

How far can we push the AdS/Ricci-flat correspondence?

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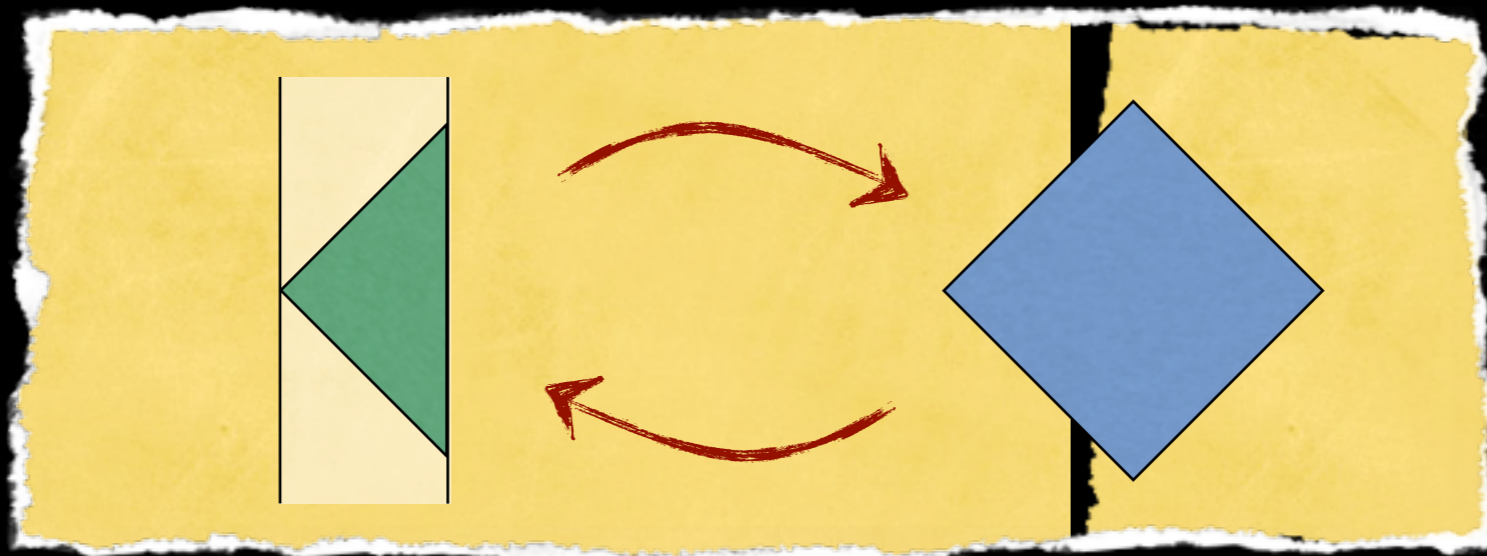
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- ✧ **Gravity is believed to be holographic:** it should be described by a non-gravitational theory in one dimension less
't Hooft '93, Susskind '94
- ✧ **This is well understood for asymptotically anti-de Sitter spacetimes:** AdS/CFT correspondence
Maldacena '97, Gubser Klebanov Polyakov '98, Witten '98, ...
- ✧ **Original arguments for holography are insensitive to asymptotics**
- ✧ **Decoupling argument extends to nonconformal branes** (non-trivial dilaton & non-AdS asymptotics) Kanitscheider et al '08
Wiseman & Withers '08
 - obtained from AdS via a **generalized dimensional reduction**
 - Holographic dictionary **inherited from AdS** Kanitscheider & Skenderis '09

I will review a generalized dimensional reduction linking **Ricci-flat** and **AdS** solutions, aiming at formulating holography for AF spacetimes.

AdS/Ricci-flat correspondence



~ a map linking AdS gravity and vacuum Einstein gravity ~

A map relating AdS and Ricci-flat solutions

MC, Camps, Goutéraux & Skenderis '12

1. Solutions to **AdS gravity** in $d+1$ dimensions of the form:

$$ds_{\Lambda}^2 = d\hat{s}_{p+2}^2(x) + e^{\frac{2\phi(x)}{d-p-1}} d\vec{y}_{d-p-1}^2$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$
$$\Lambda = -\frac{d(d-1)}{2\ell^2}$$

2. Extract $(p+2)$ -dim metric $\hat{g}(x)$ and the scalar $\phi(x)$

3. Substitute $d \rightarrow -n$ in $\hat{g}(x)$ and $\phi(x)$

4. Insert back in $ds_0^2 = e^{\frac{2\phi(x)}{n+p+1}} \left(d\hat{s}_{p+2}^2(x) + \ell^2 d\Omega_{n+1}^2 \right)$

unit S^{n+1}



Then, the metric ds_0^2 is **Ricci-flat** $\tilde{R}_{\mu\nu} = 0$

It solves **vacuum Einstein** equations in $(n+p+3)$ dimensions

Trading curvatures: from AdS to Ricci-flat

AdS gravity

$$D = d + 1$$

$$S = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} (R - 2\Lambda)$$

Reduction on \mathcal{T}^{d-p-1}

$$ds_{\Lambda}^2 = d\hat{s}_{p+2}^2 + e^{\frac{2\phi(x)}{d-p-1}} d\vec{y}_{d-p-1}^2$$

$$\alpha = \frac{d-p-2}{d-p-1}$$

$$\beta = -2\Lambda$$

$$\hat{D} = p + 2$$

$$\hat{S} = \frac{1}{16\pi \hat{G}_N} \int_{\mathcal{M}} d^{p+2}x \sqrt{-\hat{g}} e^{\phi} \left(\hat{R} + \alpha (\partial\phi)^2 + \beta \right)$$

Trading curvatures: from AdS to Ricci-flat

AdS gravity

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Reduction on \mathcal{T}^{d-p-1}

$$ds_{\Lambda}^2 = d\hat{s}_{p+2}^2 + e^{\frac{2\phi(x)}{d-p-1}} d\vec{y}_{d-p-1}^2$$

$$\alpha = \frac{d-p-2}{d-p-1}$$

$$\beta = -2\Lambda$$

Vacuum Einstein gravity

$$\tilde{D} = n + p + 3$$

$$\tilde{S} = \frac{1}{16\pi \tilde{G}_N} \int_{\mathcal{M}} d^{n+p+3}x \sqrt{-\tilde{g}} \tilde{R}$$

Reduction on \mathcal{S}^{n+1}

$$ds_0^2 = e^{\frac{2\phi(x)}{n+p+1}} (d\hat{s}_{p+2}^2 + \ell^2 d\Omega_{n+1}^2)$$

$$\alpha = \frac{n+p+2}{n+p+1}$$

$$\beta = \mathcal{R}_{\mathcal{S}^{n+1}}$$

$$\hat{D} = p + 2$$

$$\hat{S} = \frac{1}{16\pi \hat{G}_N} \int_{\mathcal{M}} d^{p+2}x \sqrt{-\hat{g}} e^{\phi} \left(\hat{R} + \alpha (\partial\phi)^2 + \beta \right)$$

Trading curvatures: from AdS to Ricci-flat

AdS gravity

$$D = d + 1$$

$$S = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} (R - 2\Lambda)$$

Vacuum Einstein gravity

$$\tilde{D} = n + p + 3$$

$$\tilde{S} = \frac{1}{16\pi \tilde{G}_N} \int_{\mathcal{M}} d^{n+p+3}x \sqrt{-\tilde{g}} \tilde{R}$$

Reduction on \mathcal{T}^{d-p-1}

$$ds_{\Lambda}^2 = d\hat{s}_{p+2}^2 + e^{\frac{2\phi(x)}{d-p-1}} d\vec{y}_{d-p-1}^2$$

Reduction on \mathcal{S}^{n+1}

$$ds_0^2 = e^{\frac{2\phi(x)}{n+p+1}} (d\hat{s}_{p+2}^2 + \ell^2 d\Omega_{n+1}^2)$$

$$\hat{D} = p + 2$$

$$\alpha = \frac{d - p - 2}{d - p - 1}$$

$$\beta = -2\Lambda$$

$$\hat{S} = \frac{1}{16\pi \hat{G}_N} \int_{\mathcal{M}} d^{p+2}x \sqrt{-\hat{g}} e^{\phi} \left(\hat{R} + \alpha (\partial\phi)^2 + \beta \right)$$

$$\alpha = \frac{n + p + 2}{n + p + 1}$$

$$\beta = \mathcal{R}_{\mathcal{S}^{n+1}}$$

$$d \leftrightarrow -n$$

$$-2\Lambda \leftrightarrow \mathcal{R}_{\mathcal{S}^{\tilde{n}+1}}$$

Dimension **d** (and **n**) enters analytically as a parameter in the equations of motion

Some remarks

1. Requires knowing the solution **for any d** (or n): we are mapping **families of AdS solutions to families of Ricci-flat solutions**
2. Analytical continuation $d \rightarrow -n$ on the lower dimensional theory: d and n should not be thought of as spacetime dimensions
3. This is an example of *Generalized Dimensional Reduction*
Kanitscheider & Skenderis '09 - Goutéraux, Smolic, Smolic, Skenderis & Taylor '11 - Goutéraux & Kiritsis '11
4. We are trading the curvature of AdS with the curvature of the sphere ($-2\Lambda \leftrightarrow \mathcal{R}_{S^{\tilde{n}+1}}$)
5. Extensions with other compactifications/cosmological constants
e.g. **AdS/dS** correspondence Di Dato & Fröb '14

The resulting Ricci-flat class of solutions has an underlying holographic structure and hidden conformal symmetry inherited from the locally asymptotically AdS class of solutions.

Some simple examples

~ what happens to simple known solutions under this map? ~

First example: AdS_{d+1} on a Torus

1. AdS spacetime in $d+1$ dimensions:

$$ds_{\Lambda}^2 = \frac{\ell^2}{r^2} (dr^2 + \eta_{ab} dx^a dx^b + d\vec{y}_{\mathcal{T}^{d-p-1}}^2)$$

2. Extract the metric and scalar:

$$ds_{\Lambda}^2 = d\hat{s}_{p+2}^2 + e^{\frac{2\phi}{d-p-1}} d\vec{y}_{d-p-1}^2 \Rightarrow \begin{cases} d\hat{s}_{p+2}^2 = \frac{\ell^2}{r^2} (dr^2 + \eta_{ab} dx^a dx^b) \\ \phi(x) = -(d-p-1) \ln \frac{r}{\ell} \end{cases}$$

3. Substitute $d \rightarrow -n$

$$\Rightarrow \begin{cases} d\hat{s}_{p+2}^2 = \frac{\ell^2}{r^2} (dr^2 + \eta_{ab} dx^a dx^b) \\ \phi(x) = (n+p+1) \ln \frac{r}{\ell} \end{cases}$$

