Quantum Error Correcting Codes and Holography

Ilias Kokkas

University of Tennessee, Knoxville

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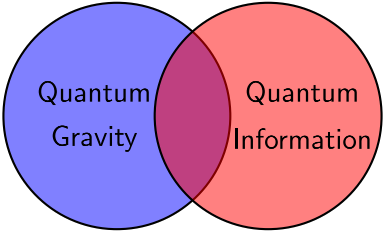
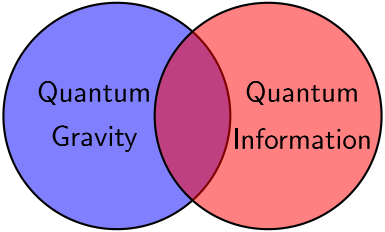

Quantum
Gravity
Quantum
Information



Black Holes: Complementarity or Firewalls?

Ahmed Almheiri,[1]* Donald Marolf,[2]*† Joseph Polchinski,[3]† and James Sully[4]*

Black holes as mirrors:
quantum information in random subsystems

Patrick Hayden
School of Computer Science, McGill University, Montreal, Quebec, H3A 2A7, Canada

John Preskill
Institute for Quantum Information, California Institute of Technology, Pasadena CA 91125, USA

Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence

Fernando Pastawski,*[a] Beni Yoshida*[a] Daniel Harlow,[b] John Preskill,[a]

Bulk Locality and Quantum Error Correction in AdS/CFT

Ahmed Almheiri,[a] Xi Dong,[a] Daniel Harlow[b]

[a] *Stanford Institute for Theoretical Physics, Department of Physics, Stanford University, Stanford, CA 94305, USA*
[b] *Princeton Center for Theoretical Science, Princeton University, Princeton NJ 08540 USA*

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1. Interaction with the environment (decoherence)
2. Imperfect gates on quantum computers

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$$\boxed{3p^2(1-p) + p^3 < p \quad \text{if } p < \frac{1}{2}}$$

Obstacles to quantum error correction

Quantum communication over a noisy channel: $\boxtimes = |\psi\rangle = a\,|0\rangle + b\,|1\rangle$

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2. More errors: small rotations on the Bloch sphere or phase flip
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We can overcome these issues!

3 qubit bit flip code

Alice wishes to send to Bob the qubit $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$

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$$|\psi_L\rangle \neq (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle)$$

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This state is produced using 2 CNOT gates and 2 ancilla qubits

$$|\psi_L\rangle = CNOT_{1,2} CNOT_{1,3} |\psi\rangle_1 |00\rangle_{23}$$

$\text{CNOT}_{c,t}$ gate			
BEFORE		AFTER	
CONTROL	TARGET	CONTROL	TARGET
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

2 Alice sends $|\psi_L\rangle$ to Bob over a bit flip channel with prob. p : $\begin{matrix} |0\rangle \rightarrow |1\rangle \\ |1\rangle \rightarrow |0\rangle \end{matrix}$

Consider for example that Bob receives $|\psi_{e1}\rangle = a\,|100\rangle + b\,|011\rangle$

Challenge: Find the error without destroying the message!

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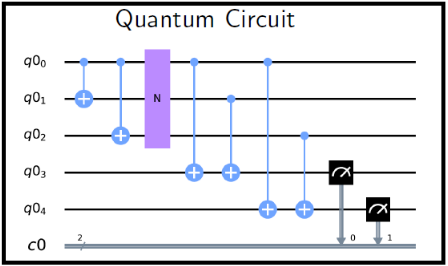

ERROR LOCATION	ANCILLA
No error	00
1st qubit	11
2nd qubit	10
3rd qubit	01

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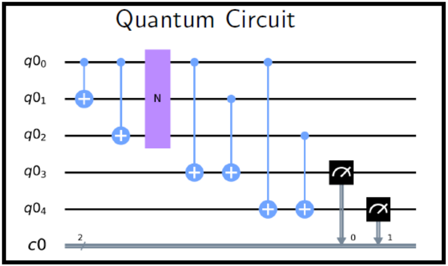

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4 Bob measures the ancilla qubits and corrects the error!

