Black Hole Infromation Paradox Rises

Krommydas Dimitrios

Under the supervision of: Anastasios Petkou

Abstract

This undergraduate thesis purpose will be to understand the *Black Hole Information Paradox* , formulate it as a *"Theorem"* and discuss it's inevitable conclusions. To do that, we will follow some basic steps:

1) Define a set of "Niceness Conditions" (which we will demand the following steps obey) such, that it will give us local hamiltonian evolution (low curveture, avoidance of Quantum Gravity effects and non-local effects)

2) Define a *"Traditional Black Hole"* (Schwarchild metric,solar mass, low curveture at it's horizon $R \sim \frac{1}{M^2}$, infromation free horizon)

3) Slice the Black Hole spacetime (1-3D slicing). The slices will be made in such way, that they will be able to cover our whole spacetime (and $r = 0$). Their time evolution will give rise to real entangled pairs.

4) Study the overal state as the Black Hole evaporates, connect the notion of information to *"Entanglement Enropy"* and see how the paradox rises.

Contents

1 Introduction **4**

2 Quantum Field Theory **5**
2.1 Simple Harmonic Oscillator 5
2.2 Creation and Annihilation operators in the S.H.O. case 6
2.3 Heisenberg's Picture 9
2.4 Quantum Field Theory, In Flat Spacetime 10
2.5 Quantum Field Theory In Curved Spacetime 13

3 Quantum Statistical Mechanics **19**
3.1 Pure States: 19
3.1.1 *Pure state* 19
3.1.2 *Inner product* 20
3.1.3 *Operators* 20
3.1.4 *Trace* 21
3.2 Mixed States 21
3.2.1 *Mixed state* 21
3.3 Entropy 22
3.4 Criteria 23
3.5 Systems consisting of Subsystems 23
3.5.1 Reduced Density Matrix 23
3.5.2 Entangled State 24

4 A Quantum Paradox **24**
4.1 Solar System Limit 25
4.2 Niceness Conditions N 25
4.3 Traditional Black Hole 26
4.4 Slicing the T.B.H. Geometry 26
4.5 Evolution to later slices 28
4.6 Scales and Limits 28
4.7 Changes between the Slices 28
4.8 Pair Production 30
4.8.1 Leading Order 32
4.8.2 Loss of unitarity 34
4.8.3 Deformation of the leading order state (small corrections:definition and effect): 35

5 The Hawking Theorem **40**

1 Introduction

First let us introduce a simplistic view on what the information paradox is.

It is a well-known phenomenon that a black hole, if it is small enough and we use the appropriate timescale, will radiate until it evaporates (*Hawking radiation*). But how can a black hole radiate? What kind of radiation is that, and what information does it carry?

To answer these questions, first we will need to present an (also simplistic) explanation on what hawking radiation is. We know that, in general relativity, there are no outgoing particles capable of escaping the horizon of a black hole, due to the high strength of the gravitational field. But if one was to consider some phenomena that exist in the world of quantum mechanics, escaping the horizon might become possible. For instance, we can assume that a pair production takes place "just" inside the horizon. The first particle will follow an ingoing trajectory until it reaches the origin singularity and the second, the outgoing one, will "*tunnel*" it's way (*quantum tunneling*) through the horizon and escape to infinity.

Now let us assume the above black hole, to be a non-rotating one (take Schwarchild solution for simplicity). The only information we have about such hole, is it's mass (assuming we know the mass of the star which collapsed to our black hole). This means that to this black hole, even though it seems intuitively wrong, we can assign just one state ($|\psi_M\rangle$). Hawking proved, the calculations will be presented later, that the radiation quanta from such a black hole will all be approximately in the same state. And that gives rise to our paradox. *If there exist two black holes, (capable of radiation) with the exact same mass, no matter what they are made of, their radiation will be the same.* Meaning, if someone were to collect all the radiation quanta emitted by those holes until their evaporation, he would not be able to separate them. In that sense we can say that the initial information about the states of matter that fell in the black hole, has vanished! This is something that contradicts with one of the fundamental postulates of quantum mechanics, information **has** to be conserved (to be more precise "*complete information about a system is encoded in its wave function up to when the wave function collapses*" is the correct formulation, but since no such collapse should take place the above is a valid conclusion).

Now let us see how this paradox arose, this time in a more precise view. *Hawking* wanted to calculate quantum effects on a curved space near a black hole. Knowing that the attempts to obtain a *Quantum gravity* theory through duality led to a dead end (i.e. a non-renormalizable theory), he thought of using a semiclassical approach. *Semiclassical gravity* is the approximation to the theory of *quantum gravity* in which one treats matter fields as being quantum and the gravitational field as being classical. Written in *Einstein equations* :

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \left\langle \hat{T}_{\mu\nu} \right\rangle_{\psi}$$

where ψ indicates the quantum state of the matter fields.

In semiclassical gravity, matter is represented by quantum matter fields that propagate according to the *theory of quantum fields in curved spacetime.* But, unless the background metric tensor has a *global timelike Killing vector*, there is no way to define a vacuum or ground state canonically and this is something we need. The concept of a vacuum is not invariant under *diffeomorphisms* (given two manifolds M and N, a differentiable map $f : M \to N$ is called a diffeomorphism if it is a bijection and its inverse $f^{-1} : M \to N$ is differentiable as well). This is because a mode decomposition of a field into positive and negative frequency modes is not invariant under diffeomorphisms. If t´(t) is a diffeomorphism, in general, the Fourier transform of $e^{[ikt'(t)]}$ will contain negative frequencies even if $k > 0$. Creation operators correspond to positive frequencies, while annihilation operators correspond to negative frequencies. This is why a state which looks like a vacuum to one observer cannot look like a vacuum state to another observer; it could even appear as a heat bath under suitable hypotheses!

Quantum gravitational effects become apparent only near *Plank scale* (i.e. small distances, high energies) or high curvature (which in gravity is essentially the same as high energy). So if we want to avoid such effects (still not entirely but for them to be small) we will have to define an appropriate **limit** where a *local*, *well defined*, *approximate evolution equation* will become possible. This equation is, of course, needed to describe the evolution of our system since we will have to deal with radiation that goes on until our black hole evaporates and we will need several later states of our system. We will name this limit *"solar system limit",* where the term signifies that we can do *normal* physics when spacetime curvatures are of the order found in our solar system.

"*Hawking's Theorem*" as Mathur formulated it, starts with a natural set of niceness conditions N, and proves that requiring locality with these conditions would lead to an 'unacceptable' physical evolution. This Theorem, (in each detailed form presented below) is a very precise statement of the contradiction found by Hawking, and bypassing the paradox needs a basic change in our understanding of Quantum physics.

2 Quantum Field Theory

2.1 Simple Harmonic Oscillator

Starting from *Schrödinger's picture*, i.e. letting the quantum states evolve in time, we will quantize *Simple Harmonic Oscillator.* The wavefunction is the set of all coefficients of the state vector $|\psi\rangle$, written in a delta-function basis of position $|x\rangle$:

$$\left|\psi_{(t)}\right\rangle = \int dx \psi\left(x,t\right)|x\rangle \tag{1}$$

By defining the operators $\hat{x} = x$ and $\hat{p} = -i\frac{\partial}{\partial x}$ as operators of position and momentum respectively, we introduce the (canonical) commutation relation:

$$[\hat{x}, \hat{p}] = i \tag{2}$$

The Hamiltonian operator of the *Simple Harmonical Oscillator* is :

$$\hat{\mathrm{H}} = -\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 + \frac{1}{2}\omega^2 x^2 \tag{3}$$

and the Schrödinger equation of motion:

$$i\frac{\partial \Psi (x,t)}{\partial t} = \hat{H}\Psi (x,t)\,, (\hbar = 1) \tag{4}$$

The fact that the Hamiltonian is time-independent lets us use the method of separable variables, so we write $\psi(x,t) = f(x)\,g(t)$. For $n < 0$ and normalizing we get:

$$\Psi_n (x,t) = e^{\left(\frac{1}{2}\right)\omega x^2} H_n\left(\sqrt{\omega} x\right) e^{-iE_n t} \tag{5}$$

where $H_n$are the Hermit polynomials:

$$H_n = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{6}$$

and :

$$E_n = \left(n + \frac{1}{2}\right)\omega \tag{7}$$

are the energy's eigenvalues. All the states are eigenstates of the Hamiltonian. A *Simple Harmonical Oscillator's* state can be written as a superposition of eigenstates:

$$\psi (x,t) = \sum_n c_n \psi_n (x,t) \tag{8}$$

$C_n$are normalization constants. Its ground state has energy:

$$E_n = \frac{1}{2}\omega \tag{9}$$

a fact that contradicts with the Classical case, in which for $x = 0$ and $p = 0$ the energy of the Oscillator is zero.

2.2 Creation and Annihilation operators in the S.H.O. case

Another solution of the S.H.O. problem can be found by introducing *Annihilation* and *Creation* operators ,$\hat{\alpha}^{\dagger}$and $\hat{\alpha}$. Let us now get an idea of what are these operator, and how they operate on the S.H.O.

We define:

$$\hat{\alpha} = \frac{1}{\sqrt{2\omega}} \left(\omega \hat{x} + i\hat{p}\right) \qquad \hat{\alpha}^{\dagger} = \frac{1}{\sqrt{2\omega}} \left(\omega \hat{x} - i\hat{p}\right) \tag{10}$$

From this definition we can easily derive:

$$\hat{x} = \frac{1}{\sqrt{2\omega}} \left(\hat{\alpha} + \hat{\alpha}^{\dagger}\right) \qquad \hat{p} = -i\sqrt{\frac{\omega}{2}} \left(\hat{\alpha} - \hat{\alpha}^{\dagger}\right) \tag{11}$$

We will now introduce the necessary canonical quantization relation we will make some computations to derive this relation.

Proof:

$$\left[\hat{\alpha}, \hat{\alpha}^{\dagger}\right] \psi = \hat{\alpha}\hat{\alpha}^{\dagger}\psi - \hat{\alpha}^{\dagger}\hat{\alpha}\psi =$$

$$= \frac{1}{2\omega} \left[\left(\omega \hat{x} + i\hat{p}\right)\left(\omega \hat{x} - i\hat{p}\right)\psi - \left(\omega \hat{x} - i\hat{p}\right)\left(\omega \hat{x} + i\hat{p}\right)\psi\right] =$$

$$= \frac{1}{2\omega} \left[\left(\omega^2 \hat{x^2} + \hat{p^2} + i\omega\hat{p}\hat{x} - i\omega\hat{x}\hat{p}\right)\psi - \left(\omega^2 \hat{x^2} + \hat{p^2} - i\omega\hat{p}\hat{x} + i\omega\hat{x}\hat{p}\right)\psi\right] =$$

$$= \frac{1}{2\omega} \left[2i\omega\hat{p}\hat{x}\psi - 2i\omega\hat{x}\hat{p}\psi\right] =$$

$$= i\left(-i\frac{\partial}{\partial x}\right)(x\psi) - ix\left(-i\frac{\partial}{\partial x}\right)\psi \iff$$

$$\iff \left[\hat{\alpha}, \hat{\alpha}^{\dagger}\right]\psi = \frac{\partial}{\partial x}(x\psi) - x\frac{\partial \psi}{\partial x} = \left(\frac{\partial}{\partial x}x\right)\psi = \psi \iff$$

$$\left[\hat{\alpha}, \hat{\alpha}^{\dagger}\right] = 1 \tag{12}$$

Our new Hamiltonian will be:

$$\hat{\mathrm{H}} = \left(\hat{\alpha}^{\dagger}\hat{\alpha} + \frac{1}{2}\right)\omega \tag{13}$$

Also we can easily derive the commutation relations:

$$\left[\hat{\mathrm{H}}, \hat{\alpha}\right] = -\omega\hat{\alpha} \text{ and } \left[\hat{\mathrm{H}}, \hat{\alpha}^{\dagger}\right] = \omega\hat{\alpha}^{\dagger} \tag{14}$$

Now observing the relations (14)and (9)we are tempted to define the *number operator* :

$$\hat{n} = \hat{\alpha}^{\dagger}\hat{\alpha} \tag{15}$$

We will take as a basis the set of all the eigenfunctions $|n\rangle$, and define the operation of number operator to be:

$$\hat{n}\left|n\right\rangle = n\left|n\right\rangle \tag{16}$$

We will show that the operation of creation and annihilation operators is the following: When the creation operator $\hat{\alpha}^{\dagger}$acts in an eigenstate (of $\hat{n}$)$|n\rangle$, it will give us another eigenstate of $\hat{n}$ but with an eigenvalue increased by 1. When the annihilation operator acts on a eigenstate (of $\hat{n}$) $|n\rangle$, we will get another eigenstate of $\hat{n}$ but with an eigenvalue decreased by 1.