4. Lift to $n+p+3$ dimensions:

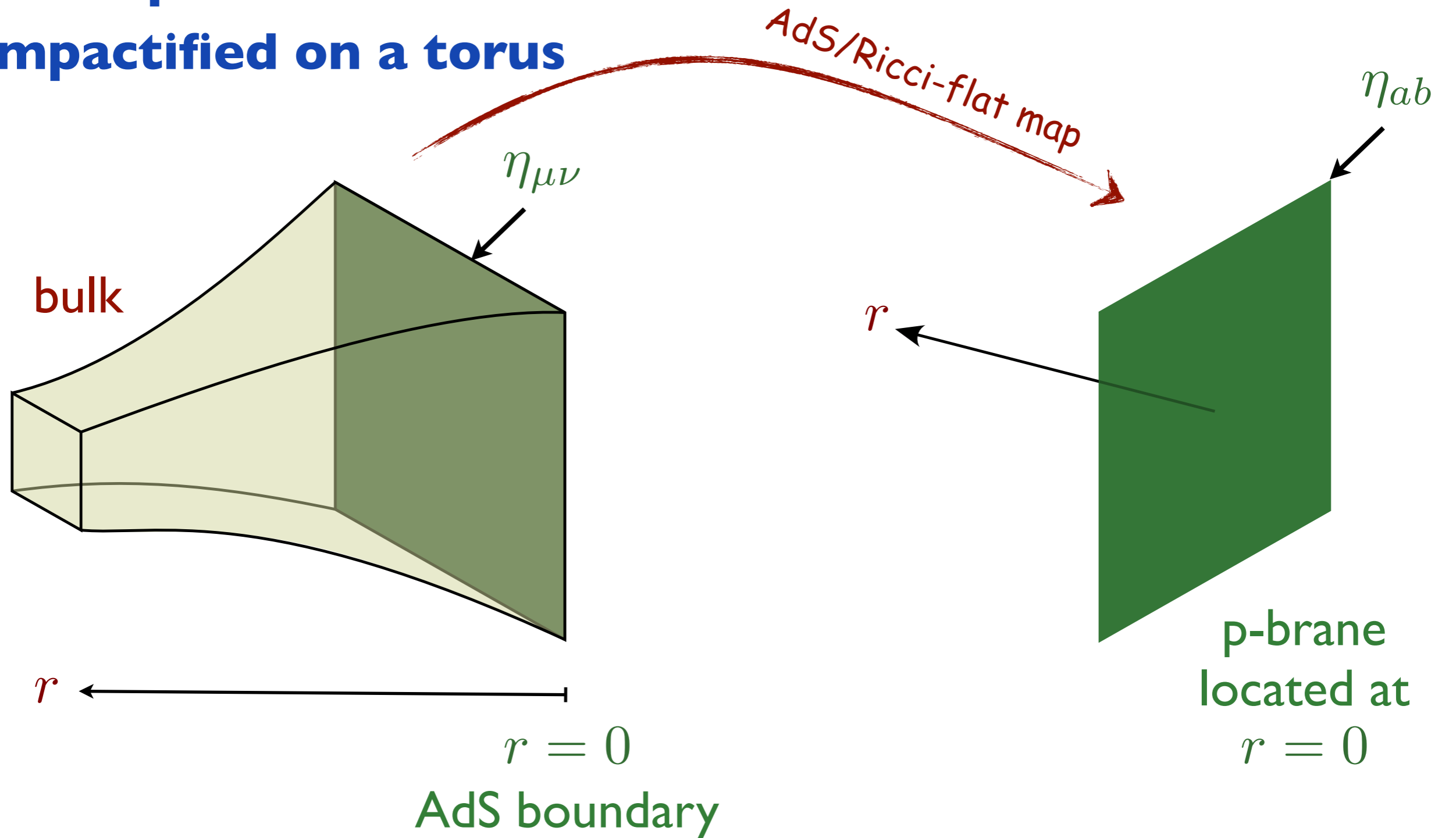
$$ds_0^2 = e^{\frac{2\phi}{n+p+1}} (d\hat{s}_{p+2}^2 + \ell^2 d\Omega_{n+1}^2)$$

$$\Rightarrow ds_0^2 = \underbrace{\eta_{ab} dx^a dx^b}_{\mathbb{R}^{1,p}} + \underbrace{dr^2 + r^2 d\Omega_{n+1}^2}_{\mathbb{R}^{n+2}}$$

**Minkowski
in $n+p+3$ dim.**

First example: AdS_{d+1} on a Torus

**AdS spacetime
compactified on a torus**



**Minkowski
spacetime**

Second example: **Excitations on top of AdS**

I. Fefferman-Graham coordinates for Einstein-AdS solutions: $(\rho = r^2)$

$$ds_{\Lambda}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \left(\eta_{\mu\nu} + \rho^{d/2} g_{(d)\mu\nu} + \dots \right) dz^{\mu} dz^{\nu}$$

Flat boundary metric

$$T_{\mu\nu} = \frac{d}{16\pi G_N} g_{(d)\mu\nu},$$

Expectation value of
the dual stress tensor

The stress tensor satisfies:

$$\partial^a T_{ab} = 0, \quad T_a{}^a = 0$$

as a consequence of the gravitational field equations
(Ward identities for the CFT on flat background)

Second example: **Excitations on top of AdS**

1. Fefferman-Graham coordinates for asymptotically AdS solutions:

$$ds_{\Lambda}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \left(\eta_{\mu\nu} + \rho^{d/2} g_{(d)\mu\nu} + \dots \right) dz^{\mu} dz^{\nu}$$

Flat boundary metric  compactify (d-p-1) of these flat directions 

2. Reduced theory: $d\hat{s}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \left(\eta_{ab} + \rho^{d/2} (\hat{g}_{(d)ab} + \rho \hat{g}_{(d+2)ab} + \dots) \right) dx^a dx^b$

$$\phi = \rho^{d/2} \hat{\phi}_{(d)} + \rho^{d/2+1} \hat{\phi}_{(d+2)} + \dots$$

Holographic dictionary for nonconformal branes:

Kanitscheider & Skenderis '09

$$\hat{T}_{ab} = \frac{d}{16\pi \hat{G}_N} \hat{g}_{(d)ab}, \quad \hat{\mathcal{O}}_{\phi} = -\frac{d(d-p-1)}{32\pi \hat{G}_N} \hat{\phi}_{(d)}$$

expectation values of the dual stress energy tensor and of the scalar operator

Ward identities: $\partial^a \hat{T}_{ab} = 0, \quad \hat{T}_a^a = (d-p-1) \hat{\mathcal{O}}_{\phi}$

the expectation value of the scalar operator breaks conformal invariance

Second example: **Excitations on top of AdS**

3. & 4. Analytical continuation and uplift to $n+p+3$ dimensions: $(\rho = 1/r^2)$

$$\begin{aligned}
 ds_0^2 &= \left(1 - \frac{16\pi\hat{G}_N}{n r^n} \left(1 + \frac{r^2}{2(n-2)} \square \right) \hat{\mathcal{O}}_\phi(x) \right) (dr^2 + \eta_{ab} dx^a dx^b + r^2 d\Omega_{n+1}^2) \\
 &\quad - \frac{16\pi\hat{G}_N}{n r^n} \left(1 + \frac{r^2}{2(n-2)} \square \right) \hat{T}_{ab}(x) dx^a dx^b + \dots \\
 &= (\eta_{AB} + h_{AB} + \dots) dx^A dx^B
 \end{aligned}$$

As a perturbation of flat spacetime it verifies:

$$\bar{h}_{AB} = h_{AB} - \frac{h}{2} \eta_{AB} \qquad \square \bar{h}_{AB} = 16\pi\hat{G}_N \Omega_{n+1} \delta_A^a \delta_B^b \hat{T}_{ab} \delta^{n+2}(r)$$

i.e. it solves linearized Einstein eqns $\square \bar{h}_{AB} = -16\pi\tilde{G}_N \tilde{T}_{AB}$

with
$$\tilde{T}_{ab} = -\frac{\hat{G}_N}{\tilde{G}_N} \Omega_{n+1} \hat{T}_{ab} \delta^{n+2}(r)$$

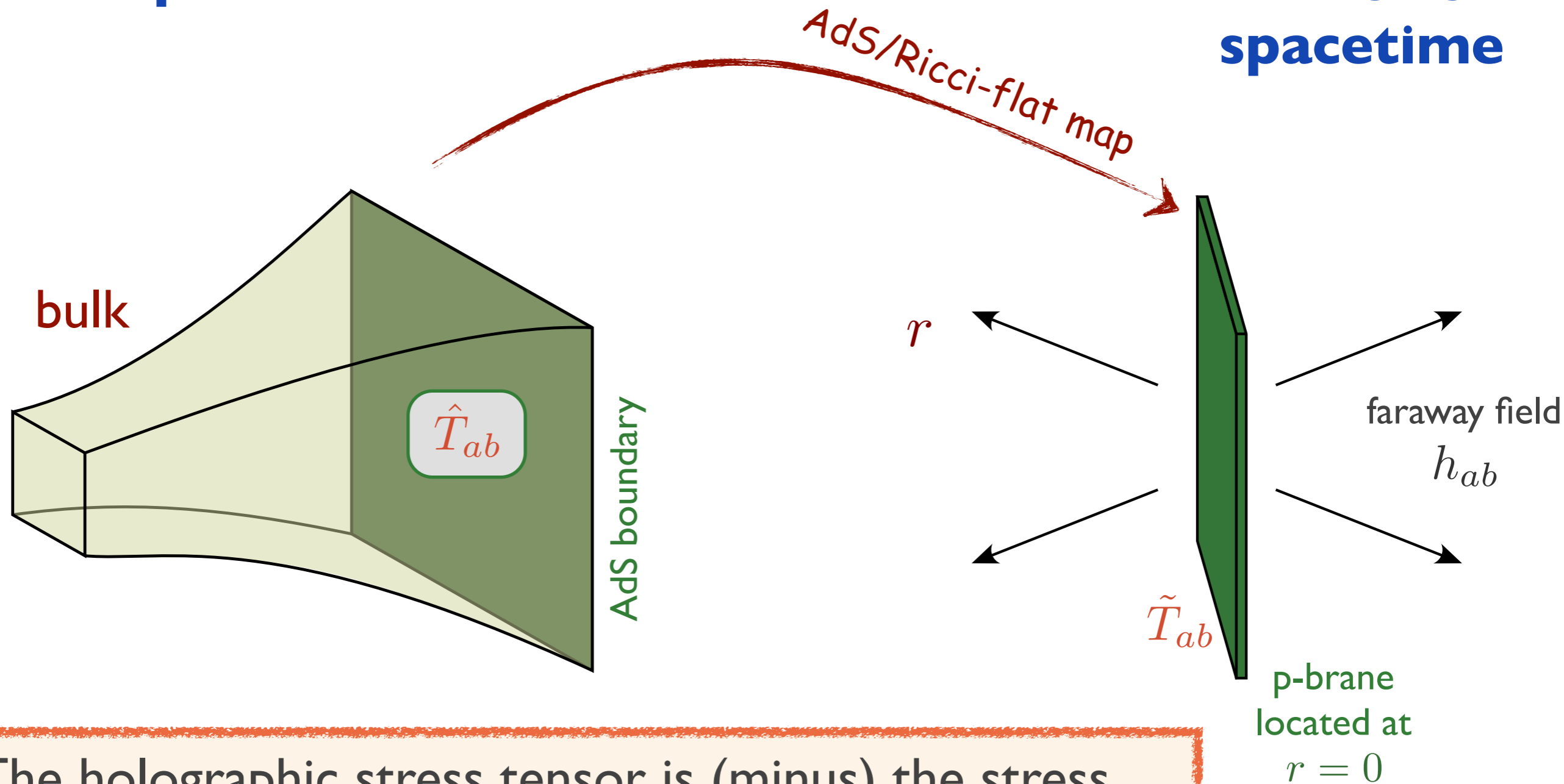
(stress tensor of a p-brane located at $r=0$)

Holographic stress tensor sources the faraway grav. field

Second example: **Excitations on top of AdS**

AdS spacetime

Minkowski spacetime



The holographic stress tensor is (minus) the stress tensor of a brane, located at the origin of Minkowski, that sources the linearized gravitational field h_{ab}

