Introduction to topological quantum field theory

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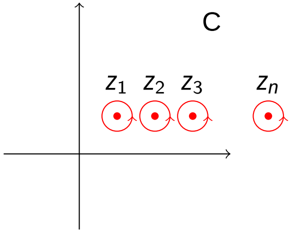
December 19, 2019

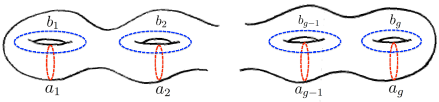
Overview

1. Topology and quantum field theory
2. Topological quantum field theory
3. The structure in 2d: Frobenius algebras

Topology

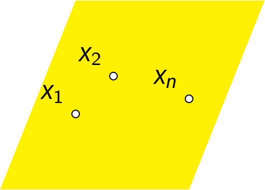
- Topology classifies geometrical objects with respect to properties that are preserved under continuous deformations.
- Examples:
  - Dimension
  - Number of \holes"
  - Number of \handles": *genus*





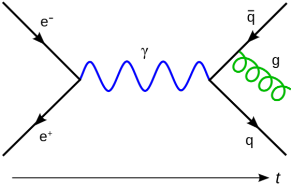
Quantum field theory

- Algebraic point of view: Quantum states & correlation functions

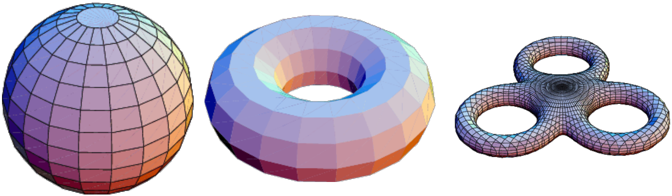


$hO(x_1) \quad O(x_n)i$

=) Amplitudes:

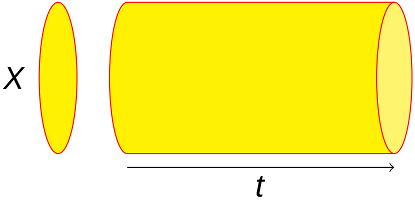


- Geometric point of view: non-trivial spacetimes & global features (= topological invariants)



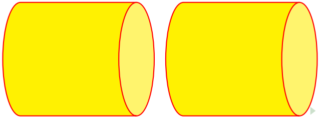
Quantum field theory & geometry

- Spacelike physical space X: Associated Hilbert space H_X
- Properties of H_X depend very much on the geometry of X.
- Time evolution: a cylinder that connects *same* spaces



- Locality: can "chop" time in small pieces and recover full quantum theory. E.g. in quantum mechanics:

$$U_t = e^{\frac{iHt}{\hbar}}, \quad U_{t_1+t_2} = U(t_2) \quad U(t_1)$$



Towards a mathematical definition

- Axiomatize QFT: Understand its central features
- Spacetime & properties *!*
  physical states, Hilbert spaces, linear operators

Abstractly

QFT : Geometry *!* Algebra

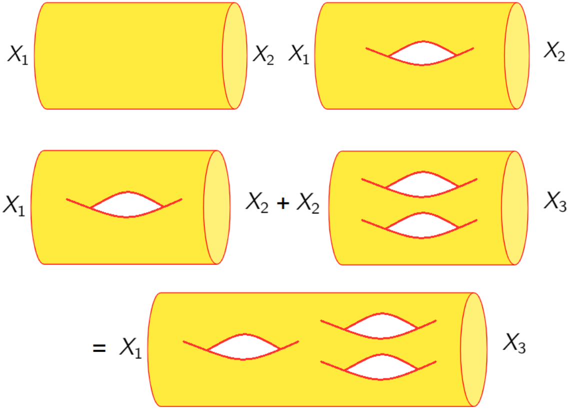
- In general very hard! Geometry includes a lot of structure.
- When geometry=topology, becomes more clear: TQFT!
- Topological quantum field theory: Space of states does not depend on the *metric*. Only the "shape" matters!

Topological QFT: the basic features

- To every different “shape” $X \mapsto$ Hilbert space of states H_X
- Disconnected spaces $X_1 \sqcup X_2 \mapsto$ Tensor product of Hilbert spaces $H_{X_1} \otimes H_{X_2}$
- No space $\emptyset \mapsto$ Complex numbers $\mathbb{C}$

Topological QFT: the basic features

- Spacetimes connecting X_1 and X_2 (cobordisms): think about them as functions sending space X_1 to space X_2
  - Linear maps H_{X_1} ! H_{X_2}: \generalized time evolution"
  - Gluing spacetimes: Composition of linear maps



Topological QFT: the rules

- Simplest case: X is 1-dimensional and closed, a *circle*!
- We consider 2d surfaces connecting circles

A 2d closed TQFT is a *rule*, which associates a vector space to every circle and a linear operator to every cobordism. Axioms:

Topological QFT: the rules

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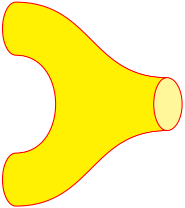
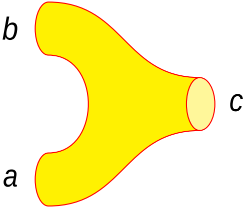
1. $\emptyset \mapsto H_{\emptyset} = \mathbb{C}|0\rangle$
2. $S^1 \mapsto H_{S^1}$ = space of states
3. $S^1 \sqcup S^1 \mapsto H_{S^1 \sqcup S^1} = H_{S^1} \otimes H_{S^1}$
4. Cylinder $\mapsto$ Identity operator $\mathbb{1}$
5. Gluing compatible surfaces $\mapsto$ Composition of linear operators

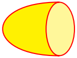
Some useful examples

- A *pair of pants*: 
  - Cobordism $S^1 \sqcup S^1 \to S^1$.
  - Linear map $H \otimes H \to H$: multiplication
  - Think of every surface as some *operation on vectors*



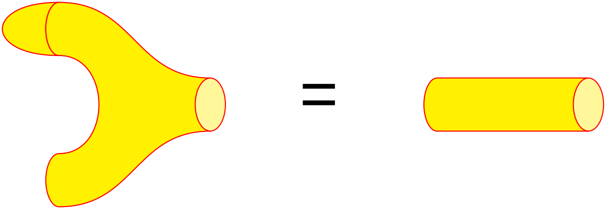
Extra structure! Algebra (vector space with multiplication)

A unit

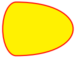
A cup: 

Cobordism? ! S^1.

Linear map$\mathbb{C}$! H : unit

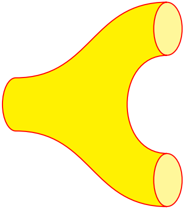


More operations

A cap: 

Cobordism S^1 ! ? .

Linear map H ! $\mathbb{C}$: counit



A pair of pants:

Cobordism S^1 ! S^1 t S^1.

Linear map H ! H H : comultiplication

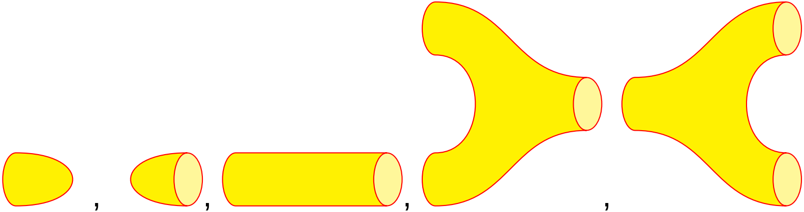
Algebra & coalgebra

Towards Frobenius algebras

Recall: every surface represents some operation

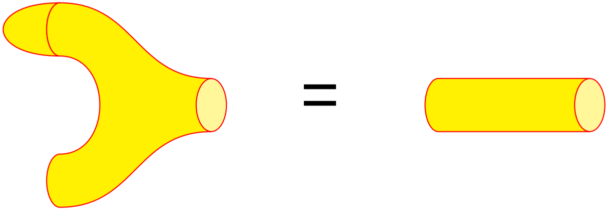
All intuitive rules from TQFT translate into algebraic formulas

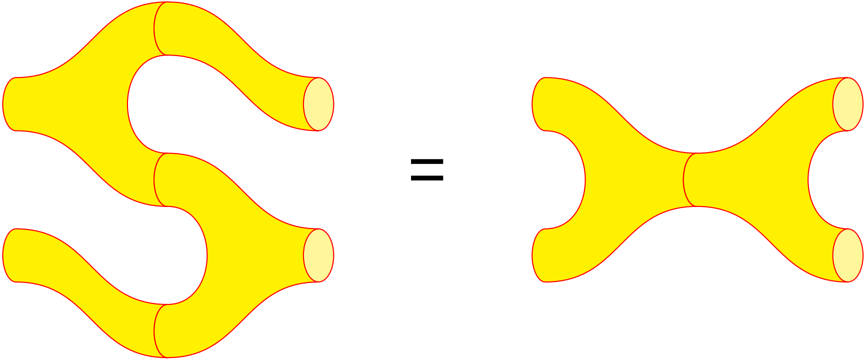
A theorem: Any 2d surface can be decomposed into element
generators



Frobenius algebra

Frobenius algebra





and more...

Why study TQFT?

- Mathematically well-defined
- Exact results for sectors of (susy) QFT
- Toy model to study topological properties of QFT
- Deep mathematics

Summary and current research

- A QFT assigns algebraic structures to geometric structures
- Classification: the data of a 2d closed TQFT are equivalent to the structure of a Frobenius algebra
- Goal: classify more complicated TQFT's (higher dimensions, more structure)

Thank you for your attention!

When the first conference speaker opens with:
'Here a quantum field theory is given by a monoidal functor

$$Z : \mathrm{Bord}_d^S \to \mathrm{Vect}$$

from a suitable monoidal category of cobordisms to a suitable monoidal category of vector spaces.'