This way, by defining $|0\rangle$as the ground state, we van create all the other state by multiple actions of creation operator.

Proof:

$$\left[\hat{\mathrm{H}}, \hat{\alpha}^{\dagger}\right] = \hat{\mathrm{H}}\hat{\alpha}^{\dagger} - \hat{\alpha}^{\dagger}\hat{\mathrm{H}} = \omega\left(\hat{n}\hat{\alpha}^{\dagger} + \frac{1}{2}\hat{\alpha}^{\dagger} - \hat{\alpha}^{\dagger}\hat{n} - \frac{1}{2}\hat{\alpha}^{\dagger}\right) =$$

$$= \omega\hat{n}\hat{\alpha}^{\dagger} - \omega\hat{\alpha}^{\dagger}\hat{n} \iff \hat{n}\hat{\alpha}^{\dagger} = \frac{1}{\omega}\left(\left[\hat{\mathrm{H}}, \hat{\alpha}^{\dagger}\right] + \omega\hat{\alpha}^{\dagger}\hat{n}\right) \iff$$

$$\iff \hat{n}\hat{\alpha}^{\dagger} = \hat{\alpha}^{\dagger}\left(1 + \hat{n}\right) \tag{17}$$

$$\left[\hat{\mathrm{H}}, \hat{\alpha}\right] = \hat{\mathrm{H}}\hat{\alpha} - \hat{\alpha}\hat{\mathrm{H}} = \omega\left(\hat{n}\hat{\alpha} + \frac{1}{2}\hat{\alpha} - \hat{\alpha}\hat{n} - \frac{1}{2}\hat{\alpha}\right) =$$

$$= \omega\hat{n}\hat{\alpha} - \omega\hat{\alpha}\hat{n} \iff \hat{n}\hat{\alpha} = \frac{1}{\omega}\left(\left[\hat{\mathrm{H}}, \hat{\alpha}\right] + \omega\hat{\alpha}\hat{n}\right) \iff$$

$$\iff \hat{n}\hat{\alpha} = \hat{\alpha}\left(\hat{n} - 1\right) \tag{18}$$

Thus:

$$\hat{n}\hat{\alpha}^{\dagger}\left|n\right\rangle = \hat{\alpha}^{\dagger}\left(1 + \hat{n}\right)\left|n\right\rangle = \left(n + 1\right)\hat{\alpha}^{\dagger}\left|n\right\rangle \tag{19}$$

$$\hat{n}\hat{\alpha}\left|n\right\rangle = \hat{\alpha}\left(\hat{n} - 1\right)\left|n\right\rangle = \left(n - 1\right)\hat{\alpha}\left|n\right\rangle \tag{20}$$

for $n \geq 0$.

We will the define the ground state, as the one the equals to zero after the annihilation operation:

$$\hat{\alpha}\left|0\right\rangle = 0 \tag{21}$$

As we stated, now we can produce all the other states from the ground state with the operating on it with $\hat{\alpha}^{\dagger}$, thus:

$$\left|n\right\rangle = \frac{1}{\sqrt{n!}}\left(\hat{\alpha}^{\dagger}\right)^{n}\left|0\right\rangle \tag{22}$$

Lets now write the eigenvalues of the annihilation and creation operators, in the S.H.O. case. From (21):

$$\hat{\alpha}|n\rangle = \hat{\alpha}\frac{1}{\sqrt{n!}}\left(\hat{\alpha}^{\dagger}\right)^{n}|0\rangle = \hat{\alpha}\hat{\alpha}^{\dagger}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{(n-1)!}}\left(\hat{\alpha}^{\dagger}\right)^{n-1}|0\rangle =$$

$$= \hat{\alpha}\hat{\alpha}^{\dagger}\frac{1}{\sqrt{n}}|n-1\rangle = \left(\left[\hat{\alpha},\hat{\alpha}^{\dagger}\right] + \hat{\alpha}^{\dagger}\hat{\alpha}\right)\frac{1}{\sqrt{n}}|n-1\rangle =$$

$$= \frac{1}{\sqrt{n}}(1+\hat{n})|n-1\rangle = \frac{1+n-1}{\sqrt{n}}|n-1\rangle = \sqrt{n}|n-1\rangle \iff$$

$$\iff \hat{\alpha}|n\rangle = \sqrt{n}|n-1\rangle \tag{23}$$

$$\hat{\alpha}^{\dagger}|n\rangle = \hat{\alpha}^{\dagger}\frac{1}{\sqrt{n!}}\left(\hat{\alpha}^{\dagger}\right)^{n}|0\rangle = \frac{\sqrt{n+1}}{\sqrt{(n+1)!}}\left(\hat{\alpha}^{\dagger}\right)^{n+1}|0\rangle = \sqrt{n+1}|n+1\rangle \iff$$

$$\iff \hat{\alpha}^{\dagger}|n\rangle\sqrt{n+1}|n+1\rangle \tag{24}$$

2.3 Heisenberg's Picture

The idea that quantum states evolve in time but the operators of the corresponding observables do not, is called *Schrödinger's picture* and its the one we have used until now. Here on out though it would be advisable to use another picture, totally equivalent to Schrödinger's, so that we can smoothly go from the first quantization to Quantum Field Theory. This picture is called *Heisenberg's picture* and in it quantum states remain them same, while the operators evolve in time.

Let Â be an arbitrary operator. The equation describing its evolution is called *Heisenberg equation:*

$$\frac{d\hat{A}(t)}{dt} = \frac{1}{i}\left[\hat{A}(t),\hat{H}\right] \tag{25}$$

and it can be solved by giving:

$$\hat{A}(t) = e^{i\frac{Ht}{\hbar}}\hat{A}e^{-i\frac{Ht}{\hbar}} = \hat{U}^{\dagger}(t)\hat{A}\hat{U}(t) \tag{26}$$

For the S.H.O. we get creation and annihilation operators evolution by:

$$\frac{d\hat{\alpha}(t)}{dt} = -i\omega\hat{\alpha}(t) \text{ and } \frac{d\hat{\alpha}^{\dagger}(t)}{dt} = i\omega\hat{\alpha}^{\dagger}(t) \text{ respectively} \tag{27}$$

and solving them we have:

$$\hat{\alpha}(t) = e^{-i\omega t}\hat{\alpha}(0) \text{ and } \hat{\alpha}^{\dagger}(t) = e^{+i\omega t}\hat{\alpha}^{\dagger}(0) \tag{28}$$

And interesting fact, is that the number operator we defined earlier, remains the same under time evolution.

Proof

$$\hat{n}(t) = \hat{\alpha}^{\dagger}(t)\,\hat{\alpha}(t) = e^{-i\omega t}\hat{\alpha}(0)\,e^{+i\omega t}\hat{\alpha}^{\dagger}(0) = \hat{\alpha}^{\dagger}(0)\,\hat{\alpha}(0) = \hat{n}(0) \tag{29}$$

2.4 Quantum Field Theory, In Flat Spacetime

In Field Theory, for an $n-dimentional$ spacetime, *Action* is the time integral:

$$S = \int d^n x \underset{\cdot}{L} \tag{30}$$

Also in Field Theory, unlike in relativistic approximation, *Klein-Gordon's equation* does not have any problems. Since it's the one describing bosons, which we will find in our problem,

and is in general more simple we will prefer using it over *Dirac's.* The *Lagrangian Density* in a $4-D$ flat spacetime is:

$$\underset{\cdot}{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial\varphi - \frac{1}{2}m^2\varphi^2 \tag{31}$$

Where $\eta^{\mu\nu}$ is the *Minkowski* metric tensor:

$$\eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{32}$$

and φ is a real scalar free field. The metric is:

$$ds^2 = -dt^2 + \left(d\left(x^i\right)\right)^2 \quad , i = 1,2,3 \tag{33}$$

We can derive the *Euler-Lagrange equation* from:

$$\frac{\partial \underset{\cdot}{L}}{\partial\varphi} = -m^2\varphi \;\; and \;\; \frac{\partial \underset{\cdot}{L}}{\partial\left(\partial_\mu\varphi\right)} = -\partial^\mu\varphi \tag{34}$$

and thus we get the Klein-Gordon equation:

$$-\partial_\mu\partial^\mu\varphi + m^2\varphi \tag{35}$$

The conjugate momentum to the field is:

$$\pi = \frac{\partial \underset{\cdot}{L}}{\partial\dot{\varphi}} \equiv \frac{\partial \underset{\cdot}{L}}{\partial\left(\partial_0\varphi\right)} = \dot{\varphi} \tag{36}$$

The connection between *Hamiltonian Density* and *Lagrangian Density* is given by the *Legendre Transformation :*

$$\underset{\cdot}{H}(\varphi,\pi)=\dot{\varphi}\pi-\underset{\cdot}{L}(\varphi,\partial_\mu\varphi) \tag{37}$$

and from it, we can derive the *Hamiltonian* $H=\int d^3x\underset{\cdot}{H}$.

Now moving to the quantization of the field. We will use our knowledge from Harmonic Oscillators quantization. Only this time instead of x,p (the conjugate position and momentum) we will use $\varphi\left(x^i\right),\pi\left(x^i\right)$ which are the values of the field in all space, for a given time. Keep in mind that this is a Quantum Field Theory, so we will need these field to become operators. This is the main reason we needed to use *Heisenberg's picture.*

Now solving the Klein-Gordon equation we get:

$$\varphi\left(x^i\right)=\varphi_0e^{ik^ix_i}=\varphi_0e^{-i\omega t+ik^jx_j} \tag{38}$$

$$for\ i=1,2,3,4\ \ and\ \ j=1,2,3$$

This solution describes a plane wave. So, as plane wave, it should satisfy the dispersion equation:

$$\omega^2=\left(k^i\right)^2+m^2 \tag{39}$$

To be able to write the field φ in the form of an operator, an idea is to present in our solution our known creation and annihilation operator. Only this time their number will be infinite. But first it is vital that we write a Klein-Gordon's general solution, forming a complete and orthonormal set of wavemodes (for every $\omega\left(k^i\right)$).

The inner product of two plane waves is $\left\langle e^{ik^i_1x_j},e^{ik^i_2x_j}\right\rangle$. Using *Gauss Theorem* it can be proven that this inner product is independent to the surface it's defined on. Thus on an arbitrary surface we get:

$$\left\langle e^{ik^i_1x_j},e^{ik^i_2x_j}\right\rangle=-i\int\limits_{\Sigma_t}\left(e^{-i\omega_1t+ik^j_1x_j}\partial_te^{i\omega_2t-ik^j_2x_j}-e^{i\omega_2t-ik^j_2x_j}\partial_te^{-i\omega_1t+ik^j_1x_j}\right)d^3x\iff$$

$$\iff\left\langle e^{ik^i_1x_i},e^{ik^i_2x_i}\right\rangle=\left(\omega_1+\omega_2\right)e^{-i(\omega_1-\omega_2)t}\left(2\pi\right)^3\delta^3\left(k^j_1-k^j_2\right) \tag{40}$$

which due to the delta-function $\delta^{(n-1)}$disappears in all cases except the one where the waves have the same k.

