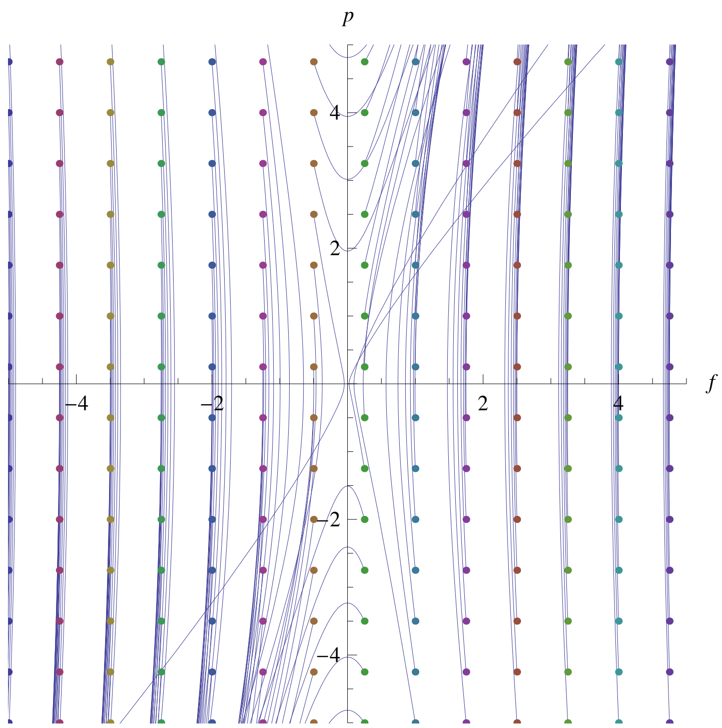

Figure B.0.6: Bessel index $\nu = \frac{11}{2}$

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