Some Aspects of Large-N Vector Models and their Higher-Spin Holography

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Outline

1. Motivations

2. $O(N)$ vector models: review
   - $O(N) \to O(N-1)$ symmetry breaking in the bosonic model
   - The fermionic $O(N)$ vector model: lightning review
   - Anomalous dimensions

3. $O(N)$/HS holography
   - The gap equations from holography
   - The singleton deformation of higher-spin theory and boundary symmetry breaking

4. Aspects of the OPE in $O(N)$ vector models
   - The conformal partial waves: free field theory
   - The skeleton graphs

5. Summary and outlook

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1. Test Holography and AdS/CFT beyond string theory.

- The $O(N)$ vector model: $O(1/N)$ anomalous dimensions of the $O(N)$-singlet higher-spin currents are [W. Rühl - private communication]:

$$J_{(s)} \sim \phi^a \partial_{\{\mu_1} ... \partial_{\mu_s\}} \phi^a \,, \quad a = 1, 2, .., N$$

$$\Delta_s = s + 1 + 4\gamma_\phi \frac{s-2}{2s-1} + \cdots \,, \; s = 2k \,, \; k = 1, 2, .. \,, \quad \gamma_\phi \sim O(1/N)$$

$$s \to \infty \,, \quad \Delta_s - s \approx 2\left(\frac{1}{2} + \gamma_\phi\right)$$

- All determined by γ_ϕ: $\rightarrow$ contrast with $\mathcal{N} = 4$ SYM. No $\ln s$ growth that would signal the presence of gauge fields. Hard to arise from rotating strings in AdS. However, fast rotating ultrashort strings (particles?) in an AdS_4 black hole yield the T-independent result [Armoni, Barbon and A.C.P. (02)]

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The conjectures

The $O(N)$ singlet sector of the bosonic vector model is dual to the simplest Vasiliev theory of AdS_4 [Klebanov and Polyakov (02)].
An analogous conjecture for the $O(N)$ fermionic vector model - slightly complicated due to parity issues - [Leigh and A. C. P. , Sezgin and Sundell (03)]

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2. Compare the bulk and boundary OPE studies: understand how the bulk "emerges" from the boundary and vice versa.

- Diagrammatic $1/N$ "skeleton" expansion elucidates the OPE structure of the boundary theories and gives interesting results.
  However, extension of such techniques to the bulk is rather mysterious.

Further questions (still impenetrable in $d \geq 3$)

3. Thermalisation of 3d vector models is well understood. The bosonic model realises the Mermin-Wagner theorem: $O(N)$ symmetry does not break for $T > 0$. Parity does break for $T > 0$.
How is this realised in terms of HSs?

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In this talk:

Vector models exhibit global and discrete symmetry breaking. The bosonic model $O(N) \to O(N-1)$. The fermionic model parity breaking.
If there is holography without strings and branes, what is the bulk counterpart of the global $O(N)$ boundary symmetry and its breaking pattern?

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- Using a bulk singleton deformation I will reproduce the boundary gap equation that describes the $O(N) \to O(N-1)$ breaking.
  Using a further boundary deformation I will reproduce the known $O(1/N)$ anomalous dimension of the elementary scalars in the boundary. The latter result raises the issue whether $O(N)$ symmetry breaking is related to higher-spin symmetry breaking.
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Outline

- N-elementary (Euclidean) scalar fields $\phi^a(x)$ subject to a constaint

$$L = \frac{1}{2}\int d^3x\,\partial_\mu\phi^a\partial_\mu\phi^a\,,\phi^a\phi^a = \frac{N}{g}\,,a = 1,2,..N\,.$$

$g \to 0$ is the free field theory limit which lies in the UV.

- Introduce a Lagrange multiplier ρ and integrate the ϕ's to obtain

$$Z = \int(\mathcal{D}\rho)e^{-NS_{eff}(\rho)}\,,\; S_{eff}(\rho) = \frac{1}{2}\mathrm{Tr}\ln(-\partial^2+\rho) - \int d^3x\frac{\rho}{2g}$$

The saddle point at large-N, with constant $\rho_0 = m^2$, yields the gap equation

$$\frac{\partial S_{eff}(\rho)}{\partial\rho}\Big|_{\rho_0} = 0 \Rightarrow \frac{1}{g} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+\rho_0}$$

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- The effective action $\mathcal{S}^N_{eff}(\sigma,\rho_0)$ for the real fluctuations σ is

