(Part of) Ph.D. Thesis

Metric-Affine Gravity and Cosmology/Aspects of Torsion and Non-Metricity in Gravity

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Outline

- Brief Intro to the Geometry of Metric-Affine Gravity
- Metric-Affine f(R) Theories of Gravity/Projective Inv. Breaking
- The peculiar $f(R) = \alpha R^2$/ Cosmological Solutions
- Theorems for the Affine-Connection
- $1 + (n-1)$ Spacetime Split with Torsion and Non-metricity
- Raychaudhuri Equation with Torsion and Non-metricity
- Conclusions

Metric, Palatini, and Metric-Affine Gravity

Metric Gravity

- $\Gamma^{\alpha}{}_{\mu\nu} \rightarrow$ *torsionless* , metric compatibility $\nabla_{\sigma} g_{\mu\nu} = 0$
- $S = S_{Gravity} + S_{Matter} = \int d^n x \sqrt{-g}\, [\mathcal{L}_G(g_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu})]$

Palatini Gravity

- $\Gamma^{\alpha}{}_{[\mu\nu]} \neq 0$, $\nabla_{\sigma} g_{\mu\nu} \neq 0$, $\Gamma^{\alpha}{}_{\mu\nu}, g_{\mu\nu}$ are left independent
- $S = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu})\right]$

Metric-Affine Gravity (generalization of Palatini)

- $S = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu})\right]$

Geometrical Objects

Two distinctively different notions on a manifold

- Metric Tensor $g_{\mu\nu}$: Defines distances,lengths and dot products

$$||\alpha||^2 := \alpha^{\mu}\alpha^{\nu}g_{\mu\nu} \ , \ \ (\alpha \cdot \beta) := \alpha^{\mu}\beta^{\nu}g_{\mu\nu}$$

- Affine-Connection $\Gamma^{\lambda}{}_{\mu\nu}$: Defines parallel transport of tensor fields on the manifold

$$\nabla_{\lambda}u^{\mu} = \partial_{\lambda}u^{\mu} + \Gamma^{\mu}{}_{\nu\lambda}u^{\nu}$$

The two need not be related a priori! Their relation may be found after solving the field equations!

Geometrical Properties

Non Riemannian geometry

- Non metricity tensor : $Q_{\alpha\mu\nu} := -\nabla_\alpha g_{\mu\nu}$
  $\rightarrow$ Dot products and lengths of vectors not preserved!
  $\frac{d}{d\lambda}(ab)|_{along\ \mathcal{C}} \neq 0$
- Cartan torsion tensor : $S_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}{}_{[\mu\nu]}$
  $\rightarrow$ Infinitesimal Parallelograms do not exist!
- Limited symmetries of Riemann Tensor. For instance

$$R_{(\mu\nu)\alpha\beta} = \nabla_{[\alpha} Q_{\beta]\mu\nu} - S_{\alpha\beta}{}^{\lambda} Q_{\lambda\mu\nu} \neq 0$$

New tensors are introduced in this context and as a result more scalar combinations can be formed

Torsion/Non-metricity related vectors

Torsion/Non-metricity related vectors

$$S_\mu = S_{\mu\lambda}{}^{\lambda} \; , \qquad \tilde{S}^\mu = \epsilon^{\mu\nu\rho\sigma} S_{\nu\rho\sigma}$$

$$Q_\mu = g^{\alpha\beta} Q_{\mu\alpha\beta} \; , \qquad \tilde{Q}_\mu = g^{\rho\alpha} Q_{\rho\alpha\mu}$$

Simplest forms of Torsion/Non-metricity

$$S_{\mu\nu}{}^{\lambda} = \frac{2}{n-1} S_{[\mu}\delta^{\lambda}_{\nu]} \; , \qquad Q_{\alpha\mu\nu} = \frac{1}{n} Q_\alpha g_{\mu\nu}$$