Second example: **Correlation functions**

To compute correlation functions we set $g_{(0)\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

and find **regular**, linear transverse traceless fluctuation in AdS

$$h_{ab}^{\text{AdS}}(k) = h_{(0)ab}(k) \frac{1}{2^{d/2-1} \Gamma(d/2)} (kr)^{d/2} K_{d/2}(kr)$$

Apply AdS/Ricci flat correspondence, with $d \rightarrow -n$

$$h_{ab}^{\text{Mink}}(k) = h_{(0)ab}(k) \frac{2^{n/2+1}}{\Gamma(-n/2)} \frac{K_{n/2}(kr)}{(kr)^{n/2}}$$

Linearized gravitational field produced by a **p-brane** with **worldvolume metric** $\eta_{ab} + h_{ab}$

Exponential fall-off at infinity: the perturbation is **normalizable** and the metric remains **asymptotically flat**

First entries in the holographic dictionary

- ✧ On AdS, the boundary condition was to choose a metric on the boundary
- ✧ This translates on the Ricci-flat side into a **choice of a metric at the location of a p -brane**
- ✧ At linear order, the holographic stress energy tensor becomes the **stress energy tensor due to this p -brane**, that sources the linearized gravitational field
- ✧ The regularity in the bulk of AdS becomes the requirement that **the Ricci-flat perturbation preserves asymptotic flatness**

A coherent picture is emerging, hinting at holography for asymptotically flat spacetimes

Generalized conformal symmetry & solution generating transformations

AdS isometries form the boundary **conformal group**

Dilatations $\delta_\lambda x^M = \lambda x^M$

Special conformal transformations $\left\{ \begin{array}{l} \delta_b z^\mu = b^\mu z^2 - 2z^\mu (z \cdot b) + r^2 b^\mu \\ \delta_b r = -2(z \cdot b)r \end{array} \right.$

On Minkowski side they act as **conformal transformation**

$$\delta g_{0AB} = 2\sigma(x)g_{0AB}$$

with $\sigma(x) = \lambda$ for dilatations

$$\sigma(x) = -2(x \cdot b) \text{ for special conformal transformations}$$

They are not isometries of Minkowski, but the resulting metric is still Ricci-flat: they act as **solution generating transformations**

The underlying generalized conformal structure constrains the physics of these Ricci-flat spacetimes

Third example: **black branes**

Planar AdS black holes:

$$ds_{\Lambda}^2 = z^2(-f(z)d\tau^2 + d\vec{x}^2 + d\vec{y}^2) + \frac{dz^2}{z^2 f(z)},$$

τ^{d-p-1}
↓

$$f(z) = 1 - \frac{1}{(bz)^d}$$

$$d \leftrightarrow -n \quad \updownarrow \quad z = \frac{1}{r}, \quad b = r_0$$

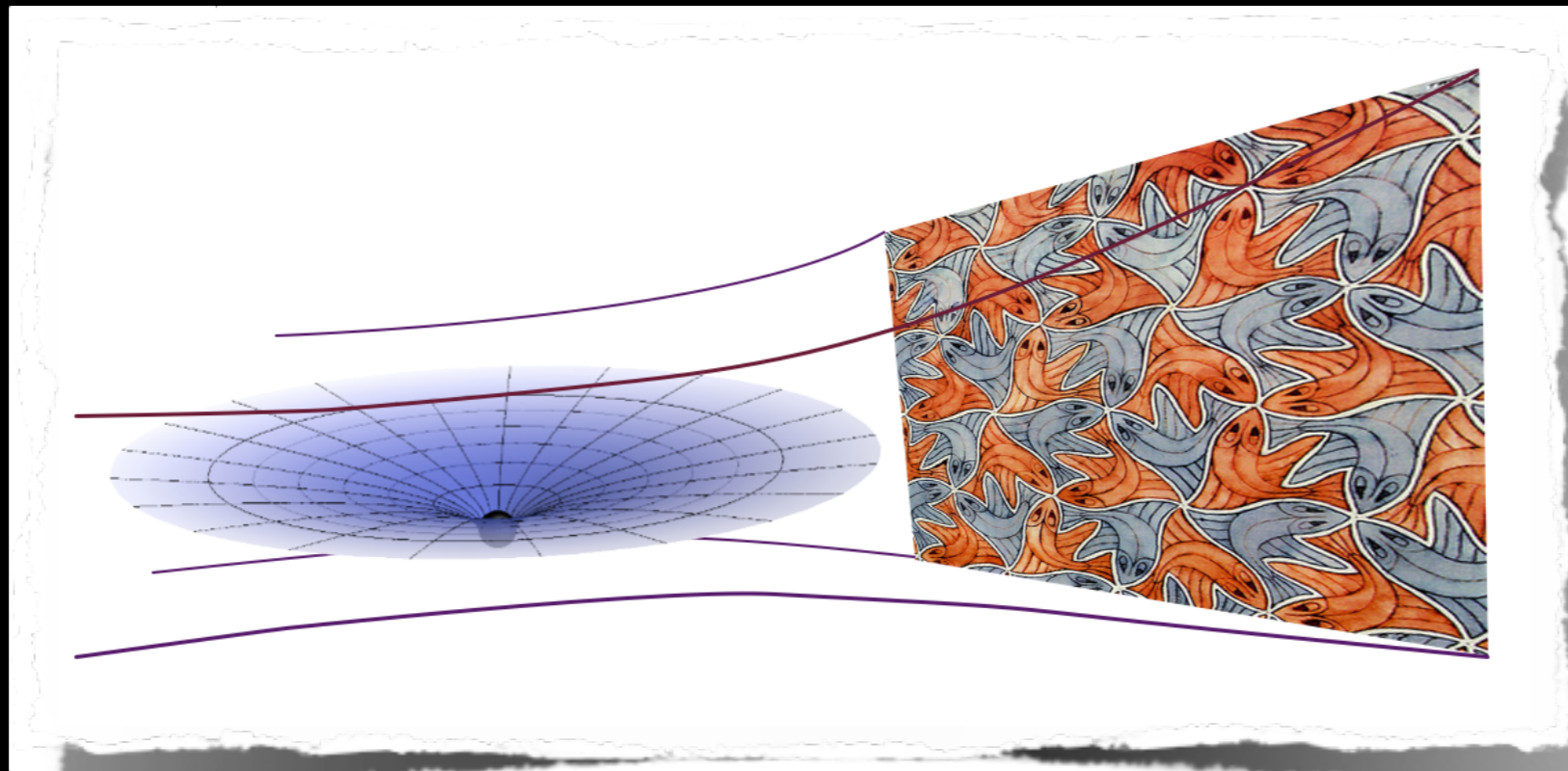
Schwarzschild black p -branes:

$$ds_0^2 = \underbrace{-f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n+1}^2}_{\text{Schwarzschild}} + d\vec{x}^2$$

S^{n+1}
↓

$$f(r) = 1 - \frac{r_0^n}{r^n}$$

Wait! This is a Fluid/Gravity workshop!

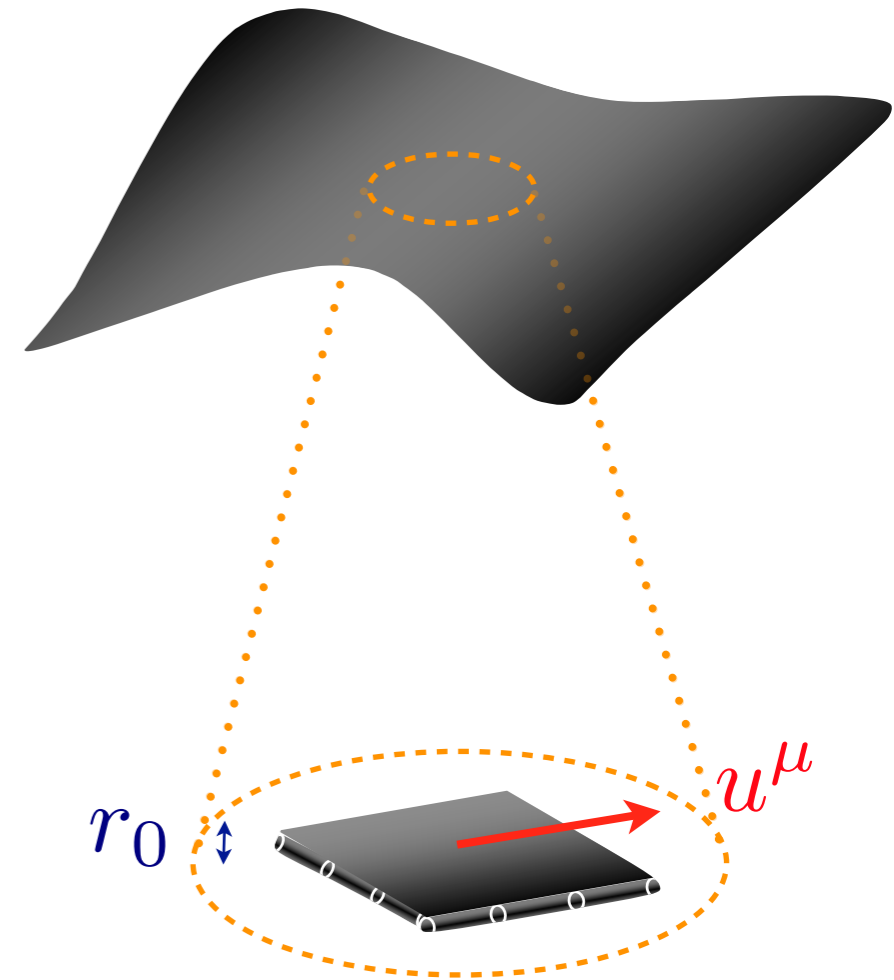
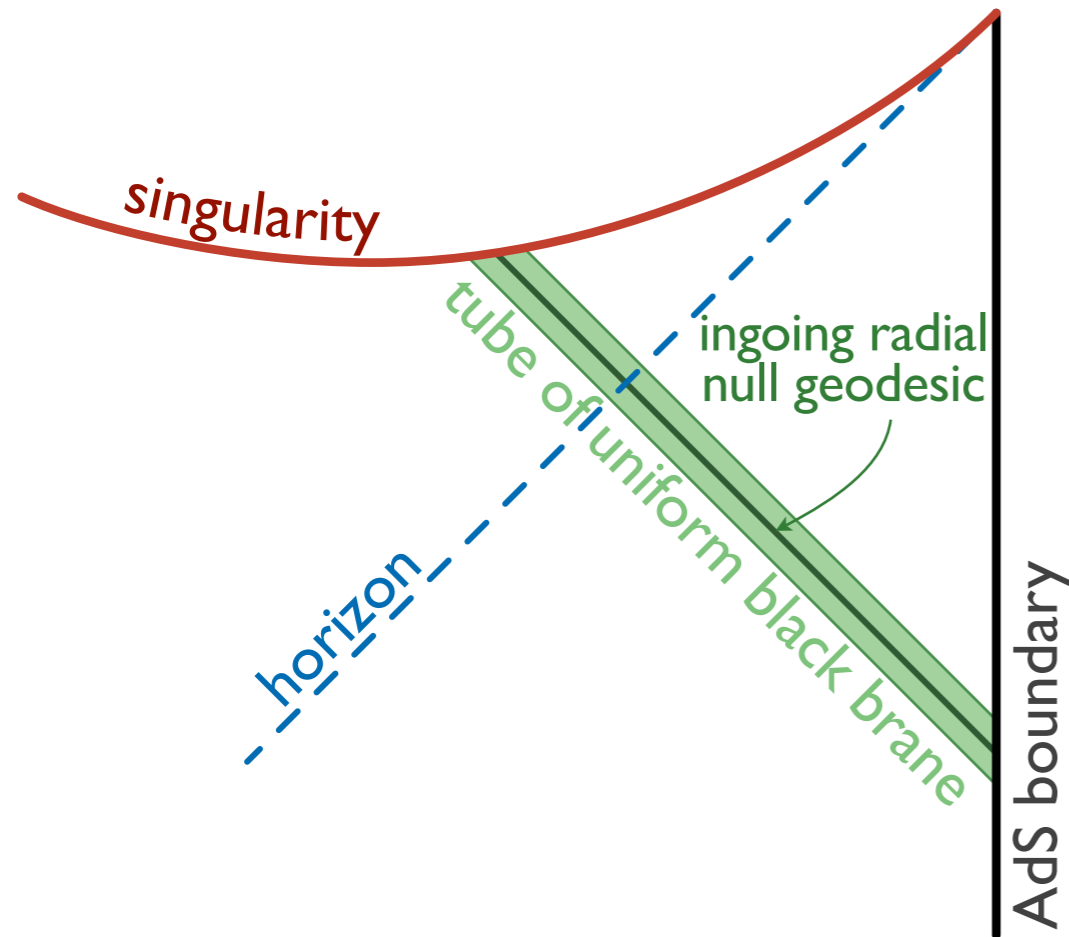