Now let us define our wavemodes as:

$$f_{k^j}\left(x^i\right)=\frac{e^{ik^ix_i}}{\left(\left(2\pi\right)^32\omega\right)^{\frac{1}{2}}} \tag{41}$$

We define us *positive* ,modes described by:

$$\partial_tf_{k^j}=-i\omega f_{k^j},\ \ \omega>0 \tag{42}$$

and *negative* ,modes described by:

$$\partial_t f^*_{k^j} = -i\omega f^*_{k^j}, \ \ \omega > 0 \tag{43}$$

From (40)we can see that the modes satisfy:

$$\left\langle f_{k^j_1}, f_{k^j_2} \right\rangle = \delta^3 \left(k^j_1 - k^j_2 \right) \tag{44}$$

and their complex conjugates:

$$\left\langle f^*_{k^j_1}, f^*_{k^j_2} \right\rangle = -\delta^3 \left(k^j_1 - k^j_2 \right) \tag{45}$$

Its also fairly obvious that:

$$\left\langle f_{k^j_1}, f^*_{k^j_2} \right\rangle = 0 \tag{46}$$

which means that the modes and their complex conjugates, are orthogonal to each other. We write the Klein-Gordon equation as:

$$\varphi \left(t, x^j \right) = \int \left[\alpha \left(k^j \right) f_{k^j} \left(t, x^j \right) + \alpha^* \left(k^j \right) f^*_{k^j} \left(t, x^j \right) \right] d^3k \tag{47}$$

To conclude our plan to write the fields and their conjugate momenta in operator form, we write the above us:

$$\hat{\varphi} \left(t, x^j \right) = \int \left[\hat{\alpha}_{k^j} f_{k^j} \left(t, x^j \right) + \hat{\alpha}^\dagger_{k^j} f^*_{k^j} \left(t, x^j \right) \right] d^3k \tag{48}$$

Finaly to have a Quantum Field Theory, we need to impose the canonical Quantization commutation relations:

$$\left[\hat{\varphi} \left(t, x^j \right), \hat{\varphi} \left(t, x^{j'} \right) \right] = 0 \tag{49}$$

$$\left[\hat{\pi} \left(t, x^j \right), \hat{\pi} \left(t, x^{j'} \right) \right] = 0 \tag{50}$$

$$\left[\hat{\varphi} \left(t, x^j \right), \hat{\pi} \left(t, x^{j'} \right) \right] = \delta^3 \left(x^j - x^{j'} \right) \tag{51}$$

where these relations are define on hypersurfaces with the same t.

From (51)we can see that the condition that needs fields $\varphi$and π to commute, holds for the whole space. The only case in which it does not hold, is that where their space coefficients coincide. Another thing we can derive from $(49 - 51)$, is that the commutation relations for $\hat{\alpha}$and $\hat{\alpha}^\dagger$are the same us in *Simple Harmonic Oscillator*. The only difference lies with the fact that there is one such relation for every j:

$$\left[\hat{\alpha}_{k^j}, \hat{\alpha}_{k^{j'}} \right] = 0 \tag{52}$$

$$\left[\hat{\alpha}^\dagger_{k^j}, \hat{\alpha}^\dagger_{k^{j'}} \right] = 0 \tag{53}$$

$$\left[\hat{\alpha}_{k^j}, \hat{\alpha}^{\dagger}_{k^{j'}}\right] = \delta^3\left(k^j - k^{j'}\right) \tag{54}$$

Similarly to the *Harmonic Oscillator,* we define the ground state, which here we call *Vacuum State*, as:

$$\hat{\alpha}_{k^j}\left|0\right\rangle = 0 \text{ for every } k^j \tag{55}$$

Also, we can create from this state all the other states, by operating on it with $\hat{\alpha}^{\dagger}_{k^j}$:

$$\left|n_{k^j}\right\rangle = \frac{1}{\sqrt{n_{k^j}!}}\left(\hat{\alpha}^{\dagger}_{k^j}\right)^{n_{k^j}}\left|0\right\rangle \tag{56}$$

The state with n_i excitations and all different k^j is written us:

$$\left|n_1, n_2 \ldots n_l\right\rangle = \frac{1}{\sqrt{n_1!n_2!\ldots n_l!}}\left(\hat{\alpha}^{\dagger}_{k^j_1}\right)^{n_{k^j_1}}\left(\hat{\alpha}^{\dagger}_{k^j_2}\right)^{n_{k^j_2}}\ldots\left(\hat{\alpha}^{\dagger}_{k^j_l}\right)^{n_{k^j_l}}\left|0\right\rangle \tag{57}$$

Creation and annihilation operator's action is:

$$\hat{\alpha}_{k^j_i}\left|n_1, n_2, \ldots, n_i, \ldots n_l\right\rangle = \sqrt{n_i}\left|n_1, n_2, \ldots, n_{i-1}, \ldots, n_l\right\rangle \tag{58}$$

and

$$\hat{\alpha}^{\dagger}_{k^j_i}\left|n_1, n_2, \ldots n_i, \ldots, n_l\right\rangle = \sqrt{n_i + 1}\left|n_1, n_2 \ldots, n_{i-1}, \ldots, n_l\right\rangle \tag{59}$$

The number operator takes the form:

$$\hat{n}_{k^j} = \hat{\alpha}^{\dagger}_{k^j}\hat{\alpha}_{k^j} \tag{60}$$

and its action is:

$$\hat{n}_{k^j_i}\left|n_1, n_2, \ldots, n_i, \ldots, n_l\right\rangle = n_i\left|n_1, n_2, \ldots, n_i, \ldots, n_l\right\rangle \tag{61}$$

$\hat{n}_{k^j}$ eigenstates define, what is called, a *Fock basis* in the Hilbert space.

2.5 Quantum Field Theory In Curved Spacetime

To have a *Quantum Field Theory* in a curved spacetime we use similar methods to the ones we used in the flat, *Minkowski* spacetime, but of course with some appropriate additions. This time though, we will face some peculiarities that occur due to the nature of the theory which describes the curved spacetime; *General Relativity.*

Starting as in the above chapter we write the appropriate *Lagrangian Density*:

$$
\underset{\cdot}{L} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_{\mu}\varphi\nabla_{\nu}\varphi - \frac{1}{2}m^2\varphi^2 - \xi R\varphi^2\right) \tag{62}
$$

$g_{\mu\nu}$ is the metric tensor of this spacetime. We write $g = det\,(g)$. Then $\sqrt{-g}$ is the incremental volume coefficient, which in *Minkowski* spacetime was 1 since there:

$$
g^{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{63}
$$

We have also introduced the coupling with the *Ricci* scalar, through ζ. We use *General Relativity's* theorem for *Minimal Coupling* and set $\zeta = 0$. Moving to quantization, we find the conjugate momenta:

$$
\pi = \frac{\partial\mathcal{L}}{\partial\left(\nabla_0\varphi\right)} \Rightarrow
$$

$$
\pi = \sqrt{-g}\nabla_0\varphi = \sqrt{-g}\partial_0\varphi = \sqrt{-g}\dot{\varphi} \tag{64}
$$

because for scalar fields covariant derivative is equivalent to a partial derivative.

The commutation relations we require are:

$$
\left[\hat{\varphi}\left(t, x^l\right), \hat{\varphi}\left(t, x^{l'}\right)\right] = 0 \tag{65}
$$

$$
\left[\hat{\pi}\left(t, x^l\right), \hat{\pi}\left(t, x^{l'}\right)\right] = 0 \tag{66}
$$

$$
\left[\hat{\varphi}\left(t, x^l\right), \hat{\pi}\left(t, x^{l'}\right)\right] = \frac{i}{\sqrt{-g}}\delta^3\left(x^l - x^{l'}\right) \tag{67}
$$

The equation of motion will be:

$$
g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\varphi - m^2\varphi \tag{68}
$$

It can be proven, again with the use of *Gauss Theorem* of divergence, that the inner product of scalar fields φ is independent from the hypersurface it has been defined on; i.e.:

$$
\langle\varphi_1, \varphi_2\rangle_{\Sigma_1} = \langle\varphi_1, \varphi_2\rangle_{\Sigma_2} \tag{69}
$$

At this point we face the first peculiarity. In curved spacetime the symmetries of *Poincare* transformation's, apart from *Lorentz Rotations* do not exist. Thus there isn't a way to define a (*Global*) *Timelike Killing Vector* globaly. So we can define many basis for our modes, and there we do not have a *good* reason to choose one particularly. Concluding, the forms of vacuum, and number operator depend on the basis we choose.We will proceed as follows.

We assume spacetime is *Asymptotically flat* and we divide it into 3 parts:

Part 1: Here spacetime is flat. We choose for the modes f that leave here and determine the number of *ingoing particles.*

Part 2: In this part, spacetime is the curved spacetime.

Part 3: In this region spacetime is also flat. The modes here are g and they determine the number of *outgoing particles.*

Now lets discuss in detail what happens in each of these regions.

Part 1:

The complete, orthonormal set of the basis for positive modes is:

$$\langle f_j, f_{j'} \rangle = \delta_{ij'} \tag{70}$$

and for negative modes:

$$\left\langle f_j^*, f_{j'}^* \right\rangle = -\delta_{ij'} \tag{71}$$

with:

$$\left\langle f_j, f_{j'}^* \right\rangle = 0 \tag{72}$$

Hence we can write down the solutions of (68) as:

$$\varphi = \sum_j \hat{\alpha}_j f_j + \hat{\alpha}_j^* f_j^* \tag{73}$$

We write the commutation relations for $\hat{\alpha}_j$ and $\hat{\alpha}_j^\dagger$:

$$[\hat{\alpha}_j, \hat{\alpha}_{j'}] = 0 \tag{74}$$

$$\left[\hat{\alpha}_j^\dagger, \hat{\alpha}_{j'}^\dagger\right] = 0 \tag{75}$$

$$\left[\hat{\alpha}_j, \hat{\alpha}_{j'}^\dagger\right] = \delta_{ij'} \tag{76}$$

Here we will prove that $(65-67)$ hold, with the use of

Proof:

$$[\hat{\varphi}, \hat{\varphi}'] = \left[\sum_j \left(\hat{a_j} f_j + \hat{a_j}^\dagger f_j^*\right), \sum_j \left(\hat{a_{j'}} f_{j'} + \hat{a_{j'}}^\dagger f_{j'}^*\right)\right] \iff$$

$$\iff [\hat{\varphi},\hat{\varphi}'] = \sum_{jj'} \left([\hat{a_j},\hat{a_{j'}}] f_j f_{j'} + \left[\hat{a_j},\hat{a_{j'}}^\dagger\right] f_j f_{j'}^* + \left[\hat{a_j}^\dagger,\hat{a_{j'}}\right] f_j^* f_{j'} + \left[\hat{a_j}^\dagger,\hat{a_{j'}}^\dagger\right] f_j^* f_{j'}^* \right) =$$

$$= \sum_{jj'} \left(\delta_{jj'} f_j f_{j'}^* - \delta_{jj'} f_j^* f_{j'} \right) \iff$$

$$\iff [\hat{\varphi},\hat{\varphi}'] = \sum_{j} \left(f_j f_j^* - f_j^* f_j \right) = 0$$

$$[\hat{\pi},\hat{\pi}'] = \left[\left(\sqrt{-g} i\omega\right) \sum_j \left(\hat{a_j}^\dagger f_j^* - \hat{a_j} f_j \right), \left(\sqrt{-g} i\omega\right) \sum_{j'} \left(\hat{a_{j'}}^\dagger f_{j'}^* - \hat{a_{j'}} f_{j'} \right) \right] \iff$$

$$\iff [\hat{\pi},\hat{\pi}'] = (-g\omega\omega') \sum_{jj'} \left([\hat{a_j},\hat{a_{j'}}] f_j f_{j'} - \left[\hat{a_j},\hat{a_{j'}}^\dagger\right] f_j f_{j'}^* - \left[\hat{a_j}^\dagger,\hat{a_{j'}}\right] f_j^* f_{j'} + \left[\hat{a_j}^\dagger,\hat{a_{j'}}^\dagger\right] f_j^* f_{j'}^* \right) \iff$$

$$[\hat{\pi},\hat{\pi}'] = (-g\omega\omega') \sum_{jj'} \left(-\delta_{jj'} f_j f_{j'}^* + \delta_{jj'} f_j^* f_{j'} \right) = 0$$

$$[\hat{\varphi},\hat{\pi}'] = \left[\hat{\varphi}, \sqrt{-g}\hat{\dot{\varphi}} \right] = \sqrt{-g} \left[\sum_j \left(\hat{a_j} f_j + \hat{a_j}^\dagger f_j^* \right), \sum_j \left(\hat{a_j} \dot{f_{j'}} + \hat{a_j}^\dagger \dot{f_j^*} \right) \right] \iff$$

$$\iff [\hat{\varphi},\hat{\pi}'] = \sum_{jj'} \sqrt{-g} \left([\hat{a_j},\hat{a_{j'}}] f_j \dot{f_{j'}} + \left[\hat{a_j},\hat{a_{j'}}^\dagger\right] f_j \dot{f_{j'}^*} + \left[\hat{a_j}^\dagger,\hat{a_{j'}}\right] f_j^* \dot{f_{j'}} + \left[\hat{a_j}^\dagger,\hat{a_{j'}}^\dagger\right] f_j^* \dot{f_{j'}^*} \right) \iff$$

$$\iff [\hat{\varphi},\hat{\pi}'] = \sum_{jj'} \sqrt{-g} \left(\left[\hat{a_j},\hat{a_{j'}}^\dagger\right] f_j \dot{f_{j'}^*} + \left[\hat{a_j}^\dagger,\hat{a_{j'}}\right] f_j^* \dot{f_{j'}} \right) \iff$$

$$\iff [\hat{\varphi},\hat{\pi}'] = -ii \sum_{jj'} \sqrt{-g} \left(\delta_{jj'} f_j \dot{f_{j'}^*} - \delta_{jj'} f_j^* \dot{f_{j'}} \right) = \frac{i}{\sqrt{-g}} \delta_{jj'}$$