- Another interesting form of non-metricity is one for which there exist fixed length vectors! Then $Q_{(\alpha\mu\nu)} = 0$ and one possible form is: $Q_{\alpha\mu\nu} = A_\alpha g_{\mu\nu} - g_{\alpha(\mu} A_{\nu)}$

Connection decomposition

Affine connection

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \frac{1}{2}g^{\alpha\lambda}(Q_{\mu\nu\alpha} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}) - g^{\alpha\lambda}(S_{\alpha\mu\nu} + S_{\alpha\nu\mu} - S_{\mu\nu\alpha})$$

where $\tilde{\Gamma}^{\lambda}{}_{\mu\nu} := \frac{1}{2}g^{\alpha\lambda}(\partial_{\mu}g_{\nu\alpha} + \partial_{\nu}g_{\alpha\mu} - \partial_{\alpha}g_{\mu\nu})$ is the Levi-Civita part of the connection. We often write

- $\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + N^{\lambda}{}_{\mu\nu}$

where $N^{\lambda}{}_{\mu\nu}$ is called the distortion.

Each quantity $\Rightarrow$ decomposed into Riemannian and non-Riemannian counterparts. Example:

$$R = \tilde{R} + \tilde{\nabla}_{\mu}(A^{\mu} - B^{\mu}) + B_{\mu}A^{\mu} - N_{\alpha\mu\nu}N^{\mu\nu\alpha}$$

Geodesics Vs Autoparalles

Geodesic Curves

$$\frac{d^2x^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

$\rightarrow$ Solution=Curve of shortest length (joining two points locally)

Autoparallel Curves

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

$\rightarrow$ Solution=Straightest Curve

Note : The two coincide in Standard GR but not in non-Riemannian Geometries.

Hypermomentum

From a physical perspective

- Not only $T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M[g_{\mu\nu}, {\Gamma^\alpha}_{\mu\nu}]}{\delta g^{\mu\nu}}$,
  but also ${\Delta_\alpha}^{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M[g_{\mu\nu}, {\Gamma^\alpha}_{\mu\nu}]}{\delta {\Gamma^\alpha}_{\mu\nu}}$
- ${\Delta_\alpha}^{\mu\nu}$ is the hypermomentum tensor

Example

- Spinless particles (scalars) : $\nabla_\mu \phi \to \partial_\mu \phi$ (no Γ - dependence) $\Rightarrow {\Delta_\alpha}^{\mu\nu} \equiv 0$
- Spin $\iff$ torsion , ${\Gamma^\alpha}_{[\mu\nu]} \neq 0 \Rightarrow {\Delta_\alpha}^{\mu\nu} \neq 0$
- Note: Spin is not the only source of torsion!

Einstein-Hilbert Action

$S_{EH}[g_{\mu\nu}, \Gamma^{\lambda}{}_{\alpha\beta}] = \int d^n x \sqrt{-g} R = \int d^n x \sqrt{-g} g^{\mu\nu} R_{(\mu\nu)}$

Independent variations w.r.t. the metric and the connection give

$$R_{(\mu\nu)} - \frac{g_{\mu\nu}}{2} R = 0$$

$$-\frac{\nabla_\lambda(\sqrt{-g} g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\sigma(\sqrt{-g} g^{\mu\sigma})\delta^\nu_\lambda}{\sqrt{-g}} + 2(S_\lambda g^{\mu\nu} - S^\mu \delta^\nu_\lambda + g^{\mu\sigma} S_{\sigma\lambda}{}^{\nu}) = 0$$

The left hand side is denoted with $P_\lambda{}^{\mu\nu}$-Palatini tensor

End Result

After some manipulation of the field eqns$\rightarrow$ GR+ An unspecified Vectorial dof: $\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} - \frac{2}{(n-1)} S_\nu \delta^\lambda_\mu$

This can be gauged away by means of a projective transformation!

