Holographic fluids and integrability

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based on works with M. Caldarelli, J. Gath, R. Leigh, A. Mukhopadhyay, A. Petkou, V. Pozzoli, K. Siampos

Highlights

Foreword

Gravitational self-duality

Weyl self-duality from the bulk to the boundary

Integrability and resummation

Illustration

Outlook

The set up

Framework: holographic fluids as hydrodynamic approximation of finite-T and finite-μ states of a boundary CFT

Original motivation: determine transport coefficients

- Start with some bulk gravitational background related with some boundary fluid in local thermodynamic equilibrium
- Perturb and analyse the response using the bulk-boundary dictionary

Here: pure gravitational backgrounds $\longrightarrow$ neutral boundary fluids

Triggering observation: *some exact bulk solutions describe*

- *non-trivial fluid stationary states*
- *on non-trivial boundary backgrounds*

$\longrightarrow$ enable to probe substantially transport properties [Mukhopadhyay, Petkou, Petropoulos, Pozzoli, Siampos, '13]

Natural question: *can one exhibit more systematically exact bulk Einstein solutions that would produce richer or designed fluid states – and provide more information on transport?*

$\longrightarrow$ answer encoded in integrability properties

Integrability

A very general framework

The question rephrased: how to find *boundary geometries* and combine them with *boundary fluid dynamics* such as these data integrate into an *exact bulk solution* ?

Formally: find an integrable phase subspace corresponding to some first integral – effective reduction from 2nd- to 1st-order equations

- Supergravity: requirement of SUSY & Bianchi identities (BPS)
- General relativity: requirement of self-duality (in the 70s all integrable systems thought to be SDYM reductions [Ward, '85])

$\longrightarrow$ reduction by half of the independent initial data

The guiding principle here: 4-dim self-duality for $2+1$-dim holographic fluids

Implementation

3 steps

1. Translate bulk self-duality into boundary data
2. Implement an integrable deviation from self-duality
3. Resum the Fefferman–Graham/derivative series expansions into exact bulk Einstein spaces

Output: reconstruction of all known exact spaces from a single piece of boundary data – type D Plebański–Demiański, type D & N Robinson–Trautman, Kundt…

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- Gravitational self-duality
- Weyl self-duality from the bulk to the boundary
- Integrability and resummation
- Illustration
- Outlook

The ancestor of holography

The "filling-in" problem – 1982

- A round S^3 can be "filled-in" by H_4

$$\mathrm{d}s^2_{H_4} = \frac{\mathrm{d}r^2}{1+r^2} + r^2 \mathrm{d}\Omega^2_{S^3} \to r^2 \mathrm{d}\Omega^2_{S^3}$$

- How to fill-in *analytically* a Berger sphere?

$$\mathrm{d}\Omega^2_{\text{Berger}} = \left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \gamma\left(\sigma^3\right)^2$$

(σ^i: Maurer–Cartan forms of $SU(2)$)

Answer: Einstein space with self-dual Weyl tensor – quaternionic space [LeBrun '82; Pedersen '86; Pedersen, Poon '90; Tod '94; Hitchin '95]

Curvature decomposition

Metric $ds^2 = \delta_{ab}\theta^a\theta^b$*, connection one-form* ω_{ab} *and curvature two-form* $\mathcal{R}_{ab} \in \mathbf{6}$ *of* $SO(4) \cong SO(3)_{sd} \otimes SO(3)_{asd}$