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\begin{aligned}
S_{eff}(\rho) &= \frac{1}{2}\mathrm{Tr}\ln(-\partial^2+\rho_0)-\frac{\rho_0}{2g}(Vol)_3+\frac{1}{N}\mathcal{S}^N_{eff}(\sigma,\rho_0) \\
\mathcal{S}^N_{eff}(\sigma,\rho_0) &= \frac{1}{2}\int\sigma(x)\Delta_3(x,y;\rho_0)\sigma(y) \\
&\quad +\frac{1}{3!\sqrt{N}}\int\sigma(x)\sigma(y)\sigma(z)P_3(x,y,z;\rho_0)+..
\end{aligned}
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The kernels $\Delta_2(x,y;\rho_0)\,,P_3(x,y,z;\rho_0)$ .. are constructed using propagators of the ϕ's only.

- The generating functional $W[\eta]$ for connected correlation functions of σ is

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e^{W[\eta]}\equiv\int(\mathcal{D}\sigma)e^{-\mathcal{S}^N_{eff}(\sigma,\rho_0)+\int\eta\sigma}
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- Using a UV cutoff Λ the gap equation becomes

$$\left(\frac{1}{g_*}-\frac{1}{g}\right)=\frac{\sqrt{|\rho_0|}}{4\pi}+O(\rho_0/\Lambda)\,,\quad \frac{1}{g_*}=\frac{\Lambda}{2\pi^2}$$

- For $g>g_*$, we are in the massive phase with $m=\sqrt{|\rho_0|}\neq 0$.
  For $g=g_*$ there is no mass scale left and we describe the critical $O(N)$ vector model.
  For $g<g_*$, $\rho_0=0$ but an arbitrary mass scale remains - the subtraction point of renormalisation - and we enter a symmetry broken phase.

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A clearer way to see the $O(N) \to O(N-1)$ symmetry breaking pattern is to separate out the N'th component of ϕ^a's, which we denote as ϕ.

- Integrating over the remaining $N-1$ elementary scalars we obtain

$$Z = \int [\mathcal{D}\phi][\mathcal{D}\rho]\, e^{-(N-1)S_{eff}(\rho,\phi)}$$

The effective action is now defined as

$$\begin{aligned} S_{eff}(\phi,\rho) &= S_{eff}^{N-1}(\rho) + \frac{1}{2(N-1)}\int d^3x\, \phi(-\partial^2+\rho)\phi \\ S_{eff}^{N-1}(\rho) &= \frac{1}{2}\mathrm{Tr}\ln(-\partial^2+\rho) - \frac{N}{(N-1)}\int d^3x\, \frac{\rho}{2g} \end{aligned}$$

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The large-N expansion is now performed as

$$\rho(x) = \rho_0 + \frac{1}{\sqrt{N-1}}\sigma(x)\,,\ \phi(x) = \phi_0 + \varphi(x)\,.$$

with ρ_0, ϕ_0 determined by the

modified gap equations

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$O(N)\to O(N-1)$ symmetry breaking pattern

- The effective action for the $O(N)$ model $\leftarrow$ the effective action of the $O(N-1)$ model by integrating-in φ with a $\int \sigma\varphi^2$ and a $\int \varphi\sigma$ interaction.
- At the critical point $\rho_0=\phi_0=0$, one integrates-in a massless elementary scalar $\varphi(x)$ with marginal interaction. This shifts $N-1\to N$.

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- The effective action for the $O(N)$ model $\leftarrow$ the effective action of the $O(N-1)$ model by integrating-in φ with a $\int\sigma\varphi^2$ and a $\int\varphi\sigma$ interaction.
- At the critical point $\rho_0=\phi_0=0$, one integrates-in a massless elementary scalar $\varphi(x)$ with marginal interaction. This shifts $N-1\to N$.

$O(N)$ vector models: review

The modified gap equation is written

$$\frac{\phi_0^2}{N-1} = \left(\frac{N}{N-1}\frac{1}{g} - \frac{1}{g_*}\right) + \frac{|m|}{4\pi} + \cdots$$

ϕ_0 and $|m|$ cannot be simultaneously nonzero and $|m| < \Lambda$.