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What if we have small errors such as :

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For an error on $1st$ qubit

$$|\psi_L\rangle \rightarrow |\psi_{s1}\rangle = a\,|000\rangle + a\epsilon\,|100\rangle + b\,|111\rangle - b\epsilon\,|011\rangle = |\psi_L\rangle + \epsilon\,|\psi_{e1}\rangle$$

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We follow the previous steps

1. $CNOT_{1,5}CNOT_{3,5}CNOT_{1,4}CNOT_{2,4}\,|\psi_{s1}\rangle\,|00\rangle = |\psi_L\rangle\,|00\rangle + \epsilon\,|\psi_{e1}\rangle\,|11\rangle$
2. Measure the ancilla qubits
3. Correct

3 qubit phase flip code

Lastly, consider a prob. p for : $\begin{array}{l} |0\rangle \rightarrow +|0\rangle \\ |1\rangle \rightarrow -|1\rangle \end{array}$

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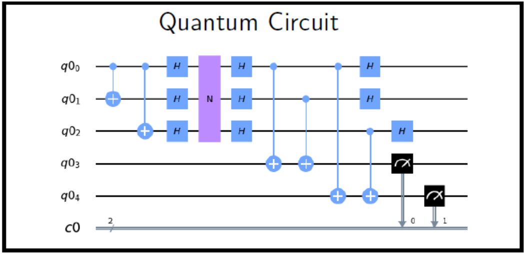

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

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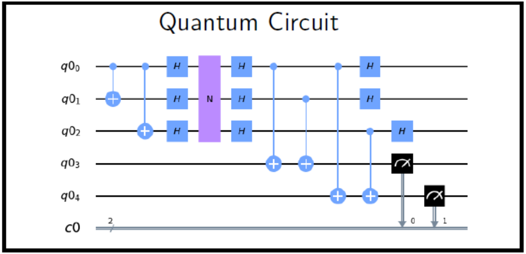

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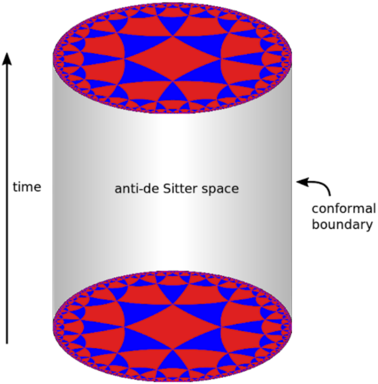
1. Alice encodes the message into $|\psi_L\rangle = a|+++\rangle + b|---\rangle$
2. Consider for example that Bob receives $|\psi_{e1}\rangle = a|-++\rangle + b|+--\rangle$
3. Bob applies $H_1 H_2 H_3$ which gives the state $a|100\rangle + b|011\rangle$
4. The rest are identical to the bit flip code
5. Before Bob corrects applies again the Hadamard gates

AdS/CFT correspondence

Conjecture: (Quantum) gravity in $AAdS_{d+1} \equiv$ CFT on $R \times S^{d-1}$

$$\exists\, f : \mathcal{H}_{AAdS} \to \mathcal{H}_{CFT}$$

We want CFT's with semiclassical dual gravity!