Projective transformations of the Connection

$$\Gamma^{\lambda}{}_{\mu\nu} \longrightarrow \Gamma^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{\mu}\xi_{\nu}$$

- The Ricci scalar is left invariant under projective transformations $R \to R$
- This implies $P_{\mu}{}^{\mu\nu} = 0(\text{identically}) \Longrightarrow$ Unspecified Vectorial dof
- Any f(R) action has this attribute and this causes problems when one tries to add matter!(More about it later on)

To conclude

$\tilde{R}_{\mu\nu} = \frac{\tilde{R}}{2} g_{\mu\nu}$, $Q_{\alpha\mu\nu} = \frac{Q_{\alpha}}{n} g_{\mu\nu}$, $S_{\mu\nu}{}^{\lambda} = \frac{2}{n-1} S_{[\mu}\delta^{\lambda}_{\nu]}$
with $Q^{\mu} = -\frac{4n}{(n-1)} S^{\mu} \Rightarrow$ Can be gauged away and finally
$Q_{\alpha\mu\nu} = 0 = S_{\alpha\mu\nu}$

Vacuum f(R) Theories

$S = \frac{1}{2\kappa} \int d^n x \sqrt{-g} f(R)$

Field Equations

$$f'(R) R_{(\mu\nu)} - \frac{f(R)}{2} g_{\mu\nu} = 0$$

$-\nabla_\lambda(\sqrt{-g} f' g^{\mu\nu}) + \nabla_\alpha(\sqrt{-g} f' g^{\mu\alpha} \delta^\nu_\lambda) + 2\sqrt{-g} f' (S_\lambda g^{\mu\nu} - S^\mu \delta^\nu_\lambda - S_\lambda{}^{\mu\nu}) = 0$

$\rightarrow$ Taking the trace of the first $f'(R) R - \frac{n}{2} f(R) = 0$

This is an algebraic equation in R and it will have number of solutions (except $f(R) = \alpha R^{n/2}$)[Ferraris]

End result

Theory $\Rightarrow$ A Class of Einstein Gravities + Cosmological Constant(s)

Adding Matter

$S = S_G + S_M = \frac{1}{2\kappa}\int d^n x\sqrt{-g} f(R) + \int d^n x \sqrt{-g}\mathcal{L}_M$

Field Equations:

$f'(R)R_{(\mu\nu)} - \frac{f(R)}{2} g_{\mu\nu} = \kappa T_{\mu\nu}$, $\quad P_\lambda{}^{\mu\nu}(h) = \kappa \Delta_\lambda{}^{\mu\nu}$

where $P_\lambda{}^{\mu\nu}(h)$ is computed wrt $h^{\mu\nu} = f'(R)g^{\mu\nu}$.

Now, since $P_\mu{}^{\mu\nu} = 0$ it follows that $\Delta_\mu{}^{\mu\nu} = 0 \Rightarrow$ Unreasonable constraint on matter fields!

Resolving the inconsistency

Must break projective invariance and fix a vectorial dof! **Proposals**

- Helh et al $Q_\mu = 0 \Rightarrow$ Works only for $f(R) = R$.
- Sotiriou and Liberati $S_\mu = 0 \Rightarrow$ Works for any $f(R)$ but excludes Vec. Tors.
- My Proposal $(\alpha S_\mu - \beta Q_\mu - n\gamma \tilde{Q}_\mu) = 0$ places Q, S on equal footing $\Rightarrow$ Works for any $f(R)$ +Bonus Feature

Special Case $f(R) = \alpha R^2$ in $4 - dim$

Consider the theory

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \alpha R^2$$

where our connection is non-metric but torsionless! The field equations are in this case

- $R_{(\mu\nu)} - \frac{R}{4} g_{\mu\nu} = 0$
- $\nabla_\alpha \left(R\sqrt{-g} g^{\mu\nu} \right) - \nabla_\beta \left(R\sqrt{-g} g^{\beta(\mu} \right) \delta_\alpha^{\nu)} = 0$

Solution of the system

$$\frac{\partial_\mu R}{R} = \frac{1}{4} Q_\mu \ , \ \ Q_{\lambda\mu\nu} = \frac{1}{4} g_{\mu\nu} Q_\lambda \ , \ \ R = \tilde{R} - \frac{3}{4} \tilde{\nabla}_\mu Q^\mu - \frac{3}{32} Q_\mu Q^\mu$$