- When $g < Ng_*/(N-1)$, $|m| = 0$ but $\phi_0 \neq 0 \Rightarrow O(N)$ is broken to $O(N-1)$. The $N-1$ Goldstone bosons are the massless elementary scalars that were integrated out.
- When the coupling is tuned to
$$g = \frac{N}{N-1}g_* > g_*$$
we have $\phi_0 = m = 0$ and we arrive at the critical $O(N)$ vector model.
- As the coupling increases to $g > Ng_*/(N-1)$, we have $\phi_0 = 0$, but then we enter the $O(N)$ symmetric phase with
$$m = \frac{2\Lambda}{\pi}\left(1 - \frac{N}{N-1}\frac{g_*}{g}\right).$$

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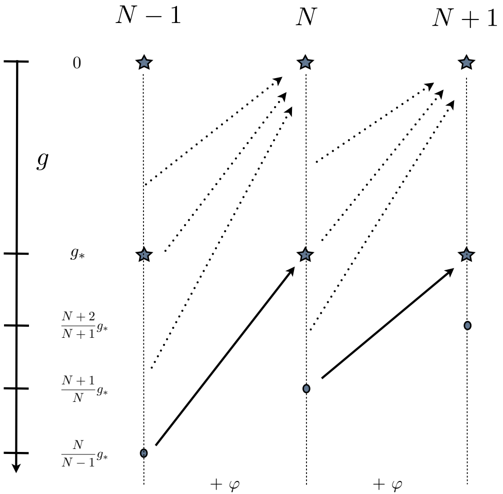

Figure : The phase diagram of the vector models. Stars denote the CFTs. The solid arrows denote marginal deformations towards the IR fixed point after the absorption of an elementary scalar φ. The dotted arrows denote irrelevant double-trace deformations leading to the UV fixed point of the symmetry enhanced theory.

- When $g < Ng_*/(N-1)$ we assign the difference

$$\frac{N}{N-1}\frac{1}{g} - \frac{1}{g_*} = \frac{\phi_0^2}{N-1} \neq 0$$

to an expectation value of ϕ_0. Then the linear interaction term $\phi_0 \int \sigma\varphi$ is nontrivial and we can shift the scalar fluctuation as

$$\varphi = \hat{\varphi} + \frac{\phi_0}{\sqrt{N-1}}\frac{1}{-\partial^2}\sigma\,,$$

$$Z \sim \int e^{-\left[\mathcal{S}_{eff}^{N-1}(\sigma,0)+\frac{1}{2}\int\hat{\varphi}D_0\hat{\varphi}+\frac{1}{2\sqrt{N-1}}\int\sigma\hat{\varphi}^2-\frac{\phi_0^2}{2(N-1)}\int\frac{1}{-\partial^2}\sigma^2+..\right]}.$$

- The last term in the exponent is a nonlocal version of the irrelevant double-trace deformation $\int\sigma^2$ which drives the theory in the UV where we expect to find the free $O(N)$ model.

- A richer picture arises in $d=5$ which can be compared to the recent results of [Giombi, Klebanov et. al. (14)].

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- The Euclidean action for the three-dimensional Majorana fermions

$$S = -\int d^3x \left[\frac{1}{2}\bar{\psi}^i \,\partial\!\!\!/\, \psi^i + \frac{G}{4N}(\bar{\psi}^i\psi^i)^2\right] , \bar{\psi} = \psi^T \sigma_2 \,, i = 1, 2, ..N.$$

- G has dimensions of inverse mass, hence the $G \to 0$ free theory lies in the IR.
- An expectation value for $\sigma \sim G\bar{\psi}^i\psi^i$ signifies parity breaking.
- With respect to a critical coupling

$$\frac{1}{G_*} = \frac{\Lambda}{\pi^2} \,,$$

we have 1) for $G < G_*$, $\sigma = 0$ and parity is unbroken, 2) for $G = G_*$ we are at the fermionic *critical* $O(N)$ fixed point that lies in the UV and 3) for $G > G_*$, $\sigma \neq 0$ and hence parity is broken.

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- The systematic $1/N$ expansion leads to the calculation of anomalous dimensions [e.g. A. Vasiliev et. al. (81-81), Gracey (91-92), Rühl et. al. (92-93), T. P. (94-96)]. From conformal invariance we have

$$\langle \phi^a(x)\phi^b(0)\rangle = \frac{C_\phi}{x^{2\Delta_\phi}}\delta^{ab}\,, \quad \langle \sigma(x)\sigma(0)\rangle = \frac{C_\sigma}{x^{2\Delta_\sigma}}$$

- We fix $d=3$ and define three critical indices γ_ϕ, κ and z of order $O(1/N)$ as

$$\Delta_\phi = \frac{1}{2}+\gamma_\phi\,, \quad \Delta_\sigma = 2-2\gamma_\phi-2\kappa\,, \quad C_\phi^2 C_\sigma = \frac{1}{\pi^4}+z$$

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- For the calculations of κ and ζ one needs to consider the 2-pt function of σ and also the renormalisation of the vertex $\sigma\phi^2$. The most updated results are already a few decades old.