~ applying the map to hydrodynamic perturbations of bhs ~

AdS fluid/gravity metric

$$d \rightarrow -n$$

Blackfold



Bhattacharyya, Hubeny, Minwalla & Rangamani '07

Bhattacharyya, Loganayagam, Mandal, Minwalla & Sharma '08

Emparan, Harmark, Niarchos & Obers '09

The fluid/gravity metric is mapped to the metric describing **long wavelength** perturbations of a **black p -brane**, whose (intrinsic) dynamics is captured by an **effective fluid (blackfold)**.

MC, Camps, Gout eraux, Skenderis '12

The Rindler/fluid correspondence

Bredberg et al '12, Compere et al '12, Eling et al '12

Black p -brane:

$$ds_0^2 = -f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n+1}^2 + d\vec{x}^2, \quad f(r) = 1 - \frac{r_0^n}{r^n}$$

When $n \rightarrow -1$ the sphere collapse to a point and we obtain
Rindler spacetime in $p+2$ dimensions:

$$ds_0^2 = -\frac{\rho^2}{4r_0^2}d\tau^2 + d\rho^2 + d\vec{x}^2, \quad \rho^2 = 4r_0^2 (1 - r/r_0)$$

The AdS boundary is mapped on a constant ρ hypersurface with induced metric η_{ab} (position of accelerating observer)

Taking carefully the $n \rightarrow -1$ limit of the mapped gravity/fluid metric, we recover the **hydrodynamic perturbations of Rindler** spacetime and the associated stress energy tensor to **second order** in derivatives.

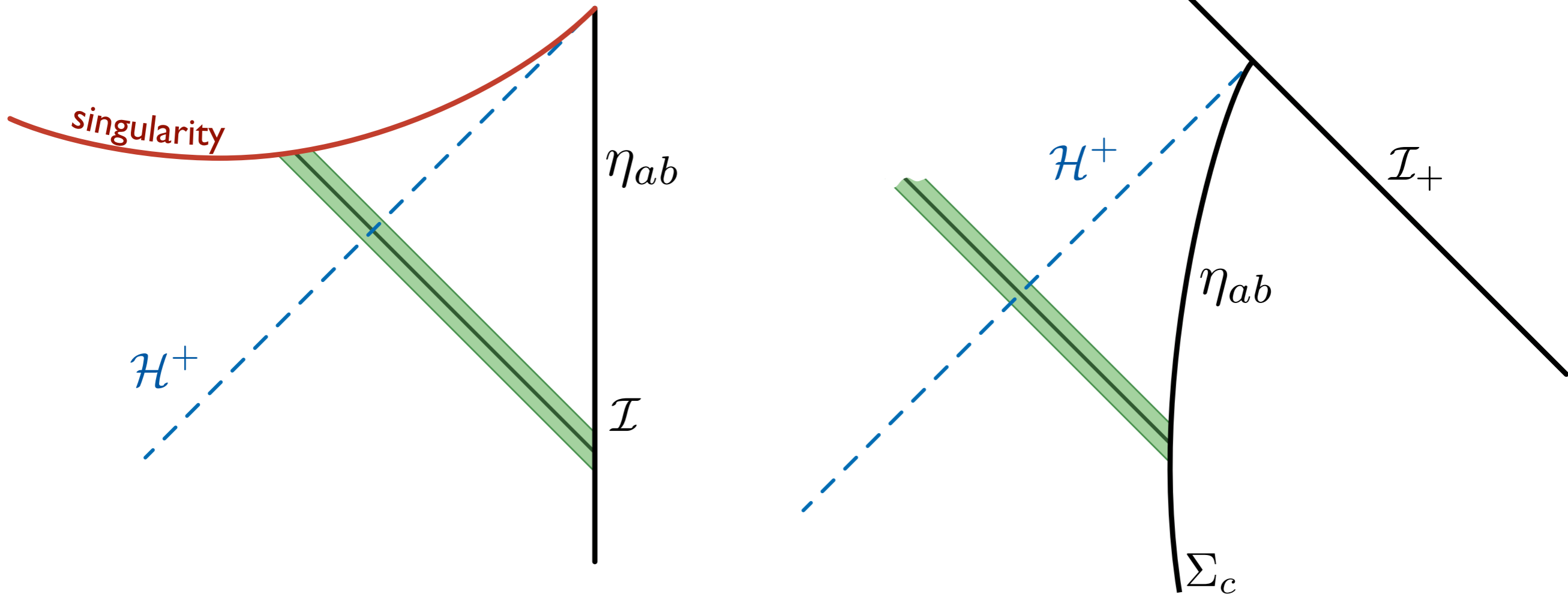
The Rindler/fluid correspondence

AdS spacetime

Rindler

$$n \rightarrow -1$$

AdS/Ricci-flat map



Dirichlet b.c. on AdS boundary

Dirichlet b.c. on timelike surface Σ_c

AdS/RF summary

- * **AdS/Ricci-flat correspondence** maps asymptotically locally **AdS** solutions on a torus to **Ricci-flat** spacetimes

These Ricci-flat spaces inherit the holographic properties of AdS

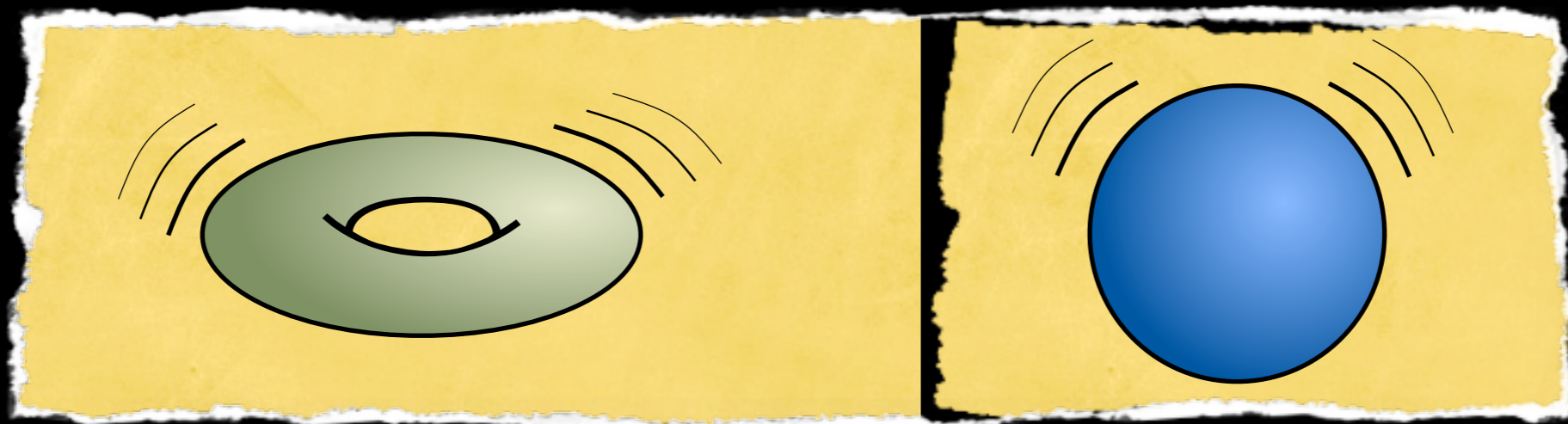
- * **Holography for asymptotically flat spacetimes?**
 - The map captures all the dynamics in the extended directions along the brane (*correlation functions, hydro mode, transport coeffs*)
 - The transverse sphere is **frozen!** These are critical modes to understand asymptotically flat holography:
we are killing all the dynamics of the Schwarzschild black hole
 - However non extremal BHs near horizon geometry is Rindler, that we linked to AdS_2 : relation to the effective chiral CFT?

- * **Extensions of the AdS/RF map**

Di Dato, Gath & Vigand Pedersen '15

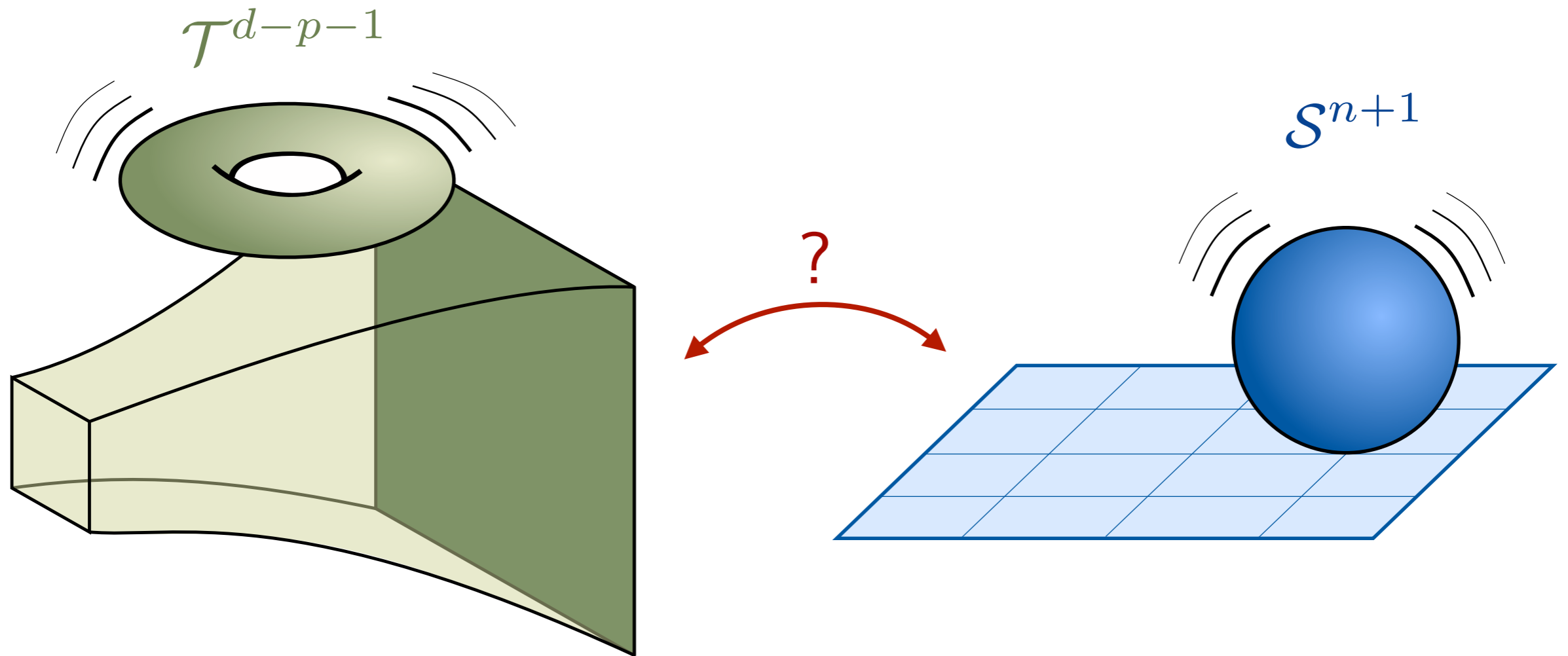
- Auxiliary reduction: extension with gauge fields (EMD)
but the sphere remains frozen, so it does not help us

How far can we push
the AdS/RF correspondence?