The vacuum state in this region is $|0\rangle_f$:

$$\hat{a_j}\,|0\rangle_f = 0 \ \text{for every} \ j \tag{77}$$

and the number operator will be:

$$\hat{n}_{fj} = \hat{a_j}^{\dagger}\hat{a_j} \tag{78}$$

An arbitrary excited state can be produced by multiple operations of $\hat{a_j}^{\dagger}$,i.e.:

$$|n_k\rangle = \frac{1}{\sqrt{n_k!}}\left(\hat{\alpha}_k^{\dagger}\right)^{n_k}|0\rangle_f \tag{79}$$

Part 2:

We follow the same procedures, but this time for modes g_j and the complex conjugatesg_j^*.

$$\hat{\varphi} = \sum_j \left(\hat{b}_j g_j + \hat{b}_i^{\dagger} g_j^*\right) \tag{80}$$

Where $\hat{b}_j$and $\hat{b}_i^{\dagger}$are the annihilation and creation operators. It must hold:

$$\left[\hat{b}_j, \hat{b}_{j'}\right] = 0 \tag{81}$$

$$\left[\hat{b}_j^{\dagger}, \hat{b}_{j'}^{\dagger}\right] = 0 \tag{82}$$

$$\left[\hat{b}_j, \hat{b}_{j'}^{\dagger}\right] = \delta_{jj'} \tag{83}$$

The vacuum state $|0\rangle_g$ will be given by:

$$\hat{b}_j\,|0\rangle_g = 0 \ \text{for every} \ j \tag{84}$$

and the number operator:

$$\hat{n}_{gj} = \hat{b}_j^{\dagger}\hat{b}_j \tag{85}$$

We notice here that the vacuum state in region 1 will not necessarily be the vacuum state in region 2. To prove that, we need some transformation that can connect the set of modes f with the set of modes g .The transformation we shall use is known as *Bogoliubov Transform* after its creator. The Matrices α_{ij} and β_{ij} are called *Bogoliubov coefficients*:

$$f_i = \sum_j \left(\alpha_{ij} g_j + \beta_{ij} g_j^*\right) \tag{86}$$

and

$$g_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^* \right) \tag{87}$$

For the bogoliubov coefficient hold:

$$\alpha_{ij} = \langle f_i, g_i \rangle \tag{88}$$

and

$$\beta_{ij} = -\langle f_i, g_i^* \rangle \tag{89}$$

Proof:

Let $g_i$be:

$$g_i = \sum_j c_{ij} f_j + d_{ij} f_j^*$$

for arbitrary c_{ij} and d_{ij}. Then:

$$\langle f_i, g_j \rangle = \sum_{j'} \left\langle f_i, c_{ij'} f_{j'} + d_{ij'} f_{j'}^* \right\rangle = \sum_{j'} \left(c_{ij'} \left\langle f_i, f_{j'} \right\rangle + d_{ij'} \left\langle f_i, f_{j'}^* \right\rangle \right) \iff$$

$$\iff \langle f_i, g_j \rangle = \sum_{j'} c_{ij'} \delta_{ij'} = c_{ij}$$

but :

$$\langle f_i, g_j \rangle = \sum_{j'} \left\langle \alpha_{ij'} g_{j'} + \beta_{ij'} g_{j'}^*, g_j \right\rangle = \sum_{j'} \left(\alpha_{ij'} \left\langle g_{j'}, g_{j'} \right\rangle + \beta_{ij'} \left\langle g_{j'}^*, g_i \right\rangle \right) \iff$$

$$\iff \langle f_i, g_j \rangle = \sum_{j'} \alpha_{ij'} \delta_{ij'} = \alpha_{ij}$$

Thus:

$$c_{ij} = \alpha_{ij} = \langle f_i, g_j \rangle$$

similarly $\beta_{ij} = -\langle f_i, g_i^* \rangle$.

It can be proven that the region 3's annihilation operator that:

$$\hat{b}_i = \sum_k \hat{a}_k \alpha_{ki} + \hat{a}_k^\dagger \beta_{ki}^* \tag{90}$$

Thus $\hat{b}_i^\dagger$ i.e. the creation operator is:

$$\hat{b}_i^\dagger = \sum_k \hat{a}_k \beta_{ki} + \hat{a}_k^\dagger \alpha_{ki}^* \tag{91}$$

Let us now assume that someone uses the g modes operators to check if $|0\rangle_f$ is the vacuum. To do that to do that he will need to compute the expectation value of operator $\hat{n}_{gi}$ on the f vacuum. That is:

$$\left\langle 0 \left|{}_f \hat{b}_i^\dagger \hat{b}_i \right| 0 \right\rangle_f = \left\langle 0 \right|_f \left(\sum_k \hat{a}_k \beta_{ki} + \hat{a}_k^\dagger \alpha_{ki}^* \right) \left(\sum_l \hat{a}_l \alpha_{li} + \hat{a}_l^\dagger \beta_{li}^* \right) \left| 0 \right\rangle_f =$$

$$= \sum_{kl} \left\langle 0 \left|{}_f \hat{a}_k \beta_{ki} \hat{a}_l^\dagger \beta_{li}^* \right| 0 \right\rangle_f = \sum_{kl} \left\langle 0 \left|{}_f \beta_{ki} \left(\hat{a}_l^\dagger \hat{a}_k + \delta_{lk} \right) \beta_{li}^* \right| 0 \right\rangle_f =$$

$$= \sum_{kl} \langle 0 \left|{}_f \beta_{ki} \delta_{lk} \beta_{li}^* \right| 0 \rangle_f = \sum_k \langle 0 \left|{}_f \beta_{ki} \beta_{ki}^* \right| 0 \rangle_f = \sum_k \beta_{ki} \beta_{ki}^* \Rightarrow$$

$$\Rightarrow \left\langle 0 \left|{}_f \hat{b}_i^\dagger \hat{b}_i \right| 0 \right\rangle_f = \left(\beta^\dagger \beta \right)_{ii} \tag{92}$$

That means that if some of those coefficients have none-zero values then particles will be seen ,by the observer using the g modes operators, populating the f vacuum.

3 Quantum Statistical Mechanics

In this chapter, we will present an essential ,to the understanding of the paradox, view of *Quantum Mechanics.* When we encounter system with a high number of subsystems, there is a need for a statistical model to describe them. Thus, follow some basic *tools* we will use, that come from this, *Quantum Statistical Mechanic*'s, field.

3.1 Pure States:

3.1.1 *Pure state*

A pure state of a quantum system is denoted by a vector (jet) $|\psi\rangle$ with unit length, i.e.$\langle\psi\ \psi\rangle| = 1$, in a complex Hilbert space H. Previously, we (and the textbook) just called this a *state*, but now we call it a *pure* state to distinguish it from a more general type of quantum state.

3.1.2 *Inner product*

We can define dual vectors (bra) $\langle\phi|$as linear maps from the Hilbert space H to the field C of complex numbers. Formally, we write:

$$\langle\phi| \left(|\psi\rangle\right) = \langle\phi| \, \psi\rangle \tag{93}$$

The object on the right-hand side denotes the inner product in H for two vectors $|\psi\rangle$ and $|\phi\rangle$. That notation for the inner product used to be just that, notation. Now that we have defined $\langle\phi|$ as a dual vector it has acquired a second meaning.

3.1.3 *Operators*

Given vectors and dual vectors we can define operators (i.e., maps from H to H) of the form

$$\hat{O} = |\psi\rangle \langle\phi| \tag{94}$$

where $\hat{O}$ acts on vectors in H and produces as result vectors in H. The hermitian conjugate of this operator is:

$$\hat{O}^{\dagger} = |\phi\rangle \langle\psi| \tag{95}$$

and it follows straight from the definition of the hermitian conjugate that:

$$\left(\left\langle n \left| \hat{O} \right| m \right\rangle\right)^{*} = \left(\left\langle m \left| \hat{O}^{\dagger} \right| n \right\rangle\right) \tag{96}$$

for all states $|n\rangle$ and $|m\rangle$ in H.

A special case of such an operator is the projection operator, or projector:

$$\hat{P}_{\psi} = |\psi\rangle \langle\psi| \tag{97}$$

this is an operator which projects a vector onto the $|\psi\rangle^{th}$ eigenstate. First the bra vector dots into the state, giving the coefficient of $|\psi\rangle$ in the state, then its multiplied by the unit vector $|\psi\rangle$, turning it back into a vector, with the right length to be a projection. This operator is hermitian, and it hold:

$$\hat{P}_{\psi}^{2} = \hat{P}_{\psi} \tag{98}$$

We can use the projection operator $\hat{P}_{\psi}$ to describe all physical quantities we can derive from the state $|\psi\rangle$. We use the symbol $\hat{\rho}$ to indicate (or emphasize) we're talking about a physical state rather than an arbitrary operator:

$$\hat{\rho} = |\psi\rangle \langle\psi| \tag{99}$$

and we call $\hat{\rho}$ the density matrix, or the density operator describing the state $|\psi\rangle$.

Additionaly,it can be proved that a pure states density matrix remains pure under *Unitary transformations*. The transformation acts on the $\hat{\rho}$ as:

$$\hat{\rho} \rightarrow \hat{U}\hat{\rho}\hat{U}^{\dagger} \tag{100}$$

Example:

if we take the two pure states:

$$|\psi_{\pm}\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$$

then the corresponding $\hat{\rho}$ are 2x2 matrices. Written in the basis $\{|0\rangle, |1\rangle\}$, have the form:

$$\hat{\rho}_{\pm} = \begin{vmatrix} \frac{1}{2} & \pm\frac{1}{2} \\ \pm\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

3.1.4 *Trace*

It is useful to define the 'Trace' operation:

$$Tr\left(\hat{K}\right) = \sum_{n} \left\langle n \middle| \hat{K} \middle| n \right\rangle \tag{101}$$

where $\hat{K}$ is an arbitrary operator, and the sum is over a set of basis vectors $\{|n\rangle\}$. If we write down a matrix representation for $\hat{K}$, i.e., a matrix with elements $\left\langle n \middle| \hat{K} \middle| m \right\rangle$then the Trace is the sum over all diagonal elements (i.e., with $m = n$).

A nice property of the Trace operation is that a basis change leaves it invariant; that is, it does not matter which basis we choose in the definition of Trace. Indeed, the Trace would be far less useful if it did depend on the basis chosen.

Also for the Trace of $\hat{\rho}$ we always have the normalization condition, which in terms of Trace is $Tr\left(\hat{\rho}\right) = 1$.

3.2 Mixed States

3.2.1 *Mixed state*

Now let us define a more general type of states, described by density operators, by introducing *mixtures* of pure states:

$$\hat{\rho} = [k = 1]N \sum p_k \left|\psi_k\right\rangle \left\langle_k \psi\right| \tag{102}$$

where $\{|\psi_k\rangle\}$ is some set of pure states, not necessarily orthogonal. The number N could be anything, and is not limited by the dimension of the Hilbert space. The N numbers (or 'weights') p_k are nonzero and satisfy the relations

$$0 < p_k \leq 1\,;\, [k = 1]N \sum p_k = 1 \tag{103}$$

The normalization of the weights p_k expresses the condition $Tr\left(\hat{\rho}\right) = 1$. The quantum state described by $\hat{\rho}$ is called a mixed state whenever $\hat{\rho}$ cannot be written as a density matrix for a pure state, or equivalently when the state cannot be described by a wavefunction.