Cosmological Solutions

For a flat FLRW Universe we have to solve the set of equations

$$(\dot{H}+H^2)+\frac{\dot{Q}}{8}+\frac{1}{8}HQ=\frac{C}{12}e^{\frac{1}{4}\int Qdt}\,,\quad \Big(Q=Q_0(t)\Big)$$
$$6(\dot{H}+2H^2)+\frac{3}{4}\dot{Q}+\frac{9}{4}HQ+\frac{3}{32}Q^2=Ce^{\frac{1}{4}\int Qdt}$$

This can be solved exactly, to yield

- $H(t)=H_0e^{\frac{1}{8}\int Qdt}-\frac{Q}{8}$
- $a(t)=a_0e^{\int\left[H_0e^{\frac{1}{8}\int Qdt}-\frac{Q}{8}\right]dt}$

(Toy model since $Q(t)=arbitrary!$)

For $Q=const.$

$$a(t)=a_0e^{\frac{8H_0}{Q}e^{\frac{Q}{8}t}-\frac{Q}{8}t}$$

Model with torsion

The same model $f(R) = \alpha R^2$ but with $S_{\mu\nu\alpha} \neq 0$, $Q_{\alpha\mu\nu} = 0$ was studied by Capozziello et al. Their solutions were

- $H(t) = H_0 e^{\frac{1}{3}\int T dt} - \frac{T}{3}$, (T=2S)
- $a(t) = a_0 e^{\int \left[H_0 e^{\frac{T}{3}\int Q dt} - \frac{T}{3} \right] dt}$

Duality

Looking back at our solution we see that one solution maps to another by exchanging

$$T \leftrightarrow \frac{3}{8} Q$$

- This duality can be seen in at least two more places

Ricci Scalars

The duality is also apparent when looking at the Ricci scalar decomposition

- $R = \tilde{R} - 2\tilde{\nabla}_\mu T^\mu - \frac{2}{3} T_\mu T^\mu \quad , \quad S_{\alpha\mu\nu} \neq 0 \;\; Q_{\alpha\mu\nu} = 0$
- $R = \tilde{R} - \frac{3}{4}\tilde{\nabla}_\mu Q^\mu - \frac{3}{32} Q_\mu Q^\mu \quad , \quad S_{\alpha\mu\nu} = 0 \;\; Q_{\alpha\mu\nu} \neq 0$

Autoparallels (For general n)

- $\ddot{x}^\alpha + \tilde{\Gamma}^\alpha{}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{2}{n-1} S^a \dot{x}^2 \quad , \quad S_{\alpha\mu\nu} \neq 0 \;\; Q_{\alpha\mu\nu} = 0$
- $\ddot{x}^\alpha + \tilde{\Gamma}^\alpha{}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2n} Q^a \dot{x}^2 \quad , \quad S_{\alpha\mu\nu} = 0 \;\; Q_{\alpha\mu\nu} \neq 0$

Duality for general dim

$$S_\mu \leftrightarrow \frac{(n-1)}{4n} Q_\mu$$

Mixed Model

If we assume that both $S_{\alpha\mu\nu} \neq 0, Q_{\alpha\mu\nu} \neq 0$ the solution is now

- $H = H_0 e^{\frac{1}{2}\int w dt} - \frac{w}{2}$
- $a(t) = a_0 e^{\int \left[H_0 e^{\frac{1}{2}\int w dt} - \frac{w}{2} \right] dt}$, where $w_\mu := \frac{1}{n} Q_\mu + \frac{4}{n-1} S_\mu = \frac{\partial_\mu R}{R}$

Affine connection

The form of the affine connection is

- $\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + \frac{1}{2}\left(\delta^\lambda_\nu w_\mu - g_{\mu\nu} w^\lambda\right)$
- Note: It may just so happen that $w_\mu = 0$ but with $S_\mu \neq 0$, $Q_\mu \neq 0$
- In this case torsion and non-metricity cancel each other and we are effectively left with a Riemannian space!