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Outline

1. Motivations
2. $O(N)$ vector models: review
   - $O(N) \to O(N-1)$ symmetry breaking in the bosonic model
   - The fermionic $O(N)$ vector model: lightning review
   - Anomalous dimensions
3. $O(N)$/HS holography
   - The gap equations from holography
   - The singleton deformation of higher-spin theory and boundary symmetry breaking
4. Aspects of the OPE in $O(N)$ vector models
   - The conformal partial waves: free field theory
   - The skeleton graphs
5. Summary and outlook

- Practically, $O(N)$/HS correspondence proceeds by considering the bulk action

$$I_{HS} = \sum_{s=0,2,4,..}^{\infty} \int d^4x\sqrt{-g}\frac{1}{2}\Phi^{(s)}\left[\Box_s - \frac{1}{L^2}(s^2-2s-2)\right]\Phi^{(s)} + O(\frac{1}{\sqrt{N}})$$

$\Phi^{(s)}$ are symmetrized and double-traceless rank-s tensors, $\Box_s$ are generalized Pauli-Fierz operators on the fixed AdS_4 background metric $g_{\mu\nu}$. There is also a "mass" term necessary to maintain HS gauge invariance.

- The quadratic part of I_{HS} yields the two-point functions of all free higher-spin currents normalized to $O(1)$. The free boundary theory is obtained by the alternative quantisation (AQ) of the conformally coupled scalar $\Phi^{(0)}$. The standard quantisation (SQ) gives the non-trivial fixed point.

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- AdS/CFT yields the renormalized boundary generating functional $W_{en}[J]$, wherefrom we get the effective action $\Gamma[\langle\mathcal{O}\rangle]$ by a Legenrde transform.
- A Lagrangian deformation of the *boundary field theory action* by a functional $f(\mathcal{O})$ corresponds - at least at large-N - to a simple deformation of the effective action

$$\Gamma_f[\sigma] = \Gamma_0[\sigma] + f(\sigma)\,, \quad \sigma = \langle\mathcal{O}\rangle\,.$$

Given such a deformation the boundary gap equation reads

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- We extend the bulk theory by second scalar with $m^2L^2 = -2$

$$I_{extHS} = I_{HS} + \int d^4x\sqrt{-g}\frac{1}{2}\Sigma\left[\Box + \frac{2}{L^2}\right]\Sigma\,.$$

- We take $\Phi^{(0)} \equiv \Phi$ in AQ, and Σ in SQ. Asymptotically, we have

$$\Phi \sim \alpha z + \beta z^2\,,\;\; \Sigma \sim \eta z + \sigma z^2$$

Φ yields a $\Delta = 1$ operator with vev α, Σ yields a $\Delta = 2$ operator with vev σ.

- We assume that these fields do not mix in the bulk. This means that the regularity conditions of the bulk equations yield $\alpha = \alpha(\beta)$ and $\sigma = \sigma(\eta)$, and determine the boundary generating functional as

$$I_{extHS} \to W[\beta,\eta] = \int \alpha(\beta)\beta - \int \sigma(\eta)\eta\,.$$

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$$f(\alpha,\sigma)=\int\left(\alpha\sigma+V(\sigma)-\frac{1}{3}\lambda(\alpha-h)^3\right)\,,\;\;V(\sigma)=-\frac{\lambda'}{g}\sigma\,.$$

with λ and λ' dimensionless and h is a parameter with dimensions of mass.

- We then obtain

$$\Gamma[\alpha,\sigma]=\int\left(\frac{1}{2}\alpha K_1\alpha-\frac{1}{2}\sigma K_1^{-1}\sigma+\sigma(\alpha-\frac{\lambda'}{g})-\frac{1}{3}\lambda(\alpha-h)^3\right)$$

where K_1 is an appropriate kernel.

- For constant α and σ, we obtain the gap equations

$$\alpha = \frac{\lambda'}{g}$$
$$\sigma = \lambda(\alpha-h)^2$$

- The first equation corresponds to the model's constraint and gives $\lambda'=\sqrt{N}$.

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where K_1 is an appropriate kernel.

- For constant α and σ, we obtain the gap equations

$$\begin{aligned}\alpha &= \frac{\lambda'}{g}\\ \sigma &= \lambda(\alpha-h)^2\end{aligned}$$

- The first equation corresponds to the model's constraint and gives $\lambda'=\sqrt{N}$.