Since $\hat{\rho}$ is hermitian, we can diagonalize it, such that:

$$\hat{\rho} = [k = 1]M \sum \lambda_k \left|\phi_k\right\rangle \left\langle_k \phi\right| \tag{104}$$

where the states $|\phi_k\rangle$ are orthogonal (unlike the above definition). The numbers λ_k satisfy:

$$0 \leq \lambda_k \leq 1\,;\, [k = 1]M \sum \lambda_k = 1 \tag{105}$$

The numbers λ_k are, in fact, nothing but the eigenvalues of $\hat{\rho}$. They sum to one because of normalization. There are exactly M = d of these numbers, where d is the dimension of the Hilbert space.

3.3 Entropy

Another important quantity is the entropy given by:

$$S\left(\hat{\rho}\right) = -Tr\left(\hat{\rho} log\left[\hat{\rho}\right]\right) \tag{106}$$

One might wonder how to calculate the log of a matrix: just diagonalize it, and take the log of the diagonal elements. That is,

$$S\left(\hat{\rho}\right) = [k = 1]M \sum \lambda_k log \lambda_k \tag{107}$$

where λ_k are the eigenvalues of $\hat{\rho}$.Note that these eigenvalues are non negative, so the entropy can always be defined. Indeed, a zero eigenvalue contributes zero to the entropy, as:

$$\lim_{x \to 0} x log x = 0$$

3.4 Criteria

There are certain criteria that help us distinguish *pure state* from *mixed states.*

Criterion 1:

For $\hat{\rho} =[k= 1]N\sum p_k \left|\psi_k\right\rangle \left\langle_k\psi\right|$ only if N $= 1$ and $p_k = 1$ we can write $\hat{\rho} = \left|\psi\right\rangle \left\langle\psi\right|$i.e. have a pure state. In all other cases the state is mixed

Criterion 2:

If $\hat{\rho}^2 = \hat{\rho}$ the state is pure. If $\hat{\rho}^2 \neq \hat{\rho}$ the state is mixed. In equivalence: If $Tr\left(\hat{\rho}^2\right) = 1$ the state is pure and if $Tr\left(\hat{\rho}^2\right) \neq 1$ the state is mixed.

Criterion 3:

When the entropy of a state is $S\left(\hat{\rho}\right) = 0$ then the state is pure, and if its $S\left(\hat{\rho}\right) > 0$ then the state is mixed.

*Thus, there is no missing information for a pure state. A pure quantum state corresponds to maximum information. It does not tell us all we could know classically (for example, momentum and position), but it is the maximum knowledge quantum mechanics allows us to have.

3.5 Systems consisting of Subsystems

3.5.1 Reduced Density Matrix

Consider a state for 2 quantum systems say A and B (we will use 2 for simplicity but we can easily understand that this applies to any finite number of systems). If we want to describe only one of them e.g. A, we need to define the appropriate density operator. This is:

$$\hat{\rho}_A = \sum_m \left\langle m\left|\hat{\rho}_{AB}\right|m\right\rangle_B \equiv Tr_B\hat{\rho}_{AB} \tag{108}$$

where $\hat{\rho}_A$ is called reduced density operator (matrix). The operation indicated by Tr_B is called 'partial trace" or a "trace over B". We can also say we "trace out" system B.

Example:

Take a pure state of the form:

$$\left|\Psi\right\rangle_{AB} = \frac{\left|0\right\rangle_A \left|1\right\rangle_B + \left|1\right\rangle_A \left|0\right\rangle_B}{\sqrt{2}}$$

What is $\hat{\rho}_A$?
Answer: First calculate:

$$|\Psi\rangle_{AB}\langle\Psi| = \frac{1}{2}\left(|0\rangle_A\langle 0| \otimes |1\rangle_B\langle 01| + |0\rangle_A\langle 1| \otimes |1\rangle_B\langle 0| + |1\rangle_A\langle 0| \otimes |0\rangle_B\langle 1| + |1\rangle_A\langle 1| \otimes |0\rangle_B\langle 0|\right)$$

where for convenience we inserted a sign $\otimes$, to indicate the different projectors before and after the sign act on different Hilbert spaces (namely, those of A and B, respectively). Then take the trace over B. Only the first and fourth term survive this:

$$\hat{\rho}_A = Tr_B\,|\Psi\rangle_{AB}\langle\Psi| = \frac{1}{2}\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|\right)$$

Now that we have $\hat{\rho}_A$ we can check, for instance, if we want if this state is pure or mixed:

$$Tr\left(\hat{\rho}_A^2\right) = \frac{1}{2} < 1$$

i.e. the state is mixed.

3.5.2 Entangled State

An *entangled state*, is a pure (overall) state that cannot be written as a product of states. Take the above example for instance:

$$|\Psi\rangle_{AB} = \frac{|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B}{\sqrt{2}}$$

as we can see, it cannot be written as a product of states.:

$$|\Psi\rangle_{AB} \neq |\psi\rangle_A|\phi\rangle_B \tag{109}$$

How can we be sure that a pure state cannot be written as a product state?
Answer: just trace out system B, and check if $\hat{\rho}_A$ is pure or not. If $\hat{\rho}_A$ is mixed then $|\Psi\rangle_{AB}$ is entangled. If its pure, then $|\Psi\rangle_{AB}$ is not entangled. Of course we can be sure just by computing $\hat{\rho}_A$ that our state is entangled because our system consists of only 2 subsystems. If there where more we would have to check all the subsystem density operators. If at least 2 where mixed, then our system would be entangled.

4 A Quantum Paradox

Our purpose in this section will be to formulate the paradox as a Theorem. To do that we will need to follow certain basic steps. Firstly we will define a set of *niceness conditions* N ,which our next steps will have to obey. Afterward we will define the *Traditional Black Hole* which we will us in euro model and also the natural vacuum. Then we will slice our spacetime using *spacelike slices* which we will evolve in time (1-3D). We will see that as these slices evolve, pair production will take place near the horizon. These pairs will be real, and collecting their part that floats to infinity will give as the Hawking radiation. Studying these particle states will lead us to the rise of the paradox. Finally, having created the needed background, we will formulate the Hawking's Theorem whose inevitable consequences we shall discuss.

4.1 Solar System Limit

In our solar system, as we well know , spacetime is curved (due the mass of Sun, planets and other celestial bodies). However, we do experiments and all kinds of calculations without taking into consideration the effects of *Quantum Gravity.* This happens because the curvature is so low, that we are not in need of Q.G. corrections. *Quantum Gravity's* effects become important, when the curvature is that of plank scale ($R \sim \frac{1}{l_p^2}$) , i.e. the masses that create it are $\gg M_{sun}$. Also we need some sense of locality to ensure that there won't be any (or at least important) action at distance between states that are very far from each other. Thus we will define a limit, such that will enable us to define a well-defined, local evolution equation.

Definition of Solar System Limit: *There must exist a set of 'niceness conditions' N containing a small parameter ε such that when ε is made arbitrarily small then physics can be described to arbitrarily high accuracy by a known, local, evolution equation. That is, under conditions N we can specify the quantum state on an initial spacelike slice, and then a Hamiltonian evolution operator gives the state on later slices. Furthermore, the influence of the state in one region on the evolution in another region must go to zero as the distance between these regions goes to infinity (locality).*

4.2 Niceness Conditions N

1) Our quantum state is defined on a spacelike slice. The intrinsic curvature ${}^{(3)}R$ of this slice should be much smaller than Planck scale everywhere: ${}^{(3)}R \ll \frac{1}{l_p^2}$

2) The spacelike slice sits in an 4-dimensional spacetime. Let us require that the slice be nicely embedded in the full spacetime; i.e., the extrinsic curvature of the slice K is small everywhere: $K \ll \frac{1}{l_p^2}$

3) The 4-curvature curvature of the full spacetime in the neighborhood of the slice should be small everywhere: ${}^{(4)}R \ll \frac{1}{l_p^2}$

4) We should require that all matter on the slice be 'good'. Thus any quanta on the slice should have wavelength much longer than Planck length $\lambda \ll \frac{1}{l_p^2}$, and the energy density U and momentum density P should be small everywhere compared to Planck density: $U \ll \frac{1}{l_p^4}$, $P \ll \frac{1}{l_p^4}$.

5) We will evolve the state on the initial slice to a later slice; all slices encountered should be 'good' as above. Further, the lapse and shift vectors needed to specify the evolution should change smoothly with position: $\frac{dN^i}{ds} \ll \frac{1}{l_p^4}$, $\frac{dN}{ds} \ll \frac{1}{l_p^4}$

4.3 Traditional Black Hole

As a *Traditional Black Hole*, we will define a Black Hole with the following characteristics. Its metric will be

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} \tag{110}$$

; i.e. the *Schwarchild* metric. The horizon is at $r = 2M$. We will assume its mass to be small, for simplicity $M_{sun} \sim 3Km$ so the curvature at the horizon will be low ${}^{(4)}R \ll \frac{1}{M^2}$. But the most important feature of the *Traditional Black Hole* is that there will be no information (particles) in the vicinity of its horizon.

Definition of Information Free Horizon: A point on the horizon will be called 'information-free' if around this point we can find a neighborhood which is the 'vacuum' in the following sense: the evolution of field modes with wavelengths $l_p \ll \lambda \lesssim M$ is given by the semiclassical evolution of quantum fields on 'empty' curved space.

This evolution implies, as we will see, that we start with modes of small wavelengths which are not populated by particles; so the state is the vacuum. Only when those modes reach a the critical length

$\lambda \sim M$ will they be populated by particles and thus pair production will take place.

4.4 Slicing the T.B.H. Geometry

The slicing as we stated above will have to satisfy the N conditions.The Traditional Black Hole has a real singularity at $r = 0$ and a co-ordinate at $r = 2M$. At $r = 0$ our niceness conditions will not hold since the curvature is very high. At $r = 2M$ g_{tt} and g_{rr} become singular; the first vanishes, and the second one diverges. Hence we need to make sure that our slicing is done in way that our metric remains singular. We are going to divide our slice in four parts. The state on the first slice is going to be the *vacuum* or the *natural vacuum* for our Quantum Field Theory. Keep in mind that there is no *global timelike killing vector* in this geometry. If we where to take a timelike killing vector, say outside the horizon, it would become *null* at the horizon and then *spacelike* inside of it. So our metric is time-dependent and when we proceed to evolve our slice, the state will change and no longer be our *natural vacuum.* This fact will lead to particle production.

Part a:

For $r > 4M$we let the slice be $t = t_1 = constant$

Part b:

Inside $r < 2M$ the spacelike slices are$r = constant$ rather than $t = constant$ since$r$ and t co-ordinates interchange. We let the slice be $r = r_1$, $\frac{M}{2} < r_1 < \frac{3M}{2}$, so that this part of the slice is not near the horizon $r = 2M$ and not near the singularity $r = 0$.

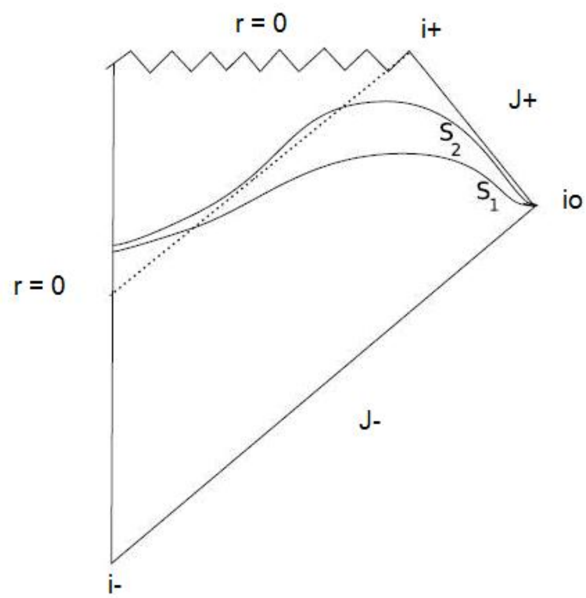
Part c:

We join these parts of the slice with a smooth *connector* segment C. The connector is also made in a way that it satisfies the N conditions.