- Reducible under $SO(3)_{\text{sd}}$ and $SO(3)_{\text{asd}}$: $\mathbf{6} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$
- Curvature two-form $(\lambda, \mu \ldots = 1, 2, 3)$

$$(\mathbf{3}, \mathbf{1}) \quad \mathcal{S}_\lambda = \frac{1}{2}\left(\mathcal{R}_{0\lambda} + \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{R}^{\mu\nu}\right)$$
$$(\mathbf{1}, \mathbf{3}) \quad \mathcal{A}_\lambda = \frac{1}{2}\left(\mathcal{R}_{0\lambda} - \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{R}^{\mu\nu}\right)$$

  and similarly for the connection one-form

- Basis for the space of two-forms $\wedge^2$

$$(\mathbf{3}, \mathbf{1}) \quad \phi^\lambda = \theta^0 \wedge \theta^\lambda + \frac{1}{2}\epsilon^\lambda{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$$
$$(\mathbf{1}, \mathbf{3}) \quad \chi^\lambda = \theta^0 \wedge \theta^\lambda - \frac{1}{2}\epsilon^\lambda{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$$

More on the Riemann tensor

Atiyah–Hitchin–Singer decomposition of $\mathcal{S}_\mu$, $\mathcal{A}_\mu$ [Cahen, Debever, Defise '67; Atiyah, Hitchin, Singer '78]

$$\begin{aligned}\mathcal{S}_\mu &= \tfrac{1}{2} W^+_{\mu\nu}\phi^\nu + \tfrac{1}{12} s\phi_\mu + \tfrac{1}{2} C^+_{\mu\nu}\chi^\nu \\ \mathcal{A}_\mu &= \tfrac{1}{2} W^-_{\mu\nu}\chi^\nu + \tfrac{1}{12} s\chi_\mu + \tfrac{1}{2} C^-_{\nu\mu}\phi^\nu\end{aligned}$$

with $W^\pm$ and $C^\pm$ 3×3 matrices encoding 19 components of the Riemann and s a function

- $s = R/2$ scalar curvature $\to 1$
- $C^+_{\mu\nu} = C^-_{\nu\mu}$ traceless Ricci $\to 9$
- $W^+_{\mu\nu}$ self-dual Weyl tensor symmetric and traceless $\to 5$
- $W^-_{\mu\nu}$ anti-self-dual Weyl tensor symmetric and traceless $\to 5$

Quaternionic spaces: $C^\pm = 0 \quad s = 2\Lambda \quad W^- = 0$ *or* $W^+ = 0 \Leftrightarrow$ *Einstein & Weyl (anti-)self-dual*

The case of Lorentzian signature: $SO(4)$ is traded for $SO(3,1)$

- Decomposition into self-dual and anti-self-dual parts possible upon *complexification*
- W^+ and W^- are complex-conjugate
  - $W^+ = 0 \Leftrightarrow W^- = 0 \Leftrightarrow$ space conformally flat
  - The 10 independent components are captured in 5 complex functions $\Psi_a, \ a = 0, \ldots, 4$ projections of W onto a null tetrad (physical meaning in terms of geodesic deviation)

The existence of 4 principal null directions, potentially degenerate with higher multiplicity, translates into special algebraic relationships among the Ψs: Petrov type I, II, III, D, N, O

Highlights

Gravity in $d = 4$ and the holographic fluid

Set an orthonormal frame $ds^2 = \eta_{ab}\theta^a\theta^b$ ($\eta : +\varepsilon++$)

- Choose a gauge with no lapse or shift $\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2r^2} + \eta_{\mu\nu}\theta^\mu\theta^\nu$
- Expand $\theta^\mu(r,x)$ for large r [Fefferman, Graham '85; subtleties: de Haro *et al* '00]

$$\theta^\mu(r,x) = kr\,E^\mu(x) + \frac{1}{kr}F^\mu_{[2]}(x) + \frac{1}{k^2r^2}F^\mu_{[3]}(x) + \cdots$$

