Gravitational Wilson lines in AdS_3/CFT_2

Dimitrios Toulikas

Institute for Theoretical Physics
University of Amsterdam

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- Faced with serious issues, it is natural to look for simpler models that share the important conceptual features of general relativity, while avoiding some of the computational difficulties.
- General Relativity in $2+1$ dimensions is one such model and it has been proven to be a useful testing ground.

General Relativity in 2+1 dimensions

- Gravity is described by the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}(R-2\Lambda) + S_m$$

Varying with respect to the metric $g_{\mu\nu}$ we get the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- The Riemann tensor can be written as

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{2}{d-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) - \frac{2}{(d-1)(d-2)}Rg_{\mu[\rho}g_{\sigma]\nu}$$

- In three dimensions the Weyl tensor $C_{\mu\nu\rho\sigma}$ vanishes identically.
  - No propagating degrees of freedom.
  - Solutions are locally Minkowski, de Sitter or anti-de Sitter depending on the sign of Λ.

Anti-de Sitter spacetime

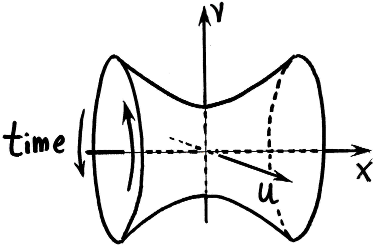
- AdS_3 is the maximally symmetric solution to the Einstein's Equations with $\Lambda < 0$.
- It can be viewed as a hyperboloid embedded into $\mathbb{R}^{(2,2)}$:

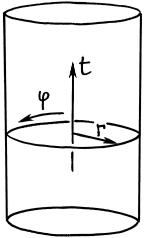
$$-(X^0)^2 + (X^1)^2 + (X^2)^2 - (X^3)^2 = -\ell^2, \qquad \Lambda = -1/\ell^2$$

- The isometry group of AdS_3 is $SO(2,2)$.
- Intrinsic coordinates:

$$X^0 = \ell \cosh\rho \cos t,\ \ X^i = \ell\,\Omega_i \sinh\rho,\ \ X^3 = \ell \cosh\rho \sin t$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \ell^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2 d\phi^2)$$





Asymptotically AdS_3 spacetimes

[Banados, Teitelboim, Zanelli '92]

- Another solution to the vacuum Einstein equations that came as a surprize is the BTZ black hole:

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + \left(d\phi - \frac{r_+ r_-}{\ell r^2} dt\right)^2$$

where $M = \frac{r_+^2 + r_-^2}{8G_3\ell^2}$ and $J = \frac{r_+ r_-}{4G_3\ell}$.

- In three dimensions, asymptotically AdS spacetimes are described by the following solution:

$$ds^2 = \ell^2 d\rho^2 + 8\pi G_3 \ell \left(\mathcal{L}(dx^+)^2 + \bar{\mathcal{L}}(dx^-)^2\right) - \left(\ell^2 e^{2\rho} + (8\pi G_3)^2 \mathcal{L}\bar{\mathcal{L}} e^{-2\rho}\right) dx^+ dx^-$$

  - Global AdS_3: $2\pi\mathcal{L} = 2\pi\bar{\mathcal{L}} = -\ell/16G_3$.
  - BTZ: $2\pi\mathcal{L} = \ell\frac{(r_+ + r_-)^2}{16G_3}$, $2\pi\bar{\mathcal{L}} = \ell\frac{(r_+ - r_-)^2}{16G_3}$.

A few words on groups and algebras

- Continuous symmetries are studied using Lie groups.
  Example: rotational symmetry of a sphere $\rightarrow SO(3)$.
- For every Lie group there is a corresponding Lie algebra that generates the group: $g = e^{\alpha^a T_a} = \mathbf{1} + \alpha^a T_a + \dots$, $g \in G$, $T_a \in \mathfrak{g}$ and $[T_a, T_b] = f_{ab}{}^c T_c$.
- We will be mainly interested in the Lie group $SL(2, \mathbb{R})$, which is the group of 2×2 real matrices with determinant one.
  The basis Lie algebra generators satisfy $[L_a, L_b] = (a - b)L_{a+b}$, $a, b = -1, 0, 1$.

3d gravity as a Chern-Simons theory

[Witten '88]

- In three dimensions we can cast gravity as a Chern-Simons gauge theory.
- A Chern-Simons theory for a gauge group G is described by the following action:

$$S_{CS}[A] = \frac{k}{4\pi}\int_M \epsilon^{\mu\nu\rho}\text{tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right)$$

  $A_\mu = A_\mu^a T_a$, $T_a \in \mathfrak{g}$.
  Action is invariant* under $A_\mu \to g(x)(A_\mu + \partial_\mu)g^{-1}(x)$,
  $g = e^{\alpha^a T_a} \in G$.
- This formulation makes manifest the topological nature of the theory but greatly obscures geometrical aspects.