Theorems

Theorem 1: Consider the action

$$S[g_{\mu\nu},\Gamma^{\lambda}{}_{\alpha\beta},\phi]=\frac{1}{2\kappa}\int d^n x\sqrt{-g}R+S_1[g_{\mu\nu},\Gamma^{\lambda}{}_{\alpha\beta},\phi]$$

where ϕ denotes any other additional fields and $S_1[g_{\mu\nu},\Gamma^{\lambda}{}_{\alpha\beta},\phi]=\int d^n x\sqrt{-g}\mathcal{L}_1(g,\Gamma,\phi)$.
Now given any general action $S_1[g,\Gamma,\phi]$ that is

- At most linear in $\Gamma^{\lambda}{}_{\mu\nu}$ and its partial derivatives
- Projective invariant

Expression for Affine Connection

Affine connection $\Rightarrow$ can then be explicitly solved:
$\Gamma^{\lambda}{}_{\mu\nu}=\tilde{\Gamma}^{\lambda}{}_{\mu\nu}-\frac{g^{\lambda\alpha}}{2}(B_{\alpha\mu\nu}-B_{\nu\alpha\mu}-B_{\mu\nu\alpha})-\frac{g^{\alpha\lambda}}{(n-2)}g_{\nu[\mu}(B_{\alpha]}-\tilde{B}_{\alpha]})$
where
$B_{\lambda}{}^{\mu\nu}:=\frac{2\kappa}{\sqrt{-g}}\frac{\delta S_1}{\delta\Gamma^{\lambda}{}_{\mu\nu}}$ and $B^{\mu}:=B_{\lambda}{}^{\mu\lambda}$, $\tilde{B}^{\mu}:=g_{\alpha\beta}B^{\mu\alpha\beta}$.

Comment 1: The projective invariance of S_1 is only necessary in order to remove the term $\frac{1}{2}\delta^{\lambda}_{\mu}\tilde{Q}_{\nu}$ from Γ. If S_1 does not respect projective invariance we just add this term.

Comment 2: If there is no gravitational sector to S_1, i.e the latter is a purely matter action $S_1 = S_M$ then $B_{\lambda}{}^{\mu\nu} = -\kappa\Delta_{\lambda}{}^{\mu\nu}$ and the connection is found to be

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \kappa\frac{g^{\lambda\alpha}}{2}(\Delta_{\alpha\mu\nu} - \Delta_{\nu\alpha\mu} - \Delta_{\mu\nu\alpha}) + \frac{g^{\alpha\lambda}}{(n-2)} g_{\nu[\mu}(\Delta_{\alpha]} - \tilde{\Delta}_{\alpha]})$$

where $\Delta^{\mu} := \Delta_{\lambda}{}^{\mu\lambda}$, $\tilde{\Delta}^{\mu} := g_{\alpha\beta}\Delta^{\mu\alpha\beta}$. Then $\Rightarrow$ The connection lacks dynamics!

Example

Consider the Theory

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + \frac{1}{2\kappa}\int d^4x F(x)\epsilon^{\mu\nu\rho\sigma}\partial_\mu S_{\nu\rho\sigma}$$

where $F(x)$ is a pseudo-scalar.

Connection Solution

Applying the results of our Theorem $\Rightarrow$

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\lambda}\partial_\rho F$$

Theorems

Theorem 2: Let

$$S[g_{\mu\nu}, \Gamma^{\lambda}{}_{\alpha\beta}, \phi] = \frac{1}{2\kappa}\int d^n x \sqrt{-g} f(R) + S_1[g_{\mu\nu}, \Gamma^{\lambda}{}_{\alpha\beta}, \phi]$$

Now given any general matter action $S_1[g, \Gamma, \phi] = S_M$ that is

- At most linear in $\Gamma^{\lambda}{}_{\mu\nu}$ and its partial derivatives
- Projective invariant