Part d:

To connect the interior part of the slice to the horizon with $r = 0$ will do a trick. We will assume our B.H. was created by the collapse of a star a long time ago. In these early times there was a shell of mass that was collapsing towards $r = 0$ in flat spacetime to create the B.H. There was no singularity then, so we can connect that part smoothly with $r = 0$.

The Penrose diagram bellow is that of a black hole formed by collapse of the 'infalling matter'. We can see the initial slice $S_1$and the evolved slice S_2.



i_0: future timelike infinity, i_-: past timelike infinity, i_+: spatial infinity, J_+: future null infinity, J_-: past null infinity.

The radial null geodesics are at +45 and -45 degrees. All null geodesics i_+,i_-, $J_+$and J_- and at i_0.

4.5 Evolution to later slices

From Wheelers Theorem in general relativity, we have what is called *many-fingered time.* This means that we can evolve every point in space with a different time variable. To put it simply, we can evolve our slices anyway we like, since they keep obeying our niceness conditions N. We will now evolve our initial slice S_1 to the later one S_2.

Part a:

At $r > 4M$ we take $t = t_1 + \Delta$.

Part b:

The $r = const$ part will be $r = r_1 + \delta$ where $\delta_1 \ll M$.We let δ1 be small, and will later take the limit where $\delta_1 \to 0$ for convenience.

Part c:

We again join the parts a,b by a smooth *connector* segment. In the limit $\delta_1 \to 0$ we can take the geometry of the connector segment C to be the same for all slices. Note the very important fact that the $r = const$ part of the later slice S_2 is longer than the $r = const$ part of S_1. This extra part of the slice is needed because the connector segment has to join the $r = const$ part to the $t = constant$ part, and the $t = constant$ part has been evolved forwards on the later slice.

Part d:

At early times we again bring the $r = const$ part smoothly down to $r = 0$, at a place where there is no singularity.

4.6 Scales and Limits

Lets have a better understanding of length scales, and then consider some limits needed to define Locality in a more precise way. The length and time scale involved in the pair creation and in fact most of the procedures we follow is $L \sim M \sim 3Km$ which is the mass of our Black Hole (M_{sun}). Another scale is the distance between the matter shell and the pair created on the slice which is $L^{'} = 10^{77} light\, years$. Ensuring that the $L \sim M \sim 3Km$ is used in every procedure our assumption of $\lambda \sim M$ for the particles created becomes more concrete. As for our limits we have

$$\frac{L}{l_p} \gg 1, \; \frac{L^{'}}{l_p} \gg 1, \; \frac{L^{'}}{L} \gg 1 \tag{111}$$

The first two inequalities say that all length scales are much longer than Planck length, and the last says that the matter M is *far away* from the place where the pairs are being created.

4.7 Changes between the Slices

Let us now understand the changes between the slices, during their evolution. In $(1 - 3D)$ (or ADM) formalism, there exist the *shift vectors* $N^i$and *lapse* *function* N that show us how the slices of the spacetime are welded together. In general we have $N = (-^{(4)}g^{00})^{-1/2}$ and $N^i =^{(4)} g_{0i}$ where $g_{\mu\nu}$ is our metric tensor. Assume a point on $S_1$with spatial co-ordinates x^i. Now move along the timelike normal till we reach a point on S_2. Let this point on S_2 have the same spatial coordinates x^i. Thus we have set the shift vector to be $N^i = 0$. With this choice we can describe the evolution as follows:

Part a:

In the$t = constant$ part of the slice we have no change in intrinsic geometry. This part of the slice just advances forward in time with a lapse function $N = (1 - \frac{2M}{r})^{-1/2}$.

Part b&d:

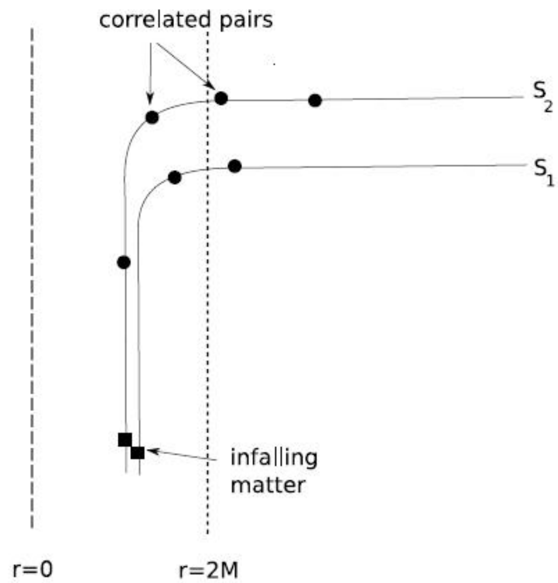
In the limit $\delta_1 \to 0$. The $r = const$ part of S_1 moves over to S_2 with no change in intrinsic geometry. The early time part which joins this segment to $r = 0$ also remains unchanged.

Part c:

The connector segment C of S_1has to *stretch* during this evolution, since the corresponding points on S_2 will have to cover both the connector C of S_2 and the extra part of the $r = const$ segment possessed by S_2.

we notice that the *stretching* takes place only in the C part. This part has space and time dimensions of order $L \sim M \sim 3Km$, i.e. the scale of our Black Hole's mass. A later slice S_n will evolve to a next S_{n+1} exactly like S_1 evolved to S_2. Each evolution from S_n to S_{n+1} can be described as follows: Divide S_n into a left part, a right part, and a middle part (which is the connector region). In the evolution to S_{n+1}, the left and right parts stay unchanged but are pushed apart, and the middle part is stretched to a longer length. The length of the middle part is $\sim M$, the proper time between the slices in this middle region is $\sim M$, and the stretching is by a factor $(1 + \alpha)$, with $\alpha \sim 1$. Thus the connector part C will grow in length with every such step by $\sim M$.

At this point we need to emphasize on the fact that the stretching takes place in the region C only. So the *Fourier modes* which used to be the vacuum state on our initial slice will also stretch to longer and longer wavelengths every time, until they reach the critical wavelength $\lambda \sim M$ and will be populated by particles. *Note that the fact that the stretching takes place only in the C region means that the only the modes of this region will stets (at least significantly). This way particles will keep being created until the Black Hole becomes very small. We cannot have such a set of slices in ordinary Minkowski space. If we try to make slices like those in the figure bellow in Minkowski space, then after some point in the evolution the later slices will not be spacelike everywhere: the *stretching* part will become null and then timelike. But it is the basic feature of the black hole that the space and time directions interchange roles inside the horizon, and we get spacelike slices having a stretching like that of the figure bellow throughout the region of interest.



In the figure above we can see that the infalling matter $|\psi\rangle_M$ is far away from the pair created. We assume $L^{'} \sim 10^{77} light\, years$ which the same scale that Hawking used.

4.8 Pair Production

Now we are going to derive the exact state of the pair as it is given by *Quantum Field Theory* in curved spacetime. We are going to look back to the *Bogoliubov* coefficients

and the action of the annihilation operator on the vacuum state f.

$$0 = \hat{a_i} |0\rangle_f = \sum_k \left[\alpha^*_{ik} \hat{b_k} - \beta^*_{ik} \hat{b^\dagger_k}\right] |0\rangle_f \tag{112}$$

solving for one mode:

$$\left(b + \gamma b^\dagger\right) |0\rangle_f = 0 \tag{113}$$

we get the solution:

$$|0\rangle_f = Ce^{\mu \hat{b^\dagger} \hat{b^\dagger}} |0\rangle_g \tag{114}$$

where C is the normalization constant and μ is a number we have yet to define. We expand the exponential in series:

$$e^{\mu \hat{b^\dagger} \hat{b^\dagger}} = \sum_n \frac{\mu^n}{n!} \left(\hat{b^\dagger} \hat{b^\dagger}\right)^n \tag{115}$$

using the commutator $\left[\hat{b},\hat{b}^\dagger = 1\right]$ we get:

$$\hat{b}\left(\hat{b}^\dagger\hat{b}^\dagger\right)^n = \left(\hat{b}^\dagger\hat{b}^\dagger\right)^n\hat{b} + 2n\hat{b}^\dagger\left(\hat{b}^\dagger\hat{b}^\dagger\right)^{n-1} \tag{116}$$

substituting the series to the exponential we have:

$$\hat{b}e^{\mu\hat{b}^\dagger\hat{b}^\dagger}\left|0\right\rangle_g = 2\mu\hat{b}^\dagger e^{\mu\hat{b}^\dagger\hat{b}^\dagger}\left|0\right\rangle_g \tag{117}$$

Now observing equation (113) we conclude that we must choose $\mu = -\frac{\gamma}{2}$. Thus we get:

$$\left|0\right\rangle_f = Ce^{-\frac{\gamma}{2}\hat{b}^\dagger\hat{b}^\dagger}\left|0\right\rangle_g \tag{118}$$

This state is one of the form:

$$\left|0\right\rangle_f = C_0\left|0\right\rangle_g + C_2\hat{b}^\dagger\hat{b}^\dagger\left|0\right\rangle_g + C_4\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\left|0\right\rangle_g + \dots \tag{119}$$

In (112) we can see that we have one part of the g vacuum, one part that has 2 particles, one part that has 4 and so on. Thus for (....) we have the solution:

$$\left|0\right\rangle_f = Ce^{-\frac{1}{2}\sum\limits_{m,n}\hat{b}^\dagger_m\gamma_{mn}\hat{b}^\dagger_n}\left|0\right\rangle_g \tag{120}$$

where γ is a symmetric matrix, containing the *Bogoliubov* coefficients.

$$\gamma = \left(\alpha^{-1}\beta + \left(\alpha^{-1}\beta\right)^T\right) \tag{121}$$

Now returning to our slices. We have seen that the evolution of a slice, leads to deformation on the connector geometry (*stretching).* This deformation leads to the change of the *Quantum state* we had in the past slices i.e. the vacuum changes, leading to particle production. The *Stretching* is characterized by a space and time scale of order $L \sim M \sim 3Km$. So the created particles will have wavelengths $\lambda \sim M$. From (...) is apparent that the particle will be created in pairs, and the number of pairs which we will choose will be $n \sim 1$.

For the state of every pair from (118) we have :

$$\left|\psi\right\rangle_{pair} = Ce^{\gamma\hat{c}^\dagger\hat{b}^\dagger}\left|0\right\rangle_c\left|0\right\rangle_b \tag{122}$$

But to understand essence of the paradox our chosen one pair is enough, i.e. the state we will use is :

$$\left|\psi\right\rangle_{pair} = \frac{1}{\sqrt{2}}\left|0\right\rangle_c\left|0\right\rangle_b + \frac{1}{\sqrt{2}}\left|1\right\rangle_c\left|1\right\rangle_b \tag{123}$$

Using *locality* which we assumed in our N conditions, we can write for the full state (Black Hole and pair) after the first pair production:

$$\left|\Psi\right\rangle = \left|\psi\right\rangle_M \otimes \left(\frac{1}{\sqrt{2}}\left|0\right\rangle_c\left|0\right\rangle_b + \frac{1}{\sqrt{2}}\left|1\right\rangle_c\left|1\right\rangle_b\right) \tag{124}$$

the $\approx$is used here instead of $=$because in our assumed *locality* we asked that the action at distance will $\to 0$ as the distance grows, so the effects of our *Shell of mass* state on the pair won't be exactly zero but close to zero. How small can we the divergence from this state be? If we write $|\psi\rangle_M$ as a spin state of only up and down possibilities, i.e. $|\psi\rangle_M = \frac{1}{\sqrt{2}}|\uparrow\rangle_M + \frac{1}{\sqrt{2}}|\downarrow\rangle_M$, then our full wavefunction will be:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle_M + \frac{1}{\sqrt{2}}|\downarrow\rangle_M \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_c|0\rangle_b + \frac{1}{\sqrt{2}}|1\rangle_c|1\rangle_b\right) \tag{125}$$

If our $|\psi\rangle_M$ has no effect on the pair state. If it has some effect, that effect will have to be :

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle_M + \frac{1}{\sqrt{2}}|\downarrow\rangle_M \otimes \left(\left[\frac{1}{\sqrt{2}} + \varepsilon\right]|0\rangle_c|0\rangle_b + \left[\frac{1}{\sqrt{2}} - \varepsilon\right]|1\rangle_c|1\rangle_b\right) \tag{126}$$

where ε is the very small parameter we defined in the *Solar System Limit.*

4.8.1 Leading Order

After the pair production takes place one particle $\{c\}$ will fall in the Black Hole, and the other $\{b\}$ will float to infinity where it will start forming the *Hawking Radiation.* At this section we will focus on the state of the produced pair, and we will see that it is entangled in a very specific way. As we are going to show later, if the state of our pair is the following, Hawking's conclusion will be inevitable.