- Read off the 2 independent $2+1$ boundary data: E^μ and $F^\mu_{[3]}$

$$\boxed{\begin{array}{l} \mathrm{d}s^2_{\text{bry.}} = \eta_{\mu\nu}E^\mu E^\nu = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\ \mathrm{T} = \frac{3k}{8\pi G}F^\mu_{[3]}e_\mu = T^\mu{}_\nu E^\nu \otimes e_\mu \end{array}}$$

these allow for the Hamiltonian bulk reconstruction

Bulk Weyl self-duality and its boundary manifestation

Expanding $W^{\pm} = 0$ *leads to* [Leigh, Petkou '07; de Haro '08; Mansi et al '08; Miskovic, Olea '09]

$$8\pi G k^2 T_{\mu\nu} \pm (i) C_{\mu\nu} = 0$$

with $C_{\mu\nu}$ *the components of the boundary Cotton*

Key property: C *and* T are

- traceless
- conserved

Away from the self-dual point so is

$$\boxed{T^{\text{ref}\pm}_{\mu\nu} = T^{\mu\nu} \pm \frac{(i)}{8\pi G k^2} C^{\mu\nu} \neq 0}$$

reflecting $W^{\pm}_{\mu\nu} = \frac{8\pi G}{k^2 r^3} T^{\text{ref}\pm}_{\mu\nu} + \cdots \neq 0$

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The question

Using FG expansion any reasonable boundary data $g_{\mu\nu}$ and $T_{\mu\nu}$ allow to reconstruct a bulk Einstein space

- *neither necessarily regular*
- *nor generally exact*

Given a boundary geometry $\mathrm{d}s^2_{\text{bry.}}$ can one determine

- the conditions it should satisfy
- the stress tensor it should be accompanied with

for the expansion to be exactly resummable?

The answer

The Lorentzian-signature $ds^2_{bry.}$ must admit 2 symmetric, traceless and conserved rank-2 tensors $T^{ref\pm}$ related by complex conjugation

The pattern: scan classes of $ds^2_{bry.}$ admitting exact $\mathrm{T}^{\mathrm{ref}\pm}$ and

- further impose on $ds^2_{bry.}$ the condition

$$\boxed{\mathrm{C} = 8\pi G k^2 \,\mathrm{Im}\,\mathrm{T}^{\mathrm{ref}+}} \qquad \text{(C)}$$

- build the bulk with the resulting $ds^2_{bry.}$ and the stress tensor

$$\boxed{\mathrm{T} = \mathrm{Re}\,\mathrm{T}^{\mathrm{ref}+}} \qquad \text{(T)}$$

1 piece of bry. data is used – subject Eq. (C), accompanied with (T)

The reference tensors $T^{ref\pm}$

Integrability in Einstein spaces is tight to Petrov types D, N, O

- n principal null directions of multiplicity m with $nm = 4$
- null shear-free geodesic congruences

$\Longrightarrow$ $W^{\pm}$ are remarkably simple and so must be $\mathrm{T}^{\mathrm{ref}\pm}$

Boundary geometries expected to lead to resummable series should

- *either possess complex-conjugate time-like geodesic congruences associated with perfect-fluid-form $T^{ref\pm}$*
- *or admit null congruences associated with pure-radiation $T^{ref\pm}$*

Remarks

On conformal perfect fluids with some time-like velocity field u

- $\mathsf{T}^{\text{perf}} = p\left(3\mathsf{u}^2 + \mathsf{d}s^2_{\text{bry.}}\right)$
- Euler equations $\begin{cases} \nabla_{\mathsf{u}} \log p + {}^{3}/_{2}\,\Theta = 0 \\ \nabla_{\perp} \log p + 3\mathsf{a} = 0 \end{cases}$

$\Longrightarrow$ geodesic and expansionless u solve them with constant p

On the actual stress tensor $T = Re T^{ref+}$

- Not expected to be perfect: $\mathsf{T} = \mathsf{T}^{\text{perf}} + \Pi$
- The fluid congruence u is read off from the perfect piece
- T^{perf} and Π *are not* separately conserved

The series expansion

Using the boundary data $ds^2_{bry.}$ and T as well as C and u the partly resummed derivative expansion reads [Bhattacharyya et al '08; Caldarelli et al '12]

$$ds^2_{bulk} = -2u(dr + rA) + r^2 k^2 ds^2_{bry.} + \frac{1}{k^2}\Sigma + \frac{u^2}{\rho^2}\left(\frac{8\pi G T_{\lambda\mu}u^\lambda u^\mu}{k^2} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^6}\right) + h.d. \qquad (R)$$