- To get the boundary gap equation we introduce boundary terms that couple the two fields together i.e. a Lagrangian deformation of the form

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$$\lambda = \frac{16\pi^2}{N}\,, \quad h = \frac{\sqrt{N}}{g_*} \;\Rightarrow\; \frac{\sqrt{N}}{g} = \frac{\sqrt{N}}{g_*} \pm \frac{\sqrt{N}}{4\pi}\sqrt{\sigma}$$

  Keeping the minus sign, this coincides with the vector model's gap equation.
- The free UV fixed point is reached taking $g, \lambda \to 0$ and the cutoff to infinity, whereby σ decouples and only α (the $\Delta = 1$ operator) survives.
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Outline

- Next, we deform the higher-spin action by a singleton field S as

$$I_{dHS} = I_{extHS} + \int d^4x\sqrt{-g}\frac{1}{2}S\left[\Box + \frac{5}{4L^2}\right]S\,,$$

with asymptotic behaviour

$$S \sim \xi z^{1/2} + \phi z^{5/2}.$$

- For S the only possible unitary quantization is to use AQ [E.G. Ohl and Uhlemann (12)] giving a free boundary operator of $\Delta = 1/2$ that decouples from the rest of the CFT.
- To force S interact with the other HS's we deform the bulk theory by

$$f_d(\alpha,\sigma,\phi) = \int\left[\alpha\sigma - \tilde{V}(\sigma) - \lambda\frac{1}{3}(\alpha - h)^3 + \tilde{\lambda}\sigma\phi^2\right]\,,\ \tilde{V}(\sigma) = \frac{\tilde{\lambda}'}{g}\sigma\,,$$

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$$\alpha + \tilde{\lambda}\phi^2 = \frac{N+1}{\sqrt{N}}\frac{1}{g}$$
$$\sigma = \frac{16\pi^2}{N}\left(\alpha - \frac{\sqrt{N}}{g_*}\right)^2$$
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- The third equation is familiar from the σ-model: there are two phases, one in which $\phi = 0$ (massive phase) and the other in which $\sigma = 0$ (broken phase).
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$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\rangle_{def} &= \langle\phi(x_1)\phi(x_2)\rangle_0 \\ &+\frac{1}{2N}\int\langle\phi(x_1)\phi(x_2)\sigma(x)\phi^2(x)\sigma(y)\phi^2(y)\rangle_0 + \cdots\end{aligned}$$

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- This is despite the fact that the deformation may be regarded as a marginal deformation of the IR $O(N)$ fixed point in the presence of an additional scalar ϕ.
- Generally, the graphical expansion for ϕ and σ generated by the deformation above is the same as the graphical expansion for ϕ^a and σ generated by the boundary field theory $\rightarrow$ hence yields the same anomalous dimensions.

The moral

At least to leading order in $1/N$, the bulk HS theory is deformed by throwing in singletons $\rightarrow$ these simply are pushed in the boundary, hence we have $N \rightarrow N+1$. We need, however, some nontrivial bulk interaction with the HSs in order to induce the necessary boundary terms to glue the extra field to the rest.

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Outline

We will focus on the following 4-pt functions

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$$\begin{aligned}\langle\phi^a(x_1)\phi^b(x_2)\phi^c(x_3)\phi^d(x_4)\rangle &\equiv \Phi^{abcd}(x_1,x_2,x_3,x_4)\\ &= \delta^{ab}\delta^{cd}\Phi_S(x_1,x_2,x_3,x_4)\\ &+ \mathcal{E}^{[ab,cd]}\Phi_A(x_1,x_2,x_3,x_4)\\ &+ \mathcal{T}^{(ab,cd)}\Phi_{st}(x_1,x_2,x_3,x_4)\end{aligned}$$
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$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \equiv \Phi_{\mathcal{O}}(x_1,x_2,x_3,x_4)$$

- These will be functions of v and Y related to the usual conformal ratios as
$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,\quad v=\frac{x_{12}^2x_{34}^2}{x_{14}^2x_{23}^2}\,,\ Y=1-\frac{v}{u}$$
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We would need the following OPEs