~ Unfreezing the torus & the sphere ~

Beyond AdS/RF: perturbations



- ✧ General linearized perturbation of AdS and Minkowski
- ✧ Full Kaluza-Klein reduction down to $p+2$ dimensions
- ✧ Compare the resulting modes
- ✧ ???

Linearized perturbations of Minkowski

$$ds_{n+p+3}^2 = \underbrace{\eta_{\mu\nu} dx^\mu dx^\nu + dr^2}_{p+2} + r^2 \underbrace{\sigma_{ij} d\theta^i d\theta^j}_{\mathcal{S}^{n+1}} + \underbrace{h_{AB} dX^A dX^B}_{\text{perturbation}}$$

Field expansion in $\text{SO}(n+2)$ representations

$$h_{ab} = h_{ab}^{I_s}(x, r) \mathbb{S}^{I_s}(\theta),$$

$$h_{ai} = B_{(v)a}^{I_v}(x, r) \mathbb{V}_i^{I_v}(\theta) + B_{(s)a}^{I_s}(x, r) \mathcal{D}_i \mathbb{S}^{I_s}(\theta),$$

$$h_{(ij)} = \hat{\phi}_t^{I_t}(x, r) \mathbb{T}_{(ij)}^{I_t}(\theta) + \phi_v^{I_v}(x, r) \mathcal{D}_{(i} \mathbb{V}_{j)}^{I_v}(\theta) + \phi_s^{I_s}(x, r) \mathcal{D}_{(i} \mathcal{D}_{j)} \mathbb{S}^{I_s}(\theta),$$

$$h^i_i \equiv \sigma^{ij} h_{ij} = \pi^{I_s}(x, r) \mathbb{S}^{I_s}(\theta)$$

Scalars

$$h_{ab}^{I_s}, \quad B_{(s)a}^{I_s}, \quad \phi_s^{I_s}, \quad \pi^{I_s}$$

Vectors

$$B_{(v)a}^{I_v}, \quad \phi_v^{I_v}$$

Tensor

$$\hat{\phi}_t^{I_t}$$

Spherical harmonics

$$\mathbb{S}^{I_s}(\theta) \quad \Lambda^{I_s}$$

$$\mathbb{V}_i^{I_v}(\theta) \quad \Lambda^{I_v}$$

$$\mathbb{T}_{(ij)}^{I_t}(\theta) \quad \Lambda^{I_t}$$

Gauge invariant variables

Skenderis & Taylor '05

Some modes are diffeomorphic to each other, or to the background solution

$$X^{A'} = X^A - \xi^A$$

$$\xi_a(x, r, \theta) = \xi_a^{I_s}(x, r) S^{I_s}, \quad \xi_i(x, r, \theta) = \xi_v^{I_v}(x, r) \mathbb{V}_i^{I_v} + \xi_s^{I_s}(x, r) \mathcal{D}_i S^{I_s}$$

$$\delta h_{ab}^{I_s} = \partial_a \xi_b^{I_s} + \partial_b \xi_a^{I_s}$$

$$\delta B_{(s)a}^{I_s} = \xi_a^{I_s} + r^2 \partial_a \xi_s^{I_s}$$

$$\delta \phi_s^{I_s} = 2r^2 \xi_s^{I_s}$$

$$\delta B_{(v)a}^{I_v} = r^2 \partial_a \xi_v^{I_v}$$

$$\delta \phi_v^{I_v} = 2r^2 \xi_v^{I_v}$$

$$\delta \pi^{I_s} = 2r^2 \Lambda^{I_s} \xi_s^{I_s} + 2(n+1)r \xi_r^{I_s}$$

$$\delta \hat{\phi}_t^{I_t} = 0$$

gauge invariant



Gauge invariant variables

Skenderis & Taylor '05

Some modes are diffeomorphic to each other, or to the background solution

$$X^{A'} = X^A - \xi^A$$

$$\xi_a(x, r, \theta) = \xi_a^{I_s}(x, r) S^{I_s}, \quad \xi_i(x, r, \theta) = \xi_v^{I_v}(x, r) \mathbb{V}_i^{I_v} + \xi_s^{I_s}(x, r) \mathcal{D}_i S^{I_s}$$

$$\delta h_{ab}^{I_s} = \partial_a \xi_b^{I_s} + \partial_b \xi_a^{I_s}$$

$$\delta B_{(s)a}^{I_s} = \xi_a^{I_s} + r^2 \partial_a \xi_s^{I_s}$$

$$\delta \phi_s^{I_s} = 2r^2 \xi_s^{I_s}$$

$$\delta B_{(v)a}^{I_v} = r^2 \partial_a \xi_v^{I_v}$$

$$\delta \phi_v^{I_v} = 2r^2 \xi_v^{I_v}$$

$$\delta \pi^{I_s} = 2r^2 \Lambda^{I_s} \xi_s^{I_s} + 2(n+1)r \xi_r^{I_s}$$

$$\delta \hat{\phi}_t^{I_t} = 0$$

gauge invariant

Define

$$\hat{B}_{(s)a}^{I_s} = B_{(s)a}^{I_s} - \frac{1}{2} \partial_a \phi_s^{I_s} + \frac{1}{r} \delta^r_a \phi_s^{I_s},$$

$$\delta \hat{B}_{(s)a}^{I_s} = \xi_a^{I_s}$$

Use $\hat{B}_{(s)a}^{I_s}$, $\phi_s^{I_s}$, $\phi_v^{I_v}$ to compensate the variations generated by the gauge parameters $\xi_a^{I_s}$, $\xi_v^{I_v}$, $\xi_s^{I_s}$

Gauge invariant variables

Skenderis & Taylor '05

Define the hatted, gauge invariant fields

$$\hat{h}_{ab}^{I_s} = h_{ab}^{I_s} - \partial_a \hat{B}_{(s)b}^{I_s} - \partial_b \hat{B}_{(s)a}^{I_s}$$

$$\hat{B}_{(v)a}^{I_v} = B_{(v)a}^{I_v} - \frac{1}{2} \partial_a \phi_v^{I_v} + \frac{1}{r} \delta^r_a \phi_v^{I_v}$$

$$\hat{\pi}^{I_s} = \pi^{I_s} - \Lambda^{I_s} \phi_s^{I_s} - 2(n+1)r \hat{B}_{(s)r}^{I_s}$$

$$\delta \hat{h}_{ab}^{I_s} = 0 \quad \delta \hat{B}_{(v)a}^{I_v} = 0 \quad \delta \hat{\pi}^{I_s} = 0 \quad \delta \hat{\phi}_t^{I_t} = 0$$

gauge invariant 

$$l = 0 : h_{ab}^{I_s}, \pi^{I_s}$$

$$l = 1 : \hat{h}_{ab}^{I_s}, \hat{\pi}^{I_s}, \hat{B}_{(v)a}^{I_v}$$

$$l \geq 2 : \hat{h}_{ab}^{I_s}, \hat{\pi}^{I_s}, \hat{B}_{(v)a}^{I_v}, \hat{\phi}_t^{I_t}$$

De Donder-Lorentz gauge

Set $B_{(s)a}^{I_s}$, $\phi_v^{I_v}$, and $\phi_s^{I_s}$ to zero, then add hats to the result!

Linearized field equations

$$E_{CD}^{(0)} \equiv \bar{g}^{AB} (h_{BC|DA} - h_{AB|CD} + h_{BD|CA} - h_{CD|BA}) = 0$$

Decompose & expand in harmonics:

$$E_{ab}^{(0)} \Big|_{\mathbb{S}^{I_s}} = 0$$

$$E_{ai}^{(0)} \Big|_{\mathbb{V}_i^{I_v}} = 0 \quad E_{ai}^{(0)} \Big|_{\mathcal{D}_i \mathbb{S}^{I_s}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathbb{T}_{ij}^{I_t}} = 0 \quad E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathbb{V}_{j)}^{I_v}} = 0 \quad E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathcal{D}_{j)} \mathbb{S}^{I_s}} = 0 \quad E_{ij}^{(0)} \Big|_{\sigma_{ij} \mathbb{S}^{I_s}} = 0$$

Linearized field equations

$$E_{CD}^{(0)} \equiv \bar{g}^{AB} (h_{BC|DA} - h_{AB|CD} + h_{BD|CA} - h_{CD|BA}) = 0$$

Decompose & project onto harmonics:

$$E_{ab}^{(0)} \Big|_{\mathbb{S}^{I_s}} = 0$$

$$E_{ai}^{(0)} \Big|_{\mathbb{V}_i^{I_v}} = 0$$

$$E_{ai}^{(0)} \Big|_{\mathcal{D}_i \mathbb{S}^{I_s}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathbb{T}_{ij}^{I_t}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathbb{V}_{j)}^{I_v}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathcal{D}_{j)} \mathbb{S}^{I_s}} = 0$$

$$E_{ij}^{(0)} \Big|_{\sigma_{ij} \mathbb{S}^{I_s}} = 0$$

Decoupled equations for $B_{(v)a}^{I_v}$ and $\hat{\phi}_t^{I_t}$

$$\left\{ \begin{array}{l} \square \hat{B}_{(v)a}^{I_v} + \frac{n-1}{r} \partial_r \hat{B}_{(v)a}^{I_v} - \frac{2}{r} \partial_a \hat{B}_{(v)r}^{I_v} - \frac{n-3}{r^2} \delta_a^r \hat{B}_{(v)r}^{I_v} + \frac{\Lambda^{I_v} - n}{r^2} \hat{B}_{(v)a}^{I_v} = 0 \\ \square \hat{\phi}_t^{I_t} + \frac{n-3}{r} \partial_r \hat{\phi}_t^{I_t} + \frac{1}{r^2} (\Lambda^{I_t} - 2n + 2) \hat{\phi}_t^{I_t} = 0 \end{array} \right.$$

Linearized field equations

$$E_{CD}^{(0)} \equiv \bar{g}^{AB} (h_{BC|DA} - h_{AB|CD} + h_{BD|CA} - h_{CD|BA}) = 0$$