- $A = a - \frac{\Theta}{2}u \quad \omega = \frac{1}{2}(du + u \wedge a)$
- $\Sigma = -2u\nabla_\nu \omega^\nu{}_\mu dx^\mu - \omega_\mu{}^\lambda \omega_{\lambda\nu} dx^\mu dx^\nu - \frac{1}{2}u^2\left(R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu\right)$
- $\rho^2 = r^2 + \frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu} \quad \eta^{\mu\nu\sigma} = \epsilon^{\mu\nu\sigma}/\sqrt{-g_{bry.}}$

Using Eqs. (C) and (T) the first terms of (R) are exact Einstein

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Examples without vorticity

$$\boxed{ds^2_{bry.} = -dt^2 + \frac{2}{k^2P^2}d\zeta d\bar{\zeta}} \qquad (nv)$$

$P(t,\zeta,\bar{\zeta})$ real & *a priori* arbitrary – define $K = 2P^2\partial_\zeta\partial_{\bar{\zeta}}\log P$

- Cotton-tensor components $C_{\mu\nu}$:

$$\begin{pmatrix} 0 & -\frac{k^2}{2}\partial_\zeta K & \frac{k^2}{2}\partial_{\bar{\zeta}} K \\ -\frac{k^2}{2}\partial_\zeta K & -\partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) & 0 \\ \frac{k^2}{2}\partial_{\bar{\zeta}} K & 0 & \partial_t\left(\frac{\partial_{\bar{\zeta}}^2 P}{P}\right) \end{pmatrix}$$

- Complex-conjugate geodesic & expansionless congruences $\mathrm{u}^+ = -\mathrm{d}t + \frac{\alpha^+}{P^2}\mathrm{d}\zeta$ and c.c.: $\alpha^\pm(t,\zeta,\bar{\zeta})$ satisfy

$$\boxed{k^2P\partial_\zeta\alpha^- = 2\left(k^2\alpha^-\partial_\zeta P + \partial_t P\right) \quad \text{plus c.c.}} \qquad \text{(h)}$$

- With M constant $\mathrm{T}^{\mathrm{ref}\pm} = \frac{Mk^2}{8\pi G}\left(3\left(\mathrm{u}^{\pm}\right)^2 + \mathrm{d}s^2_{\mathrm{bry.}}\right)$ is conserved
- Requiring $\mathrm{C} = 8\pi G k^2 \,\mathrm{Im}\mathrm{T}^{\mathrm{ref}+}$ sets 1 constraint on P

$$\boxed{\left(\partial_\zeta K\right)^2 + 6M\partial_t\left(\frac{\partial^2_\zeta P}{P}\right) = 0} \qquad \text{(D)}$$

plus 1 constraint on α^- $\partial_{\bar\zeta} K = 3Mk^2\frac{\alpha^-}{P^2}$ – combined with (h) gives

$$\boxed{P^2\partial_{\bar\zeta}\partial_\zeta K - 6M\partial_t \log P = 0} \qquad \text{(E)}$$

(plus c.c.)

The stress tensor T

- Using $\mathrm{T} = \mathrm{Re}\mathrm{T}^{\mathrm{ref}+}$ one finds the *non-perfect* $^{8\pi G}/_{k^2}\mathrm{T}$

$$\begin{pmatrix} 2M & -\frac{1}{2k^2}\partial_\zeta K & -\frac{1}{2k^2}\partial_{\bar\zeta} K \\ -\frac{1}{2k^2}\partial_\zeta K & -\frac{1}{k^4}\partial_t\left(\frac{\partial^2_\zeta P}{P}\right) & \frac{M}{k^2P^2} \\ -\frac{1}{2k^2}\partial_{\bar\zeta} K & \frac{M}{k^2P^2} & -\frac{1}{k^4}\partial_t\left(\frac{\partial^2_{\bar\zeta} P}{P}\right) \end{pmatrix}$$