Connection Solution

Affine connection can be expressed in terms of $f^{'}(T)$ and $\Delta_{\lambda}{}^{\mu\nu}$ and its form is the following

$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \frac{g^{\lambda\alpha}}{2}(H_{\alpha\mu\nu} - H_{\nu\alpha\mu} - H_{\mu\nu\alpha}) + \frac{g^{\alpha\lambda}}{(n-2)} g_{\nu[\mu}(H_{\alpha]} - \tilde{H}_{\alpha]})$

where $H_{\lambda}{}^{\mu\nu} := \frac{\kappa}{f'}\Delta_{\lambda}{}^{\mu\nu} + \frac{1}{f'}(g^{\mu\nu}\partial_{\lambda}f^{'} - \delta^{\nu}_{\lambda}\partial^{\mu}f^{'})$ and $H^{\mu} := H_{\lambda}{}^{\mu\lambda}$, $\tilde{H}^{\mu} := g_{\alpha\beta}H^{\mu\alpha\beta}$, $f' = f'(T)$, $T := g^{\mu\nu}T_{\mu\nu}$

Note:

For Palatini $f(R) \Rightarrow T_{\mu\nu}$ independent of Γ

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \frac{1}{(n-2)f'}\left(\delta^{\lambda}_{\nu}\partial_{\mu}f' - g_{\mu\nu}\partial^{\lambda}f'\right)$$

$\Rightarrow$ Connection lacks dynamics!

Theorems

Theorem 3: Generalization of Theorems 1 and 2 but now S_1 has arbitrary dependence on $\Gamma \Rightarrow$ Dynamical Connection in General!

Comment

The three Theorems can be used to find the dynamical content of connections in MAG and classify Theories

$1+(n-1)$ Spacetime split

Let $u^{\mu}=\frac{dx^{\mu}}{d\lambda}$ be a tangent vector ($4-velocity$ for $n=4$) to a curve $\mathcal{C}$. Then

$$u_{\mu}u^{\mu}\neq -1 \quad but \quad u_{\mu}u^{\mu}=-l^{2}(x)$$

since any vector's length changes because of non-metricity. This suggests that acceleration and velocity are no longer perpendicular.

Projection Tensor

The naive generalization

- $h_{\mu\nu}=g_{\mu\nu}+u_{\mu}u_{\nu}$ does not work here since $h_{\mu\nu}u^{\mu}\neq 0$. Proper definition
- $h_{\mu\nu}=g_{\mu\nu}+\frac{u_{\mu}u_{\nu}}{l^{2}}$ which now satisfies $h_{\mu\nu}u^{\mu}=0=h_{\mu\nu}u^{\nu}$ and $h_{\mu\nu}h^{\mu\nu}=n-1$ as can be easily checked

The Setup

Projections along $'time'$ and spatial space

- $\dot{T}_{\alpha_1 ... \alpha_n}{}^{\beta_1 ... \beta_m} = u^{\mu} \nabla_{\mu} T_{\alpha_1 ... \alpha_n}{}^{\beta_1 ... \beta_m}$
- $D_{\mu} T_{\alpha_1 ... \alpha_n}{}^{\beta_1 ... \beta_m} = h_{\mu}^{\lambda} h_{\alpha_1}^{\gamma_1} ... h_{\alpha_n}^{\gamma_n} h_{\delta_1}^{\beta_1} ... h_{\delta_m}^{\beta_m} \nabla_{\lambda} T_{\gamma_1 ... \gamma_n}{}^{\delta_1 ... \delta_m}$

Two kinds of Accelerations

→Path Acceleration

- $A^{\mu} \equiv \dot{u}^{\mu} \equiv u^{\lambda} \nabla_{\lambda} u^{\mu}$ ($g_{\mu\nu}$ does not commute with ∇_{α})

→Hyper Acceleration

- $a_{\mu} \equiv \dot{u}_{\mu} \equiv u^{\lambda} \nabla_{\lambda} u_{\mu}$

Note

The former vanishes for autoparallel motion while the latter does not! The two are related through $A^{\mu} = a^{\mu} + Q^{\lambda\mu\nu} u_{\lambda} u_{\nu}$