- $$\begin{aligned}\phi^a(x_1)\phi^b(x_2) &= \sum_{\Delta_s}\frac{\delta^{ab}}{(x_{12}^2)^{\Delta_\phi-\frac{1}{2}\Delta_s}}\left[1+\frac{g_{\phi\phi\mathcal{O}_s}}{C_{\mathcal{O}_s}}[\mathcal{O}_s(x_2)]\right],\\ &+\sum_{\Delta_s'}\frac{\mathcal{E}^{[ab,cd]}}{(x_{12}^2)^{\Delta_\phi-\frac{1}{2}\Delta_s'}}\frac{g_{\phi\phi\mathcal{O}_s^{[cd]}}}{C_{\mathcal{O}_s^{[cd]}}}[\mathcal{O}_s^{[cd]}(x_2)]\\ &+\sum_{\Delta_s''}\frac{\mathcal{T}^{(ab,cd)}}{(x_{12}^2)^{\Delta_\phi-\frac{1}{2}\Delta_s''}}\frac{g_{\phi\phi\mathcal{O}_s^{(cd)}}}{C_{\mathcal{O}_s^{(cd)}}}[\mathcal{O}_s^{(cd)}(x_2)],\quad x_{12}=x_1-x_2\end{aligned}$$
- The $[\mathcal{O}_s]$'s represent the full contributions (i.e. including descendants). The $C_{\mathcal{O}_s}$'s are the 2-pt function normalisation constants and the $g_{\phi\phi\mathcal{O}_s}$'s are the corresponding 3-pt function couplings. We normalized to one the 2-pt function of the ϕ^a's.
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We would also need

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  $$\mathcal{O}(x_1)\mathcal{O}(x_2) = \frac{1}{x_{12}^{2\Delta}}\left[C_{\mathcal{O}} + \frac{g_{\mathcal{O}}}{C_{\mathcal{O}}}\frac{[\mathcal{O}(x_2)]}{(x_{12}^2)^{-\frac{\Delta}{2}}} + \frac{g_{\mathcal{O}\mathcal{O}T}}{C_T}\frac{C_{\mu\nu}[T_{\mu\nu}(x_2)]}{(x_{12}^2)^{-\frac{d}{2}}} + ..\right]$$

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- Inserting the OPEs into the 4-pt function we obtain formulae like

$$\Phi(v,Y)=\sum_{\Delta_s}\frac{1}{(x_{12}^2x_{34}^2)^{\Delta_\phi}}\frac{g^2_{\phi\phi\mathcal{O}_s}}{C_{\mathcal{O}_s}}\mathcal{H}_{\Delta_s}(v,Y)$$

with $\mathcal{H}_{\Delta_s}(v,Y)$ the conformal partial wave (CPW) representing the contribution of the operator $\mathcal{O}_s$ and all its descendants into the 4-pt function.

- The CPW's are given is terms of a double series of the form

$$\mathcal{H}_{\Delta_s}(v,Y)=v^{\frac{1}{2}(\Delta_s-s)}\sum_{n,m=0}^{\infty}A_{nm}v^nY^m$$

- This becomes particularly interesting when the operators are conserved spin-s currents whose dimensions are
$\Delta_s=d-2+s$. In this case we find that the leading singular term has the form

$$\mathcal{H}_{\Delta_s}(v,Y)=A_{0s}v^{\frac{1}{2}d-1}Y^s[1+O(v)]\cdots$$

The above behaviour can be used to detect the presence of higher-spin conserved currents in a 4-pt function.

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with $\mathcal{H}_{\Delta_s}(v,Y)$ the conformal partial wave (CPW) representing the contribution of the operator $\mathcal{O}_s$ and all its descendants into the 4-pt function.

- The CPW's are given is terms of a double series of the form

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- This becomes particularly interesting when the operators are conserved spin-s currents whose dimensions are
$\Delta_s=d-2+s$. In this case we find that the leading singular term has the form

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- Assuming the presence of one only scalar operator $\mathcal{O}$ with dimension $\Delta < d$ in the OPE, we have for the first few most singular terms

$$\Phi_S(v,Y) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}}\left[1 + \frac{g^2_{\phi\phi\mathcal{O}}}{C_{\mathcal{O}}} v^{\frac{\Delta}{2}}\,{}_2F_1(\frac{\Delta}{2},\frac{\Delta}{2};\Delta;Y) + \frac{g^2_{\phi\phi T}}{4C_T} v^{\frac{d}{2}-1} Y^2 + ..\right.$$

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- It is the term in the second line on the r.h.s. that gives rise to the usual contribution of the tower of HS currents. This term comes from the box and papillon graphs that are constructed using the ϕ propagator.
- The disconnected graphs give rise to a scalar operator with dimension $\Delta = 4\Delta_\phi$, which is proportional to $(\phi^2)^2$, and a tower of higher-twist currents.