Slice 1:

The full state here will be that of the shell of mass since there is nothing else on it. We write $|\Psi\rangle = |\psi\rangle_M$.

Slice 2:

As S_1 evolves to S_2 the middle part stretches, as the left and right part stay them (are just pushed apart). As we mentioned several times, this stretching will lead to *correlated* pairs. Our full state will be:

$$|\Psi\rangle = |\psi\rangle_M \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_{c_1}|0\rangle_{b_1} + \frac{1}{\sqrt{2}}|1\rangle_{c_1}|1\rangle_{b_1}\right) \tag{127}$$

We now want to compute the entanglement entropy between systems $\{b\}$ and $M, \{c\}$. First we right the whole system's density matrix (operator). There is no entanglement between $\{b\}$ and M so the density matrix will be,

$$\hat{\rho}_{cb} = |\psi\rangle_{cb}\langle\psi| \Leftrightarrow \hat{\rho}_{cb} = \frac{1}{2}\left(|0\rangle_{c_1}|0\rangle_{b_1} + |1\rangle_{c_1}|1\rangle_{b_1}\right)\left({}_{b_1}\langle 1|\,{}_{c_1}\langle 1| + {}_{b_1}\langle 0|\,{}_{c_1}\langle 0|\right) =$$

$$= \frac{1}{2}\left(|0\rangle_{c_1} |0\rangle_{b_1}\, {}_{b_1}\langle 1|\, {}_{c_1}\langle 1| + |0\rangle_{c_1} |0\rangle_{b_1}\, {}_{b_1}\langle 0|\, {}_{c_1}\langle 0| + |1\rangle_{c_1} |1\rangle_{b_1}\, {}_{b_1}\langle 1|\, {}_{c_1}\langle 1| + |1\rangle_{c_1} |1\rangle_{b_1}\, {}_{b_1}\langle 0|\, {}_{c_1}\langle 0| \right)$$

Then we compute the reduced density operator for system $\{b\}$:

$$\hat{\rho}_b = Tr_c \hat{\rho}_{cb}$$

Now we write it down as in a matrix form for a basis $\{|0\rangle, |1\rangle\}$. That is :

$$\hat{\rho}_b = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{vmatrix}$$

So now we can easily compute the S_{ent} which is :

$$S_{ent} = -Tr\left[\rho_b ln \rho_b\right] = -Tr\left[\begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} \begin{vmatrix} -ln2 & 0 \\ 0 & -ln2 \end{vmatrix} \right] =$$

$$S_{ent} = -Tr \begin{vmatrix} -\frac{ln2}{2} & 0 \\ 0 & -\frac{ln2}{2} \end{vmatrix} = ln2 \tag{128}$$

Slice 3:

a) The state $|\psi\rangle_M$ remains the nearly the same, since nothing happens at its vicinity *(locality).*

b)Again the stretching will happen at the connector part. This leads to two facts. First, for the pair b_1, c_1that was created at the earlier step, b_1 and c_1 are pushed apart. Second, is that the stretching creates another pair b_2, c_2 in its region. Again here the hole state is given by (...), but we will use the simplified:

$$|\Psi\rangle = |\psi\rangle_M \otimes \left(\frac{1}{\sqrt{2}} |0\rangle_{c_1} |0\rangle_{b_1} + \frac{1}{\sqrt{2}} |1\rangle_{c_1} |1\rangle_{b_1} \right) \otimes \left(\frac{1}{\sqrt{2}} |0\rangle_{c_2} |0\rangle_{b_2} + \frac{1}{\sqrt{2}} |1\rangle_{c_2} |1\rangle_{b_2} \right) \tag{129}$$

Calculating ,like before, the entanglement of the $\{b_1, b_2\}$ with $M, \{c_1, c_2\}$ we get :

$$S_{ent} = 2ln2 \tag{130}$$

Slice N:

Consequently, the state in a random next slice will be:

$$|\Psi\rangle = |\psi\rangle_M \otimes \left(\frac{1}{\sqrt{2}} |0\rangle_{c_1} |0\rangle_{b_1} + \frac{1}{\sqrt{2}} |1\rangle_{c_1} |1\rangle_{b_1} \right) \otimes \left(\frac{1}{\sqrt{2}} |0\rangle_{c_2} |0\rangle_{b_2} + \frac{1}{\sqrt{2}} |1\rangle_{c_2} |1\rangle_{b_2} \right) \otimes \ldots \tag{131}$$

$$\ldots \otimes \left(\frac{1}{\sqrt{2}} |0\rangle_{c_N} |0\rangle_{b_N} + \frac{1}{\sqrt{2}} |1\rangle_{c_N} |1\rangle_{b_N} \right)$$

and the entropy of system$\{b_1, b_2, ..., b_N\}$ with system $M, \{c_1, c_2, ..., c_N\}$is :

$$S_{ent} = Nln2 \tag{132}$$

After a lot of time, when a large number of b particles will have been created, the B.H. will have shrank to a very small size. At this point there exist two possible outcomes.

Possibility 1:
As the quanta $\{b_i\}$ collect at infinity, the mass of the hole decreases. The slicing does not satisfy the niceness conditions N after the point when $M_{hole} \sim m_{plank}$ because ${}^{(4)}R \ll \frac{1}{l_p^2}$ is no longer true. We will therefore stop evolving our spacelike slices when this point is reached. The emitted radiation quanta $\{b_i\}$ have an entanglement $S_{ent} = Nln2$ with $M, \{c_i\}$.
Definition: We will say that our gravity theory contains remnants if there exists a set of objects with mass and size less than given bounds

$$m < m_{remnant} \quad , \quad l < l_{remnant} \tag{133}$$

but allowing an arbitrarily high entanglement with systems far away from the object.

Possibility 2:
The black hole evaporates away completely. The quanta $\{b_i\}$ have entanglement entropy $S_{ent} \sim Nln2 \neq 0$ .But since there is nothing left that they are entangled with, the final state is not described by any quantum wavefunction. The final state can only be described by a density matrix. Thus it seizes to be a pure state and becomes a mixed state.

4.8.2 Loss of unitarity

Possibility to is the essence of the paradox. Us we discussed in section [..] the density operator $\hat{\rho}$ remains unchanged under unitary transformations $\hat{U}\hat{\rho}\hat{U}^{\dagger} = \hat{\rho}$. Here we started with pure state density matrix and ended up with a mixed states density matrix,i.e. the density matrix changed. We are forced to admit that the evolution taking place here was non-Unitary. But the fact is that we struggled to remain in our *solar system limit ,* to have Hamiltonian evolution. In all but the last step that was true, since and we can see that until then our density matrix remained that of a pure state. So at the last step, when our Black Hole evaporates we loose Unitarity. This is basic principle of *Quantum Mechanics* and if its violated we are forced to conclude that new physics i needed to describe a Traditional Black Hole's evaporation. This notion will become more concrete when we establish the *Hawking's Theorem* and see it as an inevitable outcome.

4.8.3 Deformation of the leading order state (small corrections:definition and effect):

It is interesting to see if small corrections to the leading order state can invalidate Hawking's argument and remove all entanglement between the quanta $\{b_i\}$ and the $M, \{c_i\}$quanta in the Black Hole. If this happens, there would be no paradox, since the hole containing $M, \{c_i\}$can vanish, and we will be left with a pure state of the $\{b_i\}$ quanta, presumably carrying all the information of the initial matter $|\psi\rangle_M$.

Let the state at time step t_n be written as $|\Psi_{M,c}, \psi_b(t_n)\rangle$ where $\Psi_{M,c}$ denotes the state of the matter shell that fell in to make the black hole, and also all the c quanta that have been created at earlier steps in the evolution. ψ_b denotes all b quanta that have been created in all earlier steps. This state is entangled between the $M, \{c_i\}$ and $\{b_i\}$parts, it is not a product state. We assume nothing about its detailed structure. In the leading order evolution we would have at time step t_{n+1}:

$$|\Psi_{M,c}, \psi_b(t_n)\rangle \to |\Psi_{M,c}, \psi_b(t_n)\rangle \left[\frac{1}{\sqrt{2}}|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}} + \frac{1}{\sqrt{2}}|1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}}\right] \tag{134}$$

where the term in box brackets denotes the state of the newly created pair.

Here we will write down the most general case of "small corrections" and show, that they are not enough to change the state given by the leading order in a way which will let us escape the final mixed state.We assume that the state of the "new region" , created by the stretching, is spanned by two vectors:

$$S^{(1)} = \frac{1}{\sqrt{2}}|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}} + \frac{1}{\sqrt{2}}|1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}} \tag{135}$$

$$S^{(2)} = \frac{1}{\sqrt{2}}|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}} - \frac{1}{\sqrt{2}}|1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}} \tag{136}$$

Where we have allowed the occupation number of the mode to be either 0 or 1 and the evolution to give us only one pair per step. We could have taken more vectors, and enlarged our vector space, but the form of the following argument would not change. Now we choose a base of orthonormal states ψ_n for the $M, \{c_i\}$ and a base of orthonormal states χ_n for the $\{b_i\}$ parts so that:

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_{n,m} C_{nm}\psi_m\chi_n \tag{137}$$

With Unitary transformation we can get:

$$|\Psi_{M,c},\psi_b(t_n)\rangle = \sum_i C_i\psi_i\chi_i \tag{138}$$

Now, us we have seen in the *Statistical Quantum Mechanics* part, the density matrix can be written:

$$\rho_{ij} = |C_i|^2\,\delta_{ij} \tag{139}$$

and in this case the entanglement entropy will be

$$S_{ent} = -\sum_i |C_i|^2\,ln\left(|C_i|^2\right) \tag{140}$$

Where C_i are the eigenvalues of ρ. After step t_{n+1} :

$$\chi_i \to \chi_i \tag{141}$$

$$\psi_i \to \psi_i^{(1)}S^{(1)} + \psi_i^{(2)}S^{(2)} \tag{142}$$

i.e. ψ_i evolves to a tensor product of $\psi_i^{(i)}$ and $S_i^{(i)}$. Due to normalization principle we need:

$$\left\|\psi_i^{(1)}\right\|^2 + \left\|\psi_i^{(2)}\right\|^2 = 1 \tag{143}$$

In leading order we assumed $\psi_i^{(1)} = \psi_i$and $\psi_i^{(2)} = 0$ form the start. Here we will prove that if the corrections are not of *order unity,* our state will be that of the leading order.

The full state we create in accordance with all the above is:

$$|\Psi_{M,c},\psi_b(t_{n+1})\rangle = \sum_i C_i\left[\psi_i^{(1)}S^{(1)} + \psi_i^{(2)}S^{(2)}\right]\chi_i \tag{144}$$

now we can write:

$$|\Psi_{M,c},\psi_b(t_{n+1})\rangle = S^{(1)}\left[\sum_i C_i\psi_i^{(1)}\chi_i\right]+S^{(2)}\left[\sum_i C_i\psi_i^{(2)}\chi_i\right] \equiv S^{(1)}\Lambda^{(1)}+S^{(2)}\Lambda^{(2)} \tag{145}$$

where we have defined the states:

$$\Lambda^{(1)} = \sum_i C_i\psi_i^{(1)}\chi_i\ \ ,\ \ \Lambda^{(2)} = \sum_i C_i\psi_i^{(2)}\chi_i \tag{146}$$

and since $S^{(1)}$and $S^{(2)}$are orthonormal normalization gives:

$$\left\|\Lambda^{(1)}\right\|^2 + \left\|\Lambda^{(2)}\right\|^2 = 1 \tag{147}$$

We have now reached the point in which we will define exactly what we mean by the term *small corrections.*

Definition: We will say that corrections are small if:

$$\left\| \Lambda^{(2)} \right\|^2 < \varepsilon \quad \varepsilon \ll 1 \tag{148}$$

if such a limit does not exist, we will say that the corrections are of *order unity*

It of importance for the stability of the Hawking Theorem to prove that the entropy between the $\{b_i\}$ and $M, \{c_i\}$ grows with every step, if the demand for *small corrections* holds, and in fact it grows $\sim ln2$ as one would expect in the leading order.Let the entropy of $\{b_i\}$ after the timestep t_n be S_0. Lets divide our system to three sub-systems.