Expansion, Shear, Vorticity and the rest

- $\Theta \equiv g^{\mu\nu}\nabla_{\mu}u_{\nu}$, $\Theta_D \equiv g^{\mu\nu}D_{\mu}u_{\nu} = \Theta + \frac{a\cdot u}{l^2}$
- $\omega_{\nu\mu} \equiv D_{[\mu}u_{\nu]}$
- $\sigma_{\nu\mu} \equiv D_{<\mu}u_{\nu>} \equiv D_{(\mu}u_{\nu)} - \frac{(h^{\alpha\beta}D_{\alpha}u_{\beta})}{n-1}h_{\mu\nu}$
- $\nabla_{\mu}u_{\nu} = \omega_{\nu\mu} + \sigma_{\nu\mu} + \left(\Theta + \frac{(a\cdot u)}{l^2}\right)\frac{h_{\mu\nu}}{n-1} - \frac{\xi_{\mu}u_{\nu}+u_{\mu}a_{\nu}}{l^2} - \frac{u_{\mu}u_{\nu}(a\cdot u)}{l^4}$

where $\xi_{\mu} \equiv u^{\alpha}\nabla_{\mu}u_{\alpha}$ and $(a\cdot u) = a_{\mu}u_{\nu}g^{\mu\nu}$

Note

- $\Theta \neq \nabla_{\mu}u^{\mu}$, $\nabla_{\mu}u^{\mu} = \Theta + u^{\mu}\tilde{Q}_{\mu}$
- $\Theta = \tilde{\Theta} + \Big(-\tilde{Q}_{\mu} + 1/2Q_{\mu} + 2S_{\mu}\Big)u^{\mu}$, $\tilde{\Theta} = \tilde{\nabla}_{\mu}u^{\mu}$

The Raychaudhuri Equation

Starting by Ricci's identity and using the above definitions, we find

$$\dot{\Theta}_D + \frac{\Theta_D^2}{n-1} = -R_{\mu\nu}u^\mu u^\nu - \sigma^2 + \omega^2 + g^{\mu\nu}\nabla_\mu a_\nu$$
$$+\frac{d}{d\lambda}\left(\frac{a\cdot u}{l^2}\right) + \frac{(a\cdot u)^2}{l^4} + 2\frac{(a\cdot \xi)}{l^2} + u_\alpha Q^{\alpha\beta\mu}\nabla_\beta u_\mu - u_\alpha Q^{\mu\nu\alpha}\nabla_\nu u_\mu$$
$$+u^\mu u^\beta\left(g^{\nu\alpha}\nabla_{[\alpha}Q_{\beta]\mu\nu} - S_{\alpha\beta}{}^{\lambda}Q_{\lambda\mu}{}^{\alpha}\right) + 2u_\alpha S^{\alpha\mu\nu}\nabla_\nu u_\mu$$

Comments

- The first line is formalistically the same with the standard one!
- All the extra terms depend on non-metricity except from the last one which depends on torsion!

Most General Raychaudhuri Eq. (Irreducible Form)