- The *perfect part* is $\mathrm{T}^{\mathrm{perf}} = \frac{Mk^2}{8\pi G}\left(3\mathrm{u}^2 + \mathrm{d}s^2_{\mathrm{bry.}}\right)$ with $\mathrm{u} = -\mathrm{d}t$ a geodesic expanding congruence with zero shear and zero vorticity – *not conserved*

Resummation: using $ds^2_{bry.}$, C, T and u in Eq. (R)

$$\boxed{ds^2_{bulk} = 2dt\,dr - 2Hdt^2 + 2\frac{r^2}{P^2}d\zeta d\bar{\zeta} + h.d.} \qquad \text{(RT)}$$

with

$$2H = K + 2r\partial_t \log P - \frac{2M}{r} + k^2 r^2$$

The displayed part without h.d. is

- exact Einstein thanks to Eq. (E) $\rightarrow$ integrability condition
- Petrov type D thanks to Eq. (D) $\Leftrightarrow 3\Psi_2\Psi_4 = 2\Psi_3^2$

Robinson–Trautman type D class

- u $\longleftarrow$ 2 multiplicity-2 bulk principle null directions
- $u_\pm \longleftarrow$ 2/4 bulk tetrad elements

Alternative: same boundary Eq. (nv) $ds^2_{bry.} = -dt^2 + \frac{2}{k^2P^2}d\zeta d\bar{\zeta}$

- Pure-radiation reference tensor

$$4\pi G k^2\, \mathsf{T}^{\mathsf{ref}+} = F(t,\zeta)\, \mathsf{d}\zeta^2$$

arbitrary $F(t,\zeta) \Rightarrow \mathsf{T}^{\mathsf{ref}\pm}$ conserved

- C, T, u – resummation

Robinson–Trautman type N class

u $\longleftarrow$ 1 multiplicity-4 bulk principle null direction

Examples with vorticity

$$\boxed{ds^2_{bry.} = -\left(dt - b\right)^2 + \frac{2}{k^2 P^2} d\zeta d\bar{\zeta}} \qquad (nv)$$

P real function and $\mathrm{b} = b_\zeta \mathrm{d}\zeta + b_{\bar{\zeta}} \mathrm{d}\bar{\zeta}$ a real form – *a priori* arbitrary

- Impose $\exists$ 1 Killing $\Rightarrow$ 2nd one [Mukhopadhyay *et al* '13]
- Impose $\exists$ 2 c.c. geodesic expanding congruences $\mathrm{u}_\pm \Rightarrow$ perfect-fluid conserved $\mathrm{T}^{\mathrm{ref}\pm}$ (non-constant pressure)
- Impose $\mathrm{C} = 8\pi G k^2 \,\mathrm{Im}\mathrm{T}^{\mathrm{ref}+} \Rightarrow$ solve for P and b $\Rightarrow \mathrm{d}s^2_{\mathrm{bry.}}$
- Extract $\mathrm{T} = \mathrm{Re}\mathrm{T}^{\mathrm{ref}+} = \mathrm{T}^{\mathrm{perf}} + \Pi$
- $\mathrm{T}^{\mathrm{perf}}$ generally non-conserved – aligned with $\mathrm{u} = -\mathrm{d}t + \mathrm{b}$ shearless, expanding geodesic congruence with vorticity
- Resum – Eq. (R): exact Petrov type D Plebański–Demiański familly (mass, rotation, nut, "twist", acceleration)

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Summary: bottom-up approach and integrability

- *Idea: shape $2+1$-dim $ds^2_{bry.}$ and T for exact ascendent*
- *Pattern: design conserved $T^{ref\pm}$ of perfect or radiation conformal-fluid type*
- *Output:*
  - *Integration achieved: limited derivative expansion is exact Einstein (Plebański–Demiański, Robinson–Trautman, Kundt…)*
  - *Remarkable form of $T^{ref\pm} \Rightarrow$ special form of $W^{\pm}$: algebraic Petrov type (Kerr, Taub–NUT, C-metric, pp-waves…)*
  - *Generalizable to Einstein–Maxwell…*