Coupled equations for $\hat{h}_{ab}^{I_s}$ and $\hat{\pi}^{I_s}$

$$E_{ab}^{(0)} \Big|_{\mathcal{S}^{I_s}} = 0$$

$$E_{ai}^{(0)} \Big|_{\mathcal{V}_i^{I_v}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathcal{T}_{ij}^{I_t}} = 0$$

$$E_{ai}^{(0)} \Big|_{\mathcal{D}_i \mathcal{S}^{I_s}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathcal{V}_{j)}^{I_v}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathcal{D}_{j)} \mathcal{S}^{I_s}} = 0$$

$$E_{ij}^{(0)} \Big|_{\sigma_{ij} \mathcal{S}^{I_s}} = 0$$

Decoupled equations for $B_{(v)a}^{I_v}$ and $\hat{\phi}_t^{I_t}$

$$\left\{ \begin{array}{l} \square \hat{B}_{(v)a}^{I_v} + \frac{n-1}{r} \partial_r \hat{B}_{(v)a}^{I_v} - \frac{2}{r} \partial_a \hat{B}_{(v)r}^{I_v} - \frac{n-3}{r^2} \delta_a^r \hat{B}_{(v)r}^{I_v} + \frac{\Lambda^{I_v} - n}{r^2} \hat{B}_{(v)a}^{I_v} = 0 \\ \square \hat{\phi}_t^{I_t} + \frac{n-3}{r} \partial_r \hat{\phi}_t^{I_t} + \frac{1}{r^2} (\Lambda^{I_t} - 2n + 2) \hat{\phi}_t^{I_t} = 0 \end{array} \right.$$

Linearized field equations

$$E_{CD}^{(0)} \equiv \bar{g}^{AB} (h_{BC|DA} - h_{AB|CD} + h_{BD|CA} - h_{CD|BA}) = 0$$

Coupled equations for $\hat{h}_{ab}^{I_s}$ and $\hat{\pi}^{I_s}$

$$E_{ab}^{(0)} \Big|_{\mathcal{S}^{I_s}} = 0$$

$$E_{ai}^{(0)} \Big|_{\mathcal{V}_i^{I_v}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathcal{T}_{ij}^{I_t}} = 0$$

$$E_{ai}^{(0)} \Big|_{\mathcal{D}_i \mathcal{S}^{I_s}} = 0$$

$$E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathcal{V}_{j)}^{I_v}} = 0 \quad E_{ij}^{(0)} \Big|_{\mathcal{D}_{(i} \mathcal{D}_{j)} \mathcal{S}^{I_s}} = 0$$

$$E_{ij}^{(0)} \Big|_{\sigma_{ij} \mathcal{S}^{I_s}} = 0$$

Follow from other equations

(gauge freedom to choose $B_{(s)a}^{I_s}$, $\phi_s^{I_s}$, $\phi_v^{I_v}$)

Decoupled equations for $B_{(v)a}^{I_v}$ and $\hat{\phi}_t^{I_t}$

$$\left\{ \begin{array}{l} \square \hat{B}_{(v)a}^{I_v} + \frac{n-1}{r} \partial_r \hat{B}_{(v)a}^{I_v} - \frac{2}{r} \partial_a \hat{B}_{(v)r}^{I_v} - \frac{n-3}{r^2} \delta_a^r \hat{B}_{(v)r}^{I_v} + \frac{\Lambda^{I_v} - n}{r^2} \hat{B}_{(v)a}^{I_v} = 0 \\ \square \hat{\phi}_t^{I_t} + \frac{n-3}{r} \partial_r \hat{\phi}_t^{I_t} + \frac{1}{r^2} (\Lambda^{I_t} - 2n + 2) \hat{\phi}_t^{I_t} = 0 \end{array} \right.$$

Action for fluctuations on Minkowski

$$\begin{aligned}
 S_0 = & \int d^{p+2}x r^{n+1} \left\{ \frac{1}{2} \partial_c \hat{h}_{ab}^{I_s} \partial^a \hat{h}^{I_s bc} - \frac{1}{4} \partial_a \hat{h}_{bc}^{I_s} \partial^a \hat{h}^{I_s bc} + \frac{\Lambda^{I_s}}{4r^2} \hat{h}_{ab}^{I_s} \hat{h}^{I_s ab} + \frac{1}{4} \partial^a \hat{H}^{I_s} \partial_a \hat{H}^{I_s} \right. \\
 & - \frac{\Lambda^{I_s}}{4r^2} \hat{H}^{I_s} \hat{H}^{I_s} - \frac{1}{2} \partial^a \hat{H}^{I_s} \partial^b \hat{h}_{ab}^{I_s} - \frac{n+1}{2r} \hat{h}_{ra}^{I_s} \partial^a \hat{H}^{I_s} + \frac{1}{4r^4} \frac{n}{n+1} \partial_a \hat{\pi}^{I_s} \partial^a \hat{\pi}^{I_s} \\
 & - \frac{1}{4r^6} \left(\frac{n(n-1)}{(n+1)^2} \Lambda^{I_s} - \frac{4}{n+1} \right) \hat{\pi}^{I_s} \hat{\pi}^{I_s} - \frac{1}{2r^2} \partial^a \hat{\pi}^{I_s} \partial^b \hat{h}_{ab}^{I_s} - \frac{n-1}{2r^3} \hat{h}_{ar}^{I_s} \partial^a \hat{\pi}^{I_s} + \frac{1}{r^3} \hat{\pi}^{I_s} \partial^a \hat{h}_{ar}^{I_s} \\
 & \left. + \frac{n-1}{r^4} \hat{\pi}^{I_s} \hat{h}_{rr}^{I_s} + \frac{1}{2r^2} \partial_a \hat{H}^{I_s} \partial^a \hat{\pi}^{I_s} - \frac{1}{2r^3} \hat{\pi}^{I_s} \partial_r \hat{H}^{I_s} - \frac{\Lambda^{I_s}}{2r^4} \frac{n}{n+1} \hat{H}^{I_s} \hat{\pi}^{I_s} - \frac{1}{2r^5} \hat{\pi}^{I_s} \partial_r \hat{\pi}^{I_s} \right\} S_s^{I_s} \\
 & + \int d^{p+2}x r^{n-1} \left\{ -\frac{1}{4} \hat{G}_{(v)ab}^{I_v} \hat{G}_{(v)}^{I_v ab} + \frac{\Lambda^{I_v} - n}{2r^2} \hat{B}_{(v)a}^{I_v} \hat{B}_{(v)}^{I_v a} - \frac{2}{r} \hat{B}_{(v)}^{I_v a} \partial_a \hat{B}_{(v)r}^{I_v} + \frac{2}{r^2} \hat{B}_{(v)r}^{I_v} \hat{B}_{(v)r}^{I_v} \right\} V_s^{I_v} \\
 & + \int d^{p+2}x r^{n-3} \left\{ -\frac{1}{4} \partial_a \hat{\phi}_t^{I_t} \partial^a \hat{\phi}_t^{I_t} + \frac{\Lambda^{I_t} - 2n + 2}{4r^2} \hat{\phi}_t^{I_t} \hat{\phi}_t^{I_t} \right\} T_s^{I_t}
 \end{aligned}$$

Variations:

$\hat{h}_{ab}^{I_s}$ $\hat{\pi}^{I_s}$ $\hat{B}_{(v)a}^{I_v}$ $\hat{\phi}_t^{I_t}$ \Rightarrow independent field equations

$B_{(s)a}^{I_s}$ $\phi_s^{I_s}$ $\phi_v^{I_v}$ \Rightarrow linear comb. of the field eqns & ∇

Field eqns for $\hat{\phi}_t^{I_t}$ and $\hat{B}_{(v)a}^{I_v}$ are decoupled

Decoupling the equations

★ Eliminate the trace of $\hat{h}_{ab}^{I_s}$ using $\hat{h}^{I_s} = -\frac{1}{r^2} \frac{n-1}{n+1} \hat{\pi}^{I_s}$

★ Define the new fields $\hat{\chi}^{I_s}$ and $\hat{\varphi}^{I_s}$,

$$\hat{\pi}^{I_s} = -(n+1)r^2 \hat{\chi}^{I_s}, \quad \hat{h}_{(rr)}^{I_s} = r^2 \hat{\varphi}^{I_s} - \frac{n+p+1}{p+2} \hat{\chi}^{I_s}$$

★ The resulting equations are $\square \hat{\varphi}^{I_s} + \frac{n+1}{r} \partial_r \hat{\varphi}^{I_s} + \frac{1}{r^2} \Lambda^{I_s} \hat{\varphi}^{I_s} = 0$
 $\square \hat{\chi}^{I_s} + \frac{n+1}{r} \partial_r \hat{\chi}^{I_s} + \frac{1}{r^2} \Lambda^{I_s} \hat{\chi}^{I_s} = -2\hat{\varphi}^{I_s}$

$$\begin{aligned} \square \hat{h}_{(ab)}^{I_s} + \frac{n+1}{r} \partial_r \hat{h}_{(ab)}^{I_s} + \frac{1}{r^2} \Lambda^{I_s} \hat{h}_{(ab)}^{I_s} + \text{“}\partial_a \hat{h}_{(br)}^{I_s}\text{”} + \text{“}\delta_a^r \hat{h}_{(br)}^{I_s}\text{”} &= \\ = \frac{2(n-1)}{p+2} \eta_{ab} \hat{\varphi}^{I_s} + \frac{2n+p+1}{r} \frac{1}{p+2} \left(\delta_a^r \partial_b + \delta_b^r \partial_a + \frac{n-1}{r} \delta_a^r \delta_b^r \right) \hat{\chi}^{I_s} \end{aligned}$$

The equations decouple partially,
and are (in)homogeneous Bessel equations.

Linearized perturbations of AdS

$$ds_{d+1}^2 = \frac{\ell^2}{r^2} \left(\underbrace{\eta_{\mu\nu} dx^\mu dx^\nu + dr^2}_{p+2} + \underbrace{\delta_{ij} dy^i dy^j}_{\mathcal{T}^{d-p-1}} \right) + \underbrace{h_{AB} dX^A dX^B}_{\text{perturbation}}$$

Field expansion in Fourier modes on the torus

$$h_{ab} = h_{ab}^{\mathbf{m}_s}(y) \mathbb{S}^{\mathbf{m}_s}$$

$$h_{ai} = C_{(v)a}^{(k, \mathbf{m}_v)}(y) \mathbb{V}_i^{(k, \mathbf{m}_v)} + C_{(s)a}^{\mathbf{m}_s}(y) \partial_i \mathbb{S}^{\mathbf{m}_s}$$

$$h_{(ij)} = \hat{\psi}_t^{(k, l, \mathbf{m}_t)}(y) \mathbb{T}_{(ij)}^{(k, l, \mathbf{m}_t)} + \psi_v^{(k, \mathbf{m}_v)}(y) \partial_{(i} \mathbb{V}_{j)}^{(k, \mathbf{m}_v)} + \psi_s^{\mathbf{m}_s}(y) \partial_{(i} \partial_{j)} \mathbb{S}^{\mathbf{m}_s}$$

$$h^i_i \equiv \delta^{ij} h_{ij} = \varpi^{\mathbf{m}_s}(y) \mathbb{S}^{\mathbf{m}_s}$$

Scalars

$$h_{ab}^{\mathbf{m}_s}$$

$$C_{(s)a}^{\mathbf{m}_s}$$

$$\psi_s^{\mathbf{m}_s}$$

$$\varpi^{\mathbf{m}_s}$$

Vectors

$$C_{(v)a}^{(k, \mathbf{m}_v)}$$

$$\psi_v^{(k, \mathbf{m}_v)}$$

Tensor

$$\hat{\psi}_t^{(k, l, \mathbf{m}_t)}$$

Fields

Harmonics

$$\mathbb{S}^{\mathbf{m}_s}(\chi)$$

$$\mathbb{V}^{(k, \mathbf{m}_v)}(\chi)$$

$$\mathbb{T}^{(k, l, \mathbf{m}_t)}(\chi)$$

Gauge invariant variables (AdS)