$$\dot{\Theta} = -\frac{1}{n-1}\,\Theta^2 - R_{\mu\nu}u^\mu u^\nu - 2\left(\sigma^2 - \omega^2\right) + \mathrm{D}^\mu a_\mu + \frac{1}{\ell^2}\,a_\mu A^\mu$$
$$+\frac{2}{n-1}\left(\Theta + \frac{1}{\ell^2}\,a_\nu u^\nu\right) S_\mu u^\mu + 2S_{\mu\nu\lambda}u^\mu(\sigma^{\nu\lambda} + \omega^{\nu\lambda}) + \frac{2}{\ell^2}\,S_{\mu\nu\lambda}a^\mu u^\nu u^\lambda$$
$$-\frac{1}{\ell^2}(a_\mu u^\mu)^{\cdot} - \frac{2\Theta}{\ell^2(n-1)}\,a_\mu u^\mu + \frac{n-2}{\ell^4(n-1)}(a_\mu u^\mu)^2 + \frac{2}{\ell^2}\,a_\mu \xi^\mu - \dot{\tilde{Q}}_\mu u^\mu$$
$$+\frac{1}{n-1}\left(\Theta + \frac{1}{\ell^2}\,a_\nu u^\nu\right)(Q_\mu - \tilde{Q}_\mu)u^\mu - Q_{\mu\nu\lambda}(\sigma^{\mu\nu} + \omega^{\mu\nu})u^\lambda$$
$$-\frac{1}{\ell^2}\,Q_{\mu\nu\lambda}u^\mu u^\nu(a^\lambda + \xi^\lambda) + Q_{\mu\nu\lambda}u^\mu\sigma^{\nu\lambda} + \frac{1}{\ell^2}\,Q_{\mu\nu\lambda}(u^\mu\xi^\nu + a^\mu u^\nu)u^\lambda$$
$$+u^\mu u^\nu \nabla^\lambda Q_{\mu\nu\lambda} + Q_\mu{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu + 2S_\mu{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu\,.$$

Note: Only the terms in the first line on the right-hand side of the above have Riemannian analogues.

Cosmological Solution for Vectorial Torsion (and $Q_{\alpha\mu\nu} = 0$)

For a vectorial torsion of the form $S_{\mu\nu}{}^{\lambda} = \frac{2}{n-1} S_{[\mu}\delta^{\lambda}_{\nu]}$, assuming a flat and empty FLRW universe and an autoparallel motion ($A^{\mu} = 0$)we obtain

$$\dot{\Theta} + \frac{\Theta^2}{3} = \frac{2}{3}(u^{\mu} S_{\mu})\Theta$$

This can be solved for generic $S_{\mu} = \delta^0_{\mu} S_0(t)$ and the solution reads

- $a(t) = e^{-\frac{2}{3}\int S_0(t)dt}\left[C_1 + C_0 \int e^{\frac{2}{3}\int S_0(t)dt} dt\right]$

Cosmological Solution for Weyl non-metricity (and $S_{\alpha\mu\nu} = 0$)

Upon the same assumptions but now Weyl non-metricity and vanishing torsion, finally we get

- $a(t) = e^{-\frac{1}{8}\int Q_0(t)dt}\left[C_1 + C_0 \int e^{\frac{1}{8}\int Q_0(t)dt} dt\right]$

The duality appears again!

Collecting the two solutions

- $a(t)=e^{-\frac{2}{3}\int S_0(t)dt}\left[C_1+C_0\int e^{\frac{2}{3}\int S_0(t)dt}dt\right],(S\neq 0\ \ Q=0)$
- $a(t)=e^{-\frac{1}{8}\int Q_0(t)dt}\left[C_1+C_0\int e^{\frac{1}{8}\int Q_0(t)dt}dt\right],(S=0\ \ Q\neq 0)$

We observe an astonishing result! The two solutions look similar, in fact one maps to another by interchanging $S_\mu \longleftrightarrow \frac{3}{16}Q_\mu$. This is the same duality we saw earlier!

Conclusion: Vectorial Torsion $\leftrightarrow$ Weyl Non-Metricity

Simple Explanation

Proposition: A projective invariant Theory with vectorial torsion (and $Q=0$) is equivalent to a pr.inv. Theory with Weyl-non-metricity (and S=0). The proposition is proved in the Thesis!

Conclusions

- We proposed another way to break Proj. Inv. in Metric-Affine $f(R)$ Theories
- The duality of Torsion/non-Metricity was studied in $f(R) = \alpha R^2$/ Cosmological Solutions
- The key point and the map of the duality was found
- Theorems for the Affine-Connection
- We presented (for the first time) the Raychaudhuri Equation in spaces with both torsion and non-metricity
- We found Cosmological solutions for solely torsion and solely non-metricity and the duality reappeared!

...Thank you!!!