Dictionary to gravity

- We take the gauge group to be the isometry group of AdS_3: $G = SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Then

$$S_{EH}[e,\omega] = S_{CS}[A] - S_{CS}[\bar{A}],$$

where $A_\mu = (\omega_\mu^a + \frac{1}{\ell} e_\mu^a) L_a$, $\bar{A}_\mu = (\omega_\mu^a - \frac{1}{\ell} e_\mu^a) \bar{L}_a$ and $g_{\mu\nu} = 2\mathrm{Tr}[(A_\mu - \bar{A}_\mu)(A_\nu - \bar{A}_\nu)]$.

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- Equations of motion:

$$\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = 0,$$
$$\partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu + [\bar{A}_\mu, \bar{A}_\nu] = 0\,.$$

Gauge transformations:

$$A_\mu \to L(x)(A_\mu + \partial_\mu) L^{-1}(x),$$
$$\bar{A}_\mu \to R^{-1}(x)(\bar{A}_\mu + \partial_\mu) R(x),$$

$$L, R \in SL(2,\mathbb{R}).$$

Asymptotically AdS$_3$ spacetimes in CS formulation

Asymptotically AdS$_3$ spacetimes correspond to the connections:

$$A = b^{-1}\left(L_1 + \frac{2\pi}{k}\mathcal{L}(x^+)L_{-1}\right)bdx^+ + b^{-1}\partial_\rho b d\rho\,,$$
$$\bar{A} = -b\left(\frac{2\pi}{k}\bar{\mathcal{L}}(x^-)L_1 + L_{-1}\right)b^{-1}dx^- + b\partial_\rho b^{-1}d\rho\,,$$

where $b(\rho) = e^{\rho L_0}$ and as before:

- Global AdS$_3$: $2\pi\mathcal{L} = 2\pi\bar{\mathcal{L}} = -\ell/16G_3$.
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A few words on representations

A representation of a group is a particular map of the group elements to operators that act on a vector space.

Example from quantum mechanics: $\mathfrak{so}(3)$ algebra: $[J_i, J_j] = i\epsilon_{ijk}J_k$.

$$J_3 |j,m\rangle = m |j,m\rangle$$
$$J_\pm |j,m\rangle = (J_1 \pm iJ_2) |j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)} |j, m\pm 1\rangle$$

Spin one:

$$|1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, J_2 = \frac{1}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A representation can be characterized by the Casimir operator:

$$J^2 |j,m\rangle = (J_1^2 + J_2^2 + J_3^2) |j,m\rangle = j(j+1) |j,m\rangle .$$

Wilson line as a massive probe

[Ammon, Castro, Iqbal '13]

- The Wilson loop is defined as

$$W_{\mathcal{R}}(C) = \mathrm{Tr}_{\mathcal{R}}\left(\mathcal{P}\exp\left(-\oint_C A\right)\mathcal{P}\exp\left(-\oint_C \bar{A}\right)\right)$$

- If we want an open-ended Wilson line we need to evaluate:

$$W_{\mathcal{R}}(x_i, x_f) = \langle U_f|\mathcal{P}\exp\left(-\int_\gamma A\right)\mathcal{P}\exp\left(-\int_\gamma \bar{A}\right)|U_i\rangle$$

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- We choose an infinite-dimensional highest weight representation:

$\ell_0\,|h,0\rangle = h\,|h,0\rangle\,, \ell_1\,|h,0\rangle = 0$ and $\ell_{-1}^n \sim |h,n\rangle$ $\qquad [\ell_m,\ell_n] = (m-n)\ell_{m+n}$

$C_2\,|h,n\rangle = (2\ell_0^2 - \ell_1\ell_{-1} - \ell_{-1}\ell_1)\,|h,n\rangle = 2h(h-1)\,|h,n\rangle$

$$m^2 = c_2 + \bar{c}_2 \qquad s = \bar{h} - h$$

Path integral representation

[Ammon, Castro, Iqbal '13]