Skenderis & Taylor '05

Some modes are diffeomorphic to each other, or to the background solution $X^{A'} = X^A - \xi^A$

$$\xi_a = \xi_a^{\mathbf{m}_s}(y) \mathbb{S}^{\mathbf{m}_s}(\chi) \quad \xi_i = \xi_v^{(k, \mathbf{m}_v)}(y) \mathbb{V}_i^{(k, \mathbf{m}_v)} + \xi_s^{\mathbf{m}_s}(y) \partial_i \mathbb{S}^{\mathbf{m}_s}(\chi)$$

$$\delta h_{ab}^{\mathbf{m}_s} = \partial_a \xi_b^{\mathbf{m}_s} + \partial_b \xi_a^{\mathbf{m}_s} - \frac{2}{r} (\eta_{ab} \xi_r^{\mathbf{m}_s} - \delta_a^r \xi_b^{\mathbf{m}_s} - \delta_b^r \xi_a^{\mathbf{m}_s})$$

$$\delta C_{(s)a}^{\mathbf{m}_s} = \xi_a^{\mathbf{m}_s} + \partial_a \xi_s^{\mathbf{m}_s} + \frac{2}{r} \delta_a^r \xi_s^{\mathbf{m}_s}$$

$$\delta C_{(v)a}^{(k, \mathbf{m}_v)} = \partial_a \xi_v^{(k, \mathbf{m}_v)} + \frac{2}{r} \delta_a^r \xi_v^{(k, \mathbf{m}_v)}$$

$$\delta \varpi^{\mathbf{m}_s} = -2\mathbf{m}_s^2 \xi_s^{\mathbf{m}_s} - \frac{2}{r} (d - p - 1) \xi_r^{\mathbf{m}_s}$$

$$\delta \psi_s^{\mathbf{m}_s} = 2 \xi_s^{\mathbf{m}_s}$$

$$\delta \psi_v^{(k, \mathbf{m}_v)} = 2 \xi_v^{(k, \mathbf{m}_v)}$$

$$\delta \hat{\psi}_t^{(k, l, \mathbf{m}_t)} = 0$$

Define

gauge invariant \rightarrow

$$\hat{C}_{(s)a}^{\mathbf{m}_s} = C_{(s)a}^{\mathbf{m}_s} - \frac{1}{2} \partial_a \psi_s^{\mathbf{m}_s} - \frac{1}{r} \delta_a^r \psi_s^{\mathbf{m}_s}$$

$$\delta \hat{C}_{(s)a}^{\mathbf{m}_s} = \xi_a^{\mathbf{m}_s}$$

Use $\hat{C}_{(s)a}^{\mathbf{m}_s}$, $\psi_s^{\mathbf{m}_s}$, $\psi_v^{(k, \mathbf{m}_v)}$ to compensate the variations generated by the gauge parameters $\xi_a^{\mathbf{m}_s}$, $\xi_s^{\mathbf{m}_s}$, $\xi_v^{(k, \mathbf{m}_v)}$

Gauge invariant variables (AdS)

Skenderis & Taylor '05

Define the hatted, gauge invariant fields

$$\hat{h}_{ab}^{\mathbf{m}_s} = h_{ab}^{\mathbf{m}_s} - \partial_a \hat{C}_{(s)b}^{\mathbf{m}_s} - \partial_b \hat{C}_{(s)a}^{\mathbf{m}_s} + \frac{2}{r} \left(\eta_{ab} \hat{C}_{(s)r}^{\mathbf{m}_s} - \delta_a^r \hat{C}_{(s)b}^{\mathbf{m}_s} - \delta_b^r \hat{C}_{(s)a}^{\mathbf{m}_s} \right)$$

$$\hat{C}_{(v)a}^{(k, \mathbf{m}_v)} = C_{(v)a}^{(k, \mathbf{m}_v)} - \frac{1}{2} \partial_a \psi_v^{(k, \mathbf{m}_v)} - \frac{1}{r} \delta^r_a \psi_v^{(k, \mathbf{m}_v)}$$

$$\hat{\omega}^{\mathbf{m}_s} = \omega^{\mathbf{m}_s} + \mathbf{m}_s^2 \psi_s^{\mathbf{m}_s} + \frac{2}{r} (d - p - 1) \hat{C}_{(s)r}^{\mathbf{m}_s}$$

$$\delta \hat{h}_{ab}^{\mathbf{m}_s} = 0 \quad \delta \hat{C}_{(v)a}^{(k, \mathbf{m}_v)} = 0 \quad \delta \hat{\omega}^{\mathbf{m}_s} = 0 \quad \delta \hat{\psi}_t^{(k, l, \mathbf{m}_t)} = 0$$

gauge invariant 

$$\mathbf{m}_s^2 = 0 : \quad h_{ab}^{\mathbf{m}_s} \quad \omega^{\mathbf{m}_s} \quad \hat{\psi}_t^{(k, l, \mathbf{m}_t)}$$

$$\mathbf{m}_s^2 \geq 1 : \quad \hat{h}_{ab}^{\mathbf{m}_s} \quad \hat{\omega}^{\mathbf{m}_s} \quad \hat{\psi}_t^{(k, l, \mathbf{m}_t)}$$

$$\mathbf{m}_v^2 \geq 1 : \quad \hat{B}_{(v)a}^{I_v}$$

De Donder-Lorentz gauge

Set $C_{(s)a}^{\mathbf{m}_s}$, $\psi_v^{\mathbf{m}_v}$, and $\psi_s^{\mathbf{m}_s}$ to zero, then add hats to the result!

Linearized field equations with Λ

$$E_{MN}^{(\Lambda)} \equiv \delta R_{MN} + \frac{d}{\ell^2} h_{MN} = 0$$

Decompose & expand in Fourier modes:

$$E_{ab}^{(\Lambda)} \Big|_{\mathbb{S}^{\mathbf{m}_s}} = 0$$

$$E_{ai}^{(\Lambda)} \Big|_{\mathbb{V}_i^{\mathbf{m}_v}} = 0 \quad E_{ai}^{(\Lambda)} \Big|_{\partial_i \mathbb{S}^{\mathbf{m}_s}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\mathbb{T}_{(ij)}^{\mathbf{m}_t}} = 0 \quad E_{ij}^{(\Lambda)} \Big|_{\partial_{(i} \mathbb{V}_{j)}^{\mathbf{m}_v}} = 0 \quad E_{ij}^{(\Lambda)} \Big|_{\partial_{(i} \partial_{j)} \mathbb{S}^{\mathbf{m}_s}} = 0 \quad E_{ij}^{(\Lambda)} \Big|_{\delta_{ij} \mathbb{S}^{\mathbf{m}_s}} = 0$$

Linearized field equations with Λ

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Decompose & expand in Fourier modes:

$$E_{ab}^{(\Lambda)} \Big|_{\mathbb{S}^{\mathbf{m}_s}} = 0$$

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$$E_{ai}^{(\Lambda)} \Big|_{\partial_i \mathbb{S}^{\mathbf{m}_s}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\mathbb{T}_{(ij)}^{\mathbf{m}_t}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\partial_{(i} \mathbb{V}_{j)}^{\mathbf{m}_v}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\partial_{(i} \partial_{j)} \mathbb{S}^{\mathbf{m}_s}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\delta_{ij} \mathbb{S}^{\mathbf{m}_s}} = 0$$

Decoupled equations for $C_{(v)a}^{\mathbf{m}_v}$ and $\hat{\psi}_t^{\mathbf{m}_t}$

$$\left\{ \begin{array}{l} \square \hat{C}_{(v)a}^{\mathbf{m}_v} - \frac{d-5}{r} \partial_r \hat{C}_{(v)a}^{\mathbf{m}_v} + \frac{d-1}{r^2} \delta_a^r \hat{C}_{(v)r}^{\mathbf{m}_v} - \left(\mathbf{m}_v^2 + \frac{2(d-2)}{r^2} \right) \hat{C}_{(v)a}^{\mathbf{m}_v} = 0 \\ \square \hat{\psi}_t^{\mathbf{m}_t} - \mathbf{m}_t^2 \hat{\psi}_t^{\mathbf{m}_t} - \frac{d-5}{r} \partial_r \hat{\psi}_t^{\mathbf{m}_t} - \frac{2(d-2)}{r^2} \hat{\psi}_t^{\mathbf{m}_t} = 0 \end{array} \right.$$

Linearized field equations with Λ

$$E_{MN}^{(\Lambda)} \equiv \delta R_{MN} + \frac{d}{\ell^2} h_{MN} = 0$$

Coupled equations for $\hat{h}_{ab}^{\mathbf{m}_s}$ and $\hat{\omega}^{\mathbf{m}_s}$

$$E_{ab}^{(\Lambda)} \Big|_{\mathbb{S}^{\mathbf{m}_s}} = 0$$

$$E_{ai}^{(\Lambda)} \Big|_{\mathbb{V}_i^{\mathbf{m}_v}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\mathbb{T}_{(ij)}^{\mathbf{m}_t}} = 0$$

$$E_{ai}^{(\Lambda)} \Big|_{\partial_i \mathbb{S}^{\mathbf{m}_s}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\partial_{(i} \mathbb{V}_{j)}^{\mathbf{m}_v}} = 0 \quad E_{ij}^{(\Lambda)} \Big|_{\partial_{(i} \partial_{j)} \mathbb{S}^{\mathbf{m}_s}} = 0$$

$$E_{ij}^{(\Lambda)} \Big|_{\delta_{ij} \mathbb{S}^{\mathbf{m}_s}} = 0$$

Follow from other equations

(gauge freedom to choose $C_{(s)a}^{\mathbf{m}_s}$, $\psi_s^{\mathbf{m}_s}$, $\psi_v^{\mathbf{m}_v}$)

Decoupled equations for $C_{(v)a}^{\mathbf{m}_v}$ and $\hat{\psi}_t^{\mathbf{m}_t}$

$$\left\{ \begin{array}{l} \square \hat{C}_{(v)a}^{\mathbf{m}_v} - \frac{d-5}{r} \partial_r \hat{C}_{(v)a}^{\mathbf{m}_v} + \frac{d-1}{r^2} \delta_a^r \hat{C}_{(v)r}^{\mathbf{m}_v} - \left(\mathbf{m}_v^2 + \frac{2(d-2)}{r^2} \right) \hat{C}_{(v)a}^{\mathbf{m}_v} = 0 \\ \square \hat{\psi}_t^{\mathbf{m}_t} - \mathbf{m}_t^2 \hat{\psi}_t^{\mathbf{m}_t} - \frac{d-5}{r} \partial_r \hat{\psi}_t^{\mathbf{m}_t} - \frac{2(d-2)}{r^2} \hat{\psi}_t^{\mathbf{m}_t} = 0 \end{array} \right.$$