Sub-system A)
This system consists of all the $\{b_1, b_2, ..., b_N\}$ quanta created until and at the timestep t_n. As we have assumed, these quanta will not interact with the next pair produced.

Sub-system B)
This system is made of the contents of the Black Hole until this point,i.e. $M, \{c_1, c_2, ..., c_N\}$. The pair created at the timestep t_{n+1} will interact weakly with $M, \{c_1, c_2, ..., c_N\}$. This creates an entanglement which, until the end of this chapter we will have shown, is so small it won't affect the leading order state.

Sub-system C)
This system will be the newly created part at timestep t_{n+1}

The entropy of system A , is $S\{b\} = S_0$ and it will remain the same after the timestep t_{n+1} . That is, because as we have assumed$\{b_1, b_2, ..., b_N\}$do not interact with the new pair. The fact that the entropy of $\{b_1, b_2, ..., b_N\}$with $M, \{c_1, c_2, ..., c_N\}$ increases with each step can be interpreted in to an inequality. That is:

$$S\left(\{b\} + b_{n+1}\right) > S_0 + ln2 - 2\varepsilon \ , \ (\varepsilon \ll 1) \tag{149}$$

Proving that will suffice. To do that we follow the steps bellow:

Step 1:

If our assumption for small correction holds, we will show that the entanglement entropy between system C and the other systems is smaller than ε, i.e.:

$$S\left(c_{n+1}, b_{n+1}\right) \equiv -Tr\rho_{(c_{n+1},b_{n+1})} ln\rho_{(c_{n+1},b_{n+1})} < \varepsilon \tag{150}$$

(we use the symbol $\equiv$ instead of $=$ because of some approximation we will use in our proof)

Proof:

The pair's reduced density metric will be:

$$\rho_{(c_{n+1},b_{n+1})} = \begin{pmatrix} \left\langle \Lambda^{(1)} \middle| \Lambda^{(1)} \right\rangle & \left\langle \Lambda^{(1)} \middle| \Lambda^{(2)} \right\rangle \\ \left\langle \Lambda^{(2)} \middle| \Lambda^{(1)} \right\rangle & \left\langle \Lambda^{(2)} \middle| \Lambda^{(2)} \right\rangle \end{pmatrix} \tag{151}$$

This matrix is not diagonal, so to find its the trace of $\rho_{(c_{n+1},b_{n+1})} ln\rho_{(c_{n+1},b_{n+1})}$ we will diagonalize him. Before that we have:

$$\left\| \Lambda^{(2)} \right\|^2 = \left\langle \Lambda^{(2)} \middle| \Lambda^{(2)} \right\rangle \equiv \varepsilon_1^2 < \varepsilon^2 \tag{152}$$

and consequently

$$\left\langle \Lambda^{(1)} \middle| \Lambda^{(1)} \right\rangle \equiv 1 - \varepsilon_1^2 \tag{153}$$

from Cauchy-Schwartz inequality:

$$\left| \langle x, y \rangle \right| \leq \|x\| \cdot \|y\| \tag{154}$$

where $\langle ., . \rangle$is the inner product we get:

$$\left| \left\langle \Lambda^{(1)} \middle| \Lambda^{(2)} \right\rangle \right| = \left\| \Lambda^{(1)} \right\| \cdot \left\| \Lambda^{(2)} \right\| \equiv \varepsilon_2 < \varepsilon \tag{155}$$

Generally, from a density matrix of the form:

$$\rho = \frac{1}{2} I + \vec{a} \cdot \vec{\sigma_3} \tag{156}$$

we can go with a unitary transformation to:

$$\rho = \frac{1}{2} I + |\vec{a}| \cdot \sigma_3 \tag{157}$$

and in way diagonalize it. With regards to the above we can write for entropy:

$$S = ln2 + \frac{1}{2}\left(1 + 2\left|\vec{a}\right|\right) ln\left(1 + 2\left|\vec{a}\right|\right) - \frac{1}{2}\left(1 - 2\left|\vec{a}\right|\right) ln\left(1 - 2\left|\vec{a}\right|\right) \tag{158}$$

now with the use of (151), (152) and (153) we can get our pair's entropy which will be:

$$S\left(c_{n+1}, b_{n+1}\right) = \left(\varepsilon_1^2 - \varepsilon_2^2\right) ln\frac{e}{\varepsilon_1^2 - \varepsilon_2^2} + 0\left(\varepsilon^3\right) < \varepsilon \ , \ \ \varepsilon \ll 1 \tag{159}$$

which we will write:

$$S_p < \varepsilon \tag{160}$$

Until this point it can be seen, that the pair is weakly entangled with the rest of the system.

Step 2:

From entropy's subadditivity property, which connects the entropies of two systems A, B.

$$S(A+B) \geq |S(A) + S(B)| \tag{161}$$

and if $A = \{b\}$and $B = p$ (pair) :

$$S(\{b\} + p) \geq S_0 - \varepsilon \tag{162}$$

Step 3:

Now it is enough to show that:

$$S_{c_{n+1}} > ln2 - \varepsilon \tag{163}$$

We right the state (145) in a form which makes computing $S_{c_{n+1}}$ easy:

$$|\Psi_{M,c}, \psi_b(t_{n+1})\rangle = \left[\frac{1}{\sqrt{2}} |0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} \left(\Lambda^{(1)} + \Lambda^{(2)}\right)\right] + \left[\frac{1}{\sqrt{2}} |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}} \left(\Lambda^{(1)} - \Lambda^{(2)}\right)\right]$$

The reduced density matrix for c_{n+1} is :

$$\rho_{c_{n+1}} = \begin{pmatrix} \frac{1}{2}\left\langle\left(\Lambda^{(1)} + \Lambda^{(2)}\right)\right| \left(\Lambda^{(1)} + \Lambda^{(2)}\right)\right\rangle & 0 \\ 0 & \frac{1}{2}\left\langle\left(\Lambda^{(1)} - \Lambda^{(2)}\right)\right| \left(\Lambda^{(1)} - \Lambda^{(2)}\right)\right\rangle \end{pmatrix}$$

and with the help of (152) and (153):

$$\rho_{c_{n+1}} = \frac{1}{2} I + \begin{pmatrix} Re\left(\left\langle\Lambda^{(1)}\right| \Lambda^{(2)}\right\rangle\right) & 0 \\ 0 & -Re\left(\left\langle\Lambda^{(1)}\right| \Lambda^{(2)}\right\rangle\right) \end{pmatrix}$$

and finally :

$$S_{c_{n+1}} = ln2 - 2\left[Re\left(\left\langle\Lambda^{(1)}\right| \Lambda^{(2)}\right\rangle\right)\right]^2 \geq ln2 - 2\varepsilon^2 + 0\left(\varepsilon^3\right) \geq ln2 - \varepsilon \tag{164}$$

Now using the *Strong Subadditivity Theorem* that applies to the entropies of 3 systems:

$$S(A+B)+S(B+C)\geq S(A)+S(C) \tag{165}$$

Where $A=\{b\}\,,\ B=b_{n+1},\ C=c_{n+1}$. With the help of the above steps 1-3 we can reach the conclusion that:

$$S(\{b\}+b_{n+1})\geq S_0+ln2-\varepsilon \tag{166}$$

5 The Hawking Theorem

We finally have the background to formulate the *Hawking Theorem*: If

1) The Niceness conditions N give local Hamiltonian evolution

2) A traditional black hole (i.e. one with an information-free horizon) exists in the theory

Then formation and evaporation of such a hole will lead to mixed states/remnants.

Although we have not studied the case of remnants, this Theorem cannot tell us which of the two possible cases will be our final state. But this shouldn't make us worry, since both cases lead to a paradox.

Proof:

A) Consider the metric of the traditional black hole. This black hole admits a slicing satisfying the niceness conditions N in the domain of interest. By assumption (i) of the theorem, this implies that we have *Solar System physics* in the region around the horizon where particle pairs will be created.

B) In a region with 'solar system physics' we can identify and follow the evolution of an outgoing normal mode with wavelength $\lambda=\frac{M}{\mu}$ with $\mu>1$ a number of order unity. For concreteness, take $\mu=100$. Again using the fact that we are in the domain of standard solar system physics, we know that the state in this mode can be expanded in terms of a Fock basis of particles. Thus when $\lambda=\frac{M}{100}$ we can write:

$$|\psi\rangle_{mode}=\alpha_0\,|0\rangle+\alpha_1\,|1\rangle+\alpha_2\,|2\rangle\dots \tag{167}$$

There are two possibilities:

a)

$$\sum_{i>0} |\alpha_i|^2 \sim 1 \tag{168}$$

This means that there are particles with wavelength M 100 at the horizon. This means that the state at the horizon is not the vacuum, and so we do not have the traditional black hole, and are thus in violation of condition (ii) of the theorem.

b)

$$\sum_{i>0} |\alpha_i|^2 < \varepsilon^{'}, \ \varepsilon^{'} \ll 1 \tag{169}$$

In this case the state in the mode is the vacuum when $\lambda = \frac{M}{100}$. The requirement of solar system physics tells us that the evolution of this vacuum mode will have to be agree with the leading order evolution of vacuum modes on this geometry to within some accuracy governed by a small parameter ε. Thus there will exist an $\varepsilon \ll 1$ such that (148) is satisfied by the evolution where the wavelength grows from $\lambda = \frac{M}{100}$ to $\lambda \sim M$ and particle pairs populate this mode.

C) Since we have the niceness conditions N, the requirement of 'solar system physics' under these conditions forces us to the fact that the particle pairs in option b) above will be produced in a state close to the state $S^{(1)}$ (135).

D) The evaporation process produces $N \sim \left(\frac{M}{m_{pl}}\right)^2$ pairs before the hole reaches a size $\sim l_p$ .At this point we have a large entanglement entropy, for which we can write

$$S_{ent} > \frac{N}{2} ln2$$

since $\varepsilon \ll 1$ .Following the argument in the *leading order* section, we find that we are forced to mixed states/remnants (i.e. if the Planck sized hole evaporates away we get a radiation state 'entangled with nothing' violating quantum unitarity, and if a Planck sized remnant remains, then we have to admit remnants with arbitrarily high degeneracy in the theory).

This establishes the Hawking theorem.We have taken care to state Hawking's argument in a way that is a 'theorem', so that if we wish to bypass the conclusion that we get mixed states/remnants then we have to violate one of the assumptions stated in the theorem. Thus we can either argue that the niceness conditions N need to be supplemented by further conditions (in which case we have to say what they are), or we have to argue that we do not obtain the traditional black hole in the theory (i.e. there will not be an information free horizon).

We emphasize the essential strength of Hawking's argument in the following corollary:

Corollary 1: If the state of Hawking radiation has to be a pure state with no entanglement with the rest of the hole then the evolution of low energy modes at the horizon has to be altered by *Order Unity.*

The proof follows from Theorem 1. A small change in the state at the horizon changes this entanglement by only a small fraction, and cannot reduce it to zero. Conversely, if we wish this entanglement to be zero then we have to change the state of the created pairs to a state that is close to being orthogonal to the semiclassically expected one.

References

[1] Dr David Tong:Quantum Field Theory University of Cambridge Part III Mathematical Tripos

[2] James B. Hartle - GRAVITY An Introduction to Einstein's General Relativity. San Francisco:Addison Wesley, 2003

[3] L.H. Ford - Quantum Field Theory in Curved Spacetime, 1997

[4] Pasquale Calabrese and John Cardy - Entanglement Entropy and Quantum Field Theory, 2008

[5] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev - Entanglement in quantum critical phenomena, 2008

[6] Asher Peres, Petra F. Scudo, and Daniel R. Terno - Quantum Entropy and Special Relativity, 2002

[7] Samir D. Mathur - The information paradox: A pedagogical introduction, 2011

[8] Samir D. Mathur - What Exactly is the Information Paradox? , 2008

[9] Dr C. H. W. Barnes - Quantum Information , Cavendish Laboratory Cambridge University

[10] Lecture notes by Dr. P.K. Townsend - Black Holes ,1997