- We can represent the Wilson line as

$$W_{\mathcal{R}}(x_i, x_f) = \int \mathcal{D}U \, e^{S(U;A,\bar{A})}$$

where $S(U;A,\bar{A})_C = \sqrt{c_2} \int_C ds \sqrt{\mathsf{Tr}(U^{-1}D_s U)^2}$,

$$D_s U = \frac{d}{ds}U + A_s U - U\bar{A}_s, \; A_s = A_\mu \frac{dx^\mu}{ds}$$

- Under a gauge transformation the worldline action is invariant under:

$$U(x) \to L(x^\mu)U(s)R(x^\mu(s))$$

Appearance of the geodesic equation

- The action describing the auxiliary system can be written as

$$S_{\text{on-shell}} = \sqrt{c_2}\int_\gamma ds \sqrt{\text{Tr}\left((A_\mu - \tilde{\bar{A}}_\mu)(A_\nu - \tilde{\bar{A}}_\nu)\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}$$
$$= \sqrt{2c_2}\int_\gamma ds \sqrt{g_{\mu\nu}(x)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}$$

where $\tilde{\bar{A}}_\mu \frac{dx^\mu}{ds} \equiv U \bar{A}_\mu \frac{dx^\mu}{ds} U^{-1} - \frac{d}{ds} U U^{-1}$.

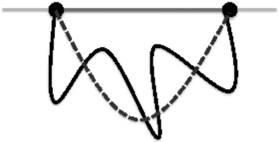
- Therefore:

$$W_{\mathcal{R}}(x_i, x_f) \sim \exp\left(-\sqrt{2c_2} L(x_i, x_f)\right)$$

Results

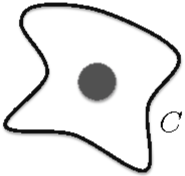
- The Wilson line reproduces the geodesic length:

$$W_{\mathcal{R}}(x_i, x_f) \sim \exp(-\sqrt{2c_2}L(x_i, x_f))$$



- For black holes, the Wilson loop computes their entropy:

$$S = -\log(W_{\mathcal{R}}(C)) = \frac{A}{4G}$$



Going away from the $m \gg 1$ limit

[Castro, Iqbal, Llabres '18]

- $W_{\mathcal{R}}(x_i, x_f) \sim \exp(-\sqrt{2c_2}L(x_i, x_f))$ relied on taking $m \gg 1$. We can go away from that limit using certain "rotated Ishibashi states": $|U\rangle \to |LUR\rangle$.
- We want to write a gravitational Wilson line as an overlap between suitable $|U\rangle$ states.

$$\begin{aligned} W_{\mathcal{R}}(x_f, x_i) &= \langle\Sigma| G\left(\mathcal{P}e^{-\int_{x_i}^{x_f} A}\right) \overline{G}\left(\mathcal{P}e^{-\int_{x_i}^{x_f} \bar{A}}\right)|\Sigma\rangle \\ &= \langle\Sigma| G\left(L(x_f)L^{-1}(x_i)\right) \overline{G}\left(R^{-1}(x_f)R(x_i)\right)|\Sigma\rangle \end{aligned}$$

- We define $|U(x)\rangle \equiv G\left(L(x_0)L^{-1}(x)\right) \overline{G}\left(R^{-1}(x_0)R(x)\right)|\Sigma\rangle$. Then

$$W_{\mathcal{R}}(x_f, x_i) = \langle U(x_f)|U(x_i)\rangle = \frac{e^{-\alpha(x_i, x_f)h}}{1 - e^{-\alpha(x_i, x_f)}}$$

where $L(x_f)L^{-1}(x_i)\widetilde{R}^{-1}(x_i)\widetilde{R}(x_f) = V\exp(-\alpha L_0)V^{-1}$.

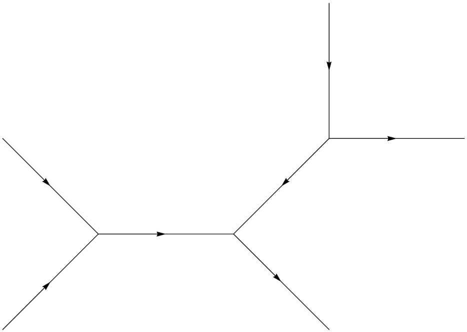
- Global AdS_3: $W_{\mathcal{R}}(x_f, x_i) = \frac{e^{-2hD(x_i, x_f)}}{1 - e^{-2D(x_i, x_f)}}$

$\to$ Bulk-to-bulk propagator of a scalar field.

Interactions

[Bhatta, Raman, Suryanarayana '16]

- We can consider networks of the form:



- Each line corresponds to a certain Wilson line in an infinite-dimensional highest weight representation labeled by $(h, \bar{h})$.
- They compute global conformal partial waves in CFT_2.

THANK YOU

Figures

- The figures of slide 4 were taken from: B. Oblak, "BMS Particles in Three Dimensions", arXiv:1610.08526 [hep-th].
- The figures of slide 14 were taken from: M. Ammon, A. Castro, and N. Iqbal, "Wilson Lines and Entanglement Entropy in Higher Spin Gravity", JHEP **1310** (2013) 110, arXiv:1306.4338 [hep-th].