Action for fluctuations on AdS

$$\begin{aligned}
 S_\Lambda = & \int d^{p+2} x \frac{\ell^{d-5}}{r^{d-5}} \left\{ \left[-\frac{1}{4} \partial_a \hat{\psi}_t^{\mathbf{m}_t} \partial^a \hat{\psi}_t^{\mathbf{m}_t} - \frac{1}{4} \left(\mathbf{m}_t^2 + \frac{2(d-2)}{r^2} \right) (\hat{\psi}_t^{\mathbf{m}_t})^2 \right] T_t^{\mathbf{m}_t} \right. \\
 & + \left[-\frac{1}{4} \hat{F}_{(v)ab}^{\mathbf{m}_v} \hat{F}_{(v)}^{\mathbf{m}_v ab} + \frac{2}{r} \hat{C}_{(v)a}^{\mathbf{m}_v} \partial^a \hat{C}_{(v)r}^{\mathbf{m}_v} - \frac{1}{2} \left(\mathbf{m}_v^2 + \frac{2(d-2)}{r^2} \right) \hat{C}_{(v)}^{\mathbf{m}_v a} \hat{C}_{(v)a}^{\mathbf{m}_v} + \frac{2}{r^2} \hat{C}_{(v)r}^{\mathbf{m}_v} \hat{C}_{(v)r}^{\mathbf{m}_v} \right] V_t^{\mathbf{m}_v} \\
 & + \left[\frac{1}{2} \partial_a \hat{h}_{bc}^{\mathbf{m}_s} \partial^b \hat{h}^{\mathbf{m}_s ac} - \frac{1}{4} \partial_a \hat{h}_{bc}^{\mathbf{m}_s} \partial^a \hat{h}^{\mathbf{m}_s bc} + \frac{2}{r} \hat{h}^{\mathbf{m}_s ab} \partial_a \hat{h}_{br}^{\mathbf{m}_s} - \frac{1}{4} \left(\mathbf{m}_s^2 + \frac{2(d-2)}{r^2} \right) \hat{h}_{ab}^{\mathbf{m}_s} \hat{h}^{\mathbf{m}_s ab} \right. \\
 & + \frac{2}{r^2} \hat{h}^{\mathbf{m}_s a}{}_r \hat{h}_{ar}^{\mathbf{m}_s} - \frac{1}{2} \partial^b \hat{h}_{ab}^{\mathbf{m}_s} \partial^a \left(\hat{H}^{\mathbf{m}_s} + \hat{\omega}^{\mathbf{m}_s} \right) + \frac{d-3}{2r} \hat{h}_{ar}^{\mathbf{m}_s} \partial^a \left(\hat{H}^{\mathbf{m}_s} + \hat{\omega}^{\mathbf{m}_s} \right) - \frac{1}{r} \partial^a \hat{h}_{ar}^{\mathbf{m}_s} \left(\hat{H}^{\mathbf{m}_s} + \hat{\omega}^{\mathbf{m}_s} \right) \\
 & + \frac{d-3}{r^2} \hat{h}_{rr}^{\mathbf{m}_s} \left(\hat{H}^{\mathbf{m}_s} + \hat{\omega}^{\mathbf{m}_s} \right) + \frac{1}{4} \frac{d-p-2}{d-p-1} \partial_a \hat{\omega}^{\mathbf{m}_s} \partial^a \hat{\omega}^{\mathbf{m}_s} + \frac{1}{2r} \hat{\omega}^{\mathbf{m}_s} \partial_r \hat{\omega}^{\mathbf{m}_s} + \frac{1}{4} \left(\mathbf{m}_s^2 + \frac{d}{r^2} \right) \hat{H}^{\mathbf{m}_s} \hat{H}^{\mathbf{m}_s} \\
 & + \frac{1}{4} \partial_a \hat{H}^{\mathbf{m}_s} \partial^a \hat{H}^{\mathbf{m}_s} + \frac{1}{2r} \hat{H}^{\mathbf{m}_s} \partial_r \hat{H}^{\mathbf{m}_s} + \frac{1}{4} \left(\frac{(d-p-2)(d-p-3)}{(d-p-1)^2} \mathbf{m}_s^2 - \frac{1}{r^2} \left(\frac{2(d-2)}{d-p-1} - d \right) \right) \hat{\omega}^{\mathbf{m}_s} \hat{\omega}^{\mathbf{m}_s} \\
 & \left. + \frac{1}{2} \partial^a \hat{H}^{\mathbf{m}_s} \partial_a \hat{\omega}^{\mathbf{m}_s} + \frac{1}{2} \left(\frac{d-p-2}{d-p-1} \mathbf{m}_s^2 + \frac{d}{r^2} \right) \hat{H}^{\mathbf{m}_s} \hat{\omega}^{\mathbf{m}_s} + \frac{1}{2r} \hat{H}^{\mathbf{m}_s} \partial_r \hat{\omega}^{\mathbf{m}_s} + \frac{1}{2r} \hat{\omega}^{\mathbf{m}_s} \partial_r \hat{H}^{\mathbf{m}_s} \right] S_t^{\mathbf{m}_s} \left. \right\}
 \end{aligned}$$

Variations:

$\hat{h}_{ab}^{\mathbf{m}_s}$ $\hat{\omega}^{\mathbf{m}_s}$ $\hat{C}_{(v)a}^{\mathbf{m}_v}$ $\hat{\psi}_t^{\mathbf{m}_t}$ \Rightarrow independent field equations

$C_{(s)a}^{\mathbf{m}_s}$ $\psi_s^{\mathbf{m}_s}$ $\psi_v^{\mathbf{m}_v}$ \Rightarrow linear comb. of the field eqns & ∇

Field eqns for $\hat{\psi}_t^{\mathbf{m}_t}$ and $\hat{C}_{(v)a}^{\mathbf{m}_v}$ are decoupled

Decoupling the AdS equations

★ Eliminate the trace of $\hat{h}_{ab}^{\mathbf{m}_s}$ using $\hat{h}^{\mathbf{m}_s} = -\frac{d-p-3}{d-p-1}\hat{\omega}^{\mathbf{m}_s}$

★ Define the fields $\hat{\chi}^{\mathbf{m}_s}$, $\hat{\varphi}^{\mathbf{m}_s}$, $\tilde{h}_{(ab)}^{\mathbf{m}_s}$: $\hat{\omega}^{\mathbf{m}_s} = (d-p-1)(\hat{\varphi}^{\mathbf{m}_s} - \hat{\chi}^{\mathbf{m}_s})$,

$$\hat{h}_{(rr)}^{\mathbf{m}_s} = \frac{d-p-3}{p+2}\hat{\varphi}^{\mathbf{m}_s} + \frac{p+1}{p+2}(d-1)\hat{\chi}^{\mathbf{m}_s}$$

$$\hat{h}_{ab}^{\mathbf{m}_s} = \tilde{h}_{(ab)}^{\mathbf{m}_s} - \frac{d-p-3}{p+1}(\eta_{ab} - \delta_a^r \delta_b^r)\hat{\varphi}^{\mathbf{m}_s} - (\eta_{ab} - (d-1)\delta_a^r \delta_b^r)\hat{\chi}^{\mathbf{m}_s}$$

(i.e. such that $\eta^{ab}\tilde{h}_{(ab)}^{\mathbf{m}_s} = 0$ and $\tilde{h}_{(rr)}^{\mathbf{m}_s} = 0$)

⇒ Resulting field eqns: $\square\hat{\chi}^{\mathbf{m}_s} - \frac{d-5}{r}\partial_r\hat{\chi}^{\mathbf{m}_s} - \mathbf{m}_s^2\hat{\chi}^{\mathbf{m}_s} = 0$

$$\square\hat{\varphi}^{\mathbf{m}_s} - \frac{d-5}{r}\partial_r\hat{\varphi}^{\mathbf{m}_s} - \left(\mathbf{m}_s^2 + \frac{2(d-2)}{r^2}\right)\hat{\varphi}^{\mathbf{m}_s} = 0$$

$$\square\tilde{h}_{(ab)}^{\mathbf{m}_s} - \frac{d-5}{r}\partial_r\tilde{h}_{(ab)}^{\mathbf{m}_s} - \left(\mathbf{m}_s^2 + \frac{2(d-2)}{r^2}\right)\tilde{h}_{(ab)}^{\mathbf{m}_s} + \frac{d-1}{r^2}\left(\delta_a^r\tilde{h}_{(br)}^{\mathbf{m}_s} + \delta_b^r\tilde{h}_{(ar)}^{\mathbf{m}_s}\right) = 0$$

The equations decouple completely, and can be explicitly solved for all modes in terms of Bessel functions.

Comparing AdS and Minkowski modes

★ The zero-modes h_{ab}^0 , $\pi(x; n)$ and h_{ab}^Λ , $\varpi(x; n)$ are mapped into each other.

$$h_{(ab)}^0(x; n) = \frac{r^2}{\ell^2} h_{(ab)}^\Lambda(x; -n) \quad \pi(x; n) = \frac{r^4}{\ell^2} \frac{n+1}{n+p+1} \varpi(x; -n)$$

$$H^0(x; n) = \frac{r^2}{\ell^2} \left[H^\Lambda(x; -n) + \frac{p+2}{n+p+1} \varpi(x; -n) \right]$$

★ Naive map of all modes (scalar sector, up to total derivatives),

$$\ell^{d-1} \tilde{S}_0 - S_{AdS} = \int d^{p+2}x \left(\frac{\ell}{r} \right)^{d-5} \left\{ \frac{1}{4} \left(\mathbf{m}^2 + \frac{\Lambda}{r^2} \right) \left(\hat{h}_{ab} \hat{h}^{ab} - \hat{H}^2 \right) \right. \\ \left. - \frac{1}{4} \frac{d-p-2}{d-p-1} \left(\frac{d-p-3}{d-p-1} \mathbf{m}^2 + \frac{\Lambda}{r^2} \right) \hat{\varpi}^2 - \frac{1}{2} \left(\frac{d-p-2}{d-p-1} \mathbf{m}^2 + \frac{\Lambda}{r^2} \right) \hat{H} \hat{\varpi} \right\}$$

- match for $\Lambda^{I_s} = 0$ and $\mathbf{m}_s^2 = 0$
- the two actions are very similar
- however, qualitatively different behaviour for non-zero modes
- vector and tensor modes follow the same pattern

Comparing AdS and Minkowski modes

- ★ Solve explicitly for all AdS perturbation modes, in terms of Bessel functions,

$$r^{\frac{d}{2}-2} J_\nu(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \quad r^{\frac{d}{2}-2} Y_\nu(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{with } \nu = \frac{d}{2}, \quad \frac{d}{2} - 1, \quad \frac{d}{2} + 2$$

$$k^a k_a = k_r^2 + \mathbf{k}^2 = -\mathbf{m}_s^2 \quad (\text{resp. } -\mathbf{m}_v^2 / -\mathbf{m}_t^2 \text{ for vector and tensor pert.})$$

- ★ Minkowski perturbation modes also in terms of Bessel functions (+part. sol),

$$r^q J_{l+\frac{n}{2}}(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \quad r^q Y_{l+\frac{n}{2}}(k_r r) e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{with } q = -\frac{n}{2}, \quad 1 - \frac{n}{2}, \quad 2 - \frac{n}{2}$$

$$k^a k_a = k_r^2 + \mathbf{k}^2 = 0$$

- ★ A rescaling by a factor r^l together with a mode-dependent analytical continuation of the dimensions can tackle the radial part, but the wavevector is timelike for AdS and null for Minkowski

- ★ There are however similarities and structures appearing, it remains to understand how to exploit them at best to extract information on asymptotically flat holography.

Conclusions



~ How far can we push the AdS/RF correspondence? ~

* **AdS/Ricci-flat correspondence** maps asymptotically locally **AdS** solutions on a torus to **Ricci-flat** spacetimes

These Ricci-flat spaces inherit the holographic properties of AdS

* **Unfreezing the sphere and torus is not easy**

- The modes on the sphere and the torus are not mapped into each other, at least naively
- There are many similarities though, and maybe it will be possible to exploit them to learn about AF holography

* **An alternative path:**

- Keep all KK modes from the sphere reduction of Minkowski
- Uplift them to AdS, and combine them into extra matter fields living in AdS
- Holographically renormalize and translate to the original AF fields

~ *Thank you!* ~