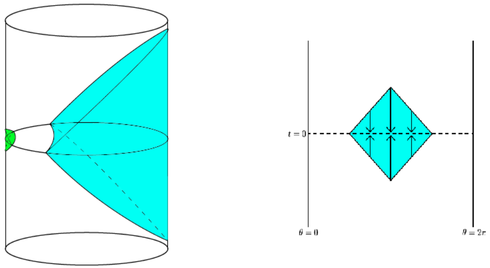
Extrapolate dictionary

$$\lim_{r\to\infty} r^{\Delta}\phi_i(r,t,\Omega) = \mathcal{O}_i(t,\Omega)$$

Bulk reconstruction

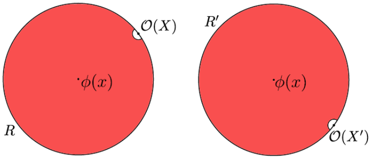
Write a bulk operator $\phi(x)$ in terms of CFT operators $\mathcal{O}(X)$

1. Solve the wave equation for the scalar field $\nabla^2\phi = m^2\phi$ in the bulk
2. Use the extrapolate dictionary for boundary conditions
3. End up with $\phi(x)|_{x\in W[R]} = \int_{D[R]} dX K(x;X)\mathcal{O}(X)$



Radial locality puzzle

Many ways to reconstruct the same bulk operator



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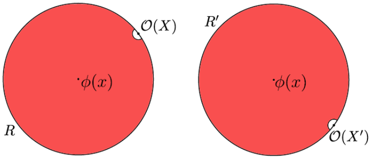

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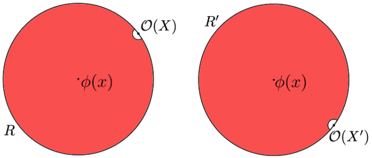

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Time slice axiom

Any operator which commutes with all local operators on a time slice in a QFT must be proportional to the identity.

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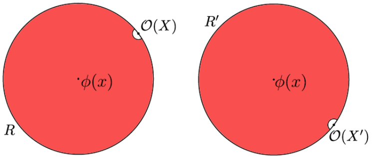

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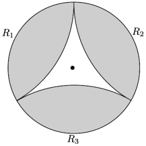

3 qutrit code

Consider the state $|\psi\rangle = \sum_{k=0}^{2} a(k) |k\rangle$ and the operator $O|k\rangle = \sum_{l=0}^{2} O_{lk} |l\rangle$

Encode this into the codespace of the 3 qutrits $|\psi_L\rangle = \sum_{k=0}^{2} a(k) |k_L\rangle$

Codespace basis

$|0_L\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$, $|1_L\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle)$, $|2_L\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle)$

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Protects against single qutrit erasure

$$U_{12}\,|k_L\rangle = |k\rangle_1\,|\chi\rangle_{23} \Rightarrow U_{12}\,|\psi_L\rangle = |\psi\rangle_1\,|\chi\rangle_{23} \quad ; \quad |\chi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right)$$

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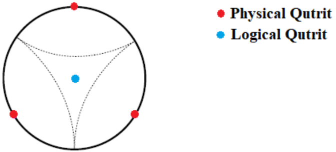
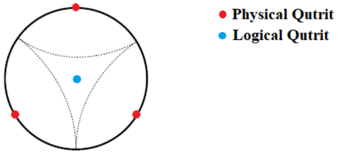
What about the logical operators? They should act like $O_L\,|k_L\rangle = \sum_{l=0}^{2} O_{lk}\,|l_L\rangle$
For this codespace logical operators have support only on 2 qutrits!

$$O_L = U_{12}^{\dagger} O_1 U_{12} = U_{23}^{\dagger} O_2 U_{23} = U_{31}^{\dagger} O_3 U_{31}$$

Reminiscent of the bulk operator reconstruction!



To construct the O_L we need access to 2 qutrits!



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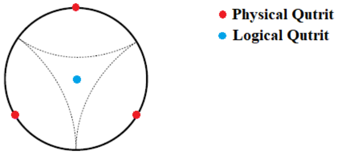
$$\langle\psi_L|[O_L, X_3]|\phi_L\rangle = \langle\psi_L|[U_{12}^{\dagger} O_1 U_{12}, X_3]|\phi_L\rangle = 0$$



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The commutator has vanishing matrix elements for a subspace of $\mathcal{H}_{CFT}$

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- F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence, J. High Energy Phys. 06 (2015) 149.
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Thank you!