Consequence for holographic fluids: transport properties

- Exact solutions provide already rich information on transport coefficients (a fortiori when T is non-perfect) [Mukhopadhyay et al '13; de Freitas, Reall '14; Bakas, Skenderis '14]
- Perturbation of exact Einstein spaces as a deeper probe for transport can be made more systematic – captured in the known h.d. terms of the ds^2_{bulk} expansion

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Illustration of LeBrun's filling-in

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A classic example

Bianchi IX AdS Schwarzschild–Taub–NUT

- Einstein space with $\Lambda = -3k^2$, mass M, nut charge n

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{V(r)} + \left(r^2 - n^2\right)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) + V(r)\left(\mathrm{d}\tau + 4n\sin^2\frac{\vartheta}{2}\mathrm{d}\varphi\right)^2$$

$$V(r) = \tfrac{1}{r^2-n^2}\left[r^2 + n^2 - 2Mr + k^2\left(r^4 - 6n^2r^2 - 3n^4\right)\right]$$

- Weyl (anti-)self-dual (*i.e.* quaternionic) iff

$$M = \pm n(1 - 4k^2n^2)$$

$\Longleftrightarrow$ no conical singularity at $r = n$

The boundary geometry: $ds^2 \underset{r\to\infty}{\to} \frac{dr^2}{k^2 r^2} + k^2 r^2 ds^2_{bry.}$

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{bry.}} &= \left(\mathrm{d}\tau + 4n\sin^2\frac{\vartheta}{2}\mathrm{d}\varphi\right)^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \\ &= \frac{1}{k^2}\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right) + 4n^2\left(\sigma^3\right)^2 \end{aligned}$$

with $\tau = -2n(\psi + \varphi)$ and $0 \le \vartheta \le \pi, 0 \le \varphi \le 2\pi, 0 \le \psi \le 4\pi$

$$\begin{cases} \sigma^1 = \sin\vartheta\sin\psi\,\mathrm{d}\varphi + \cos\psi\,\mathrm{d}\vartheta \\ \sigma^2 = \sin\vartheta\cos\psi\,\mathrm{d}\varphi - \sin\psi\,\mathrm{d}\vartheta \\ \sigma^3 = \cos\vartheta\,\mathrm{d}\varphi + \mathrm{d}\psi. \end{cases}$$

Conclusion: $\mathrm{d}s^2_{\mathrm{bry.}}$ is a Berger sphere

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Gravity in $d = 4$

Palatini formulation and $3+1$ *split* [Leigh, Petkou '07; Mansi, Petkou, Tagliabue '08]

$$I_{EH} = -\frac{1}{32\pi G}\int_{\mathcal{M}} \epsilon_{abcd}\left(\mathcal{R}^{ab} + \frac{k^2}{2}\theta^a \wedge \theta^b\right) \wedge \theta^c \wedge \theta^d$$

θ^a an orthonormal frame $\mathrm{d}s^2 = \eta_{ab}\theta^a\theta^b$ ($\eta : +\varepsilon++$)
gauge: no lapse, no shift

- Coframe: $\theta^r = \frac{\mathrm{d}r}{kr}$ and θ^μ

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2 r^2} + \eta_{\mu\nu}\theta^\mu\theta^\nu$$

- Connection: $\omega^{r\mu} = \mathcal{K}^\mu$ and $\omega^{\mu\nu} = -\epsilon^{\mu\nu\rho}\mathcal{B}_\rho$ or (a)sd combination $1/2(\mathcal{K}^\mu \pm \mathcal{B}^\mu)$ for $\varepsilon = +$