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Outline

1. Motivations
2. $O(N)$ vector models: review
   - $O(N) \to O(N-1)$ symmetry breaking in the bosonic model
   - The fermionic $O(N)$ vector model: lightning review
   - Anomalous dimensions
3. $O(N)$/HS holography
   - The gap equations from holography
   - The singleton deformation of higher-spin theory and boundary symmetry breaking
4. Aspects of the OPE in $O(N)$ vector models
   - The conformal partial waves: free field theory
   - The skeleton graphs
5. Summary and outlook

- To deform the free theory we use an expansion in"skeleton graphs" built using just three ingredients: the (unit normalised) 2-pt functions of the operators $\phi^a(x)$, $O(x)$ (with dimension $\tilde{\Delta}$) and the 3-pt function

$$\langle\phi^a(x_1)\phi^b(x_2)O(x_3)\rangle = g_*\frac{1}{(x_{12}^2)^{\Delta_\phi-\frac{\tilde{\Delta}}{2}}(x_{13}^2x_{24}^2)^{\frac{\tilde{\Delta}}{2}}}\delta^{ab}\,.$$

- The parameters $\tilde{\Delta}$ and g_*, as well as all other parameters (i.e. coupling and scaling dimensions) will be determined by studying the consistency of the skeleton expansion with the OPE.
- We also need to "amputate" using the inverse 2-pt functions

$$\begin{aligned}\delta^{ab}\Gamma(x_1,x_2,x) &\equiv \int d^dx_3\langle\phi^a(x_1)\phi^b(x_2)O(x_3)\rangle\langle O(x_3)O(x)\rangle^{-1}\\ &= g_*\frac{f(\Delta_\phi,\tilde{\Delta},d)}{(x_{12}^2)^{\Delta_\phi-\frac{\tilde{\Delta}}{2}}(x_{13}^2x_{24}^2)^{\frac{d-\tilde{\Delta}}{2}}}\delta^{ab}\end{aligned}$$

with x the internal point of a graph, and $f(\Delta_\phi,\tilde{\Delta},d)$ are ratio's of Γ-functions.

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with x the internal point of a graph, and $f(\Delta_\phi,\tilde{\Delta},d)$ are ratio's of Γ-functions.

- This construction an important simplification from the usual $1/N$ diagrammatic expansion of the vector model in that the full vertices and 2-pt functions are used.
- The skeleton expansion for Φ_S will involve tree-exchange graphs with a single $O(x)$ internal line, ladder graphs with internal $O(x)$ and $\phi^a(x)$ lines etc...
- The leading exchange graph in the direct channel $x_{12}^2, x_{34}^2 \Rightarrow 0$ yields the remarkable formula

$$g_*^2 F(\Delta_\phi, \tilde{\Delta}, d) \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \left[\mathcal{H}_{\tilde{\Delta}}(v, Y) + C(\tilde{\Delta}) \mathcal{H}_{d-\tilde{\Delta}}(v, Y)\right]$$

  This is remarkable since it corresponds to the CPWs of *both* the operator $O(x)$ but also its shadow operator with dimension $d - \tilde{\Delta}$.
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- A given skeleton graph with $2n$ vertices has the shadow symmetry property

$$G(u,Y;\Delta)=[C(d-\Delta)]^nG(u,Y;d-\Delta)$$

- It is believed that the above property is related to the analyticity of the graph under crossing. Then, the full crossing symmetric 4-pt function can be obtained by adding to the direct channel the crossed terms.
- The crossed, box (and possibly all higher order) evaluate to the generic form

$$G(x_1,x_3,x_2,x_4)=\frac{1}{(x_{12}^2x_{34}^2)^{\Delta_\phi}}v^{\Delta_\phi}\sum_{n,m=0}^{\infty}\frac{v^nY^m}{n!m!}[-a_{nm}\ln v+b_{nm}]$$

- A given skeleton graph with $2n$ vertices has the shadow symmetry property

$$G(u,Y;\Delta) = [C(d-\Delta)]^n G(u,Y;d-\Delta)$$

- It is believed that the above property is related to the analyticity of the graph under crossing. Then, the full crossing symmetric 4-pt function can be obtained by adding to the direct channel the crossed terms.
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Examples:

- Modifying the free result for $\Phi_S(v,Y)$ by the $\mathcal{O}$-exchange graph would imply the presence of three scalar operators with dimensions $< d$! To avoid that, we choose to cancel the free operator with $\Delta = 2\Delta_\phi$ with one of the two terms in the exchange graph. In fact, $C(\tilde{\Delta}) < 0$ for $2 < d < 4$ and $\tilde{\Delta} < d$. This way we fix $g_*^2 \sim O(1/N)$ and also $d - \tilde{\Delta} = 2\Delta_\phi \Rightarrow \tilde{\Delta} = 2$.
- Modifying $\Phi_A(v,Y)$ we find

$$\begin{aligned}\Phi_A(v,Y) &= \Delta_\phi v^{\Delta_\phi} Y[1+..] + g_*^2 v^{\Delta_\phi} Y[-A_{00}\ln v + B_{00} + ..] \\ &= \frac{g_J^2}{C_J} v^{\frac{d}{2}-1} Y[1+..]\end{aligned}$$