Hamiltonian evolution of $\theta^\mu, \mathcal{K}^\mu, \mathcal{B}_\rho$ from boundary data – what are the independent boundary data? Answer in asymptotically AdS: Fefferman–Graham expansion for large r *[Fefferman, Graham '85; subtleties: de Haro, Skenderis, Solodukhin, '00]*

$$\begin{aligned}
\theta^\mu(r,x) &= kr\,E^\mu(x) + \frac{1}{kr}F^\mu_{[2]}(x) + \frac{1}{k^2r^2}F^\mu_{[3]}(x) + \cdots \\
\mathcal{K}^\mu(r,x) &= -k^2r\,E^\mu(x) + \frac{1}{r}F^\mu_{[2]}(x) + \frac{2}{kr^2}F^\mu_{[3]}(x) + \cdots \\
\mathcal{B}^\mu(r,x) &= B^\mu(x) + \frac{1}{k^2r^2}B^\mu_{[2]}(x) + \cdots
\end{aligned}$$

Independent $2+1$ boundary data: E^μ and $F^\mu_{[3]}$

The holographic fluid

Interpretation of the boundary data

- E^μ: boundary orthonormal coframe – allows to determine

$$\mathrm{d}s^2_{\text{bry.}} = \eta_{\mu\nu}E^\mu E^\nu = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

  - $F^\mu_{[2]} = -1/2k^2 S^{\mu\nu} e_\nu$: Schouten
  - $B^\mu_{[2]} = 1/2k^2 C^{\mu\nu} e_\nu$: Cotton
  - ...

- $F^\mu_{[3]}$: stress current one-form – allows to construct the vev of the boundary stress tensor

$$\mathrm{T} = \frac{3k}{8\pi G} F^\mu_{[3]} e_\mu = T^\mu{}_\nu E^\nu \otimes e_\mu$$

Macroscopic object carrying microscopic data from the bulk

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The Robinson–Trautman type N class

$$ds^2_{bry.} = -dt^2 + \frac{2}{k^2 P^2} d\zeta d\bar{\zeta}$$

Now pure-radiation reference tensor

$$4\pi G k^2\, T^{ref+} = F(t,\zeta)\, d\zeta^2$$

arbitrary $F(t,\zeta) \Rightarrow T^{ref\pm}$ *conserved*

- Requiring $\mathsf{C} = 8\pi G k^2\, \mathrm{Im} \mathrm{T}^{\mathrm{ref}+}$ sets 1 constraint on P

$$\boxed{\partial_\zeta K = 0} \qquad \text{(N)}$$

plus

$$\partial_t \left(\frac{\partial_\zeta^2 P}{P} \right) + F(t,\zeta) = 0 \qquad \text{(F)}$$

(plus c.c.)

- Eq. (N) sets $K = K(t)$ and determines $P(t,\zeta,\bar{\zeta})$
- Eq. (F) determines $F(t,\zeta)$ – *no constraint*

- Using $\mathrm{T} = \mathrm{Re}\mathrm{T}^{\mathrm{ref}+}$ one finds the *non-perfect* stress tensor

$$8\pi G k^2\, \mathrm{T} = F(t,\zeta)\, \mathrm{d}\zeta^2 + \bar{F}(t,\bar{\zeta})\, \mathrm{d}\bar{\zeta}^2$$

Using $ds^2_{bry.}$, C, T and $u = -dt$ in Eq. (R) gives (RT) with $M = 0$

- Petrov type N thanks to

$$M = 0 \;\Leftrightarrow\; \Psi_2 = 0$$
$$(N) \;\Leftrightarrow\; \Psi_3 = 0$$

- Always exact Einstein

Note: $P(t,\zeta,\bar{\zeta}) = \frac{1+\varepsilon/2\, g\, \bar{g}}{\sqrt{2f\, \partial_\zeta g\, \partial_{\bar{\zeta}} \bar{g}}}$ with $\varepsilon = 0, \pm 1$ and $f(t), g(t,\zeta)$ arbitrary functions – $F(t,\zeta)$ expressed in terms of $g(t,\zeta)$ and its derivatives

Robinson–Trautman type N class

u $\longleftarrow$ 1 multiplicity-4 bulk principle null direction