- We need to kill the $\ln v$ terms in the first line, which is done if we assume that

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\gamma_\phi\,, \;\Rightarrow \gamma_\phi = \frac{2\Gamma(d-2)}{\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2}-2)}$$

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- To $\Phi_{\phi\mathcal{O}}(v, Y)$ we need exchange graphs involving the elementary scalar ϕ^a. These give *both* the CPW of ϕ^a but also of its shadow with $\Delta = 5/2$. One would also think that both contributions are $O(1/N)$. Quite remarkably, the latter contribution is singular, needs to be regularised and eventually gives rise to a $O(1)$ term in the 4-pt function. This is necessary to correctly match with the OPE and make sure that the $\Delta = 5/2$ operator does not appear!

The issue with AdS graphs:

- A scalar field exchange graph in AdS in the direct channel gives

$$\frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}}\left[\mathcal{H}_\Delta(v,Y)+\sum_{n,m=0}^{\infty}\frac{v^n Y^m}{n!m!}[-a_{nm}\ln v+b_{nm}]\right]$$

namely, the shadow contribution is missing. Nevertheless, one can show that such a graph is still analytic under a crossing transformation. This is due to some highly non-trivial Kummer-like relationships for ${}_3F_2$ functions!

- A complete holographic description of $O(N)$ vector models should account for their rich vacuum structure i.e. the $O(N) \to O(N-1)$ symmetry breaking pattern in the bosonic case. We have argued that this can be done if the bulk theory absorbs singletons field by shifting its parameter $N \to N+1$. This is the bulk dual of the global symmetry breaking/enhancement mechanism in the boundary. [SEE ALSO GELFOND AND VASILIEV (13)]
- The boundary singleton interaction generates the same $1/N$ graphical expansion for the elementary scalar and "spin-zero current" as in the standard field theoretic treatment of the $O(N)$ model. Hence, the singleton deformation breaks higher-spin symmetry and yields the well-known anomalous dimensions for the elementary and "spin-zero" scalars of the $O(N)$ model, at least to leading order in $1/N$.

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- Is it important to understand better the boundary marginal coupling of the singleton to higher-spin currents. For example, given the singleton field ϕ, one may consider boundary couplings of the form

$$S_{HS} \sim \lambda' \int t^{\mu_1 \cdots \mu_s} \phi \partial_{\mu_1} ... \partial_{\mu_s} \phi \,,$$

  where $t^{\mu_1 \cdot\cdot \mu_s}$ is the *leading* coefficient in the asymptotic behaviour of a bulk spin-s gauge field $\rightarrow$ *higher-spin dressing of the $O(N)$ model*.
- For $s \geq 2$ there are more than one possible terms. Generally, this has no effect on the vacuum structure, if that is determined by space-time constant configurations.
- It is expected that such couplings would lead to a graphical expansion for the 2-pt functions of the boundary higher-spin currents which would enable one to calculate their $1/N$ anomalous dimensions. Reproducing the result would then be a crucial test for our proposal.

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- Our results can also be applied to the holographic description of three-dimensional fermionic and supersymmetric models with higher-spin duals. Notice that such models describe parity symmetry breaking, and it would be interesting to understand the bulk counterpart of it.
- The singleton deformation could also play an important role in the study of possible black-hole solutions for higher-spin theory on AdS_4. For example, since a continuous symmetry cannot be broken at finite temperature in 2+1 dimensions, we expect that bosonic singleton absorption would not be possible for higher-spin theories in black-hole backgrounds, while fermionic singleton absorption would be allowed.

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- It is a formidable task to compare boundary skeleton expansion and the bulk Witten graphs, although they arguably describe the same theory.
  Hint:
  Boundary skeleton graphs do not have HS exchanges: I can built a HS theory using a single scalar vertex. But they have shadow-symmetry properties, and this is the part "speaking" to HS coming from the free theory.
  Bulk graphs do not have shadow-symmetry: but to built the theory one would need all HS exchanges. Namely, they include the "free part" that was actually "cancelled" by the shadow term in the skeleton graphs.