Holographic Lifshitz scale anomalies

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Motivation

- Lifshitz symmetry is realized in QFTs close to a second order phase transition, e.g. quantum critical points. Relevant for real materials such as high T_c superconductors.
- Quantum anomalies are physical aspects of QFTs dictating the form of contact terms in correlation functions and impacting on physical observables such as transport coefficients, even in flat space!
- Local Weyl anomalies are also related to RG flows and the c/a-theorem.
- A significant part of our understanding of Lifshitz theories has come from holography.

Outline

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Wess-Zumino consistency conditions

- In the local RG description of QFT [Osborn '93] couplings are promoted to *local* fields:

$$S[\{g(x)\};\{\varphi(x)\}]$$

- Global symmetries of the classical action S are promoted to *gauge symmetries* under which the local couplings transform: space-time symmetries are promoted to local diffeomorphisms and global internal symmetries to Yang-Mills/Maxwell gauge symmetries.
- The classical action is invariant under an infinitesimal gauge transformation parameterized by χ^α, i.e.

$$\delta_{\chi^\alpha} S[\{g(x)\};\{\phi(x)\}] = 0$$

but the quantum effective action $W[\{g(x)\};\{\phi(x)\}]$ generically will transform anomalously under such transformations:

$$\delta_{\chi^\alpha} W[\{g(x)\}] = A_{\chi^\alpha}[\{g(x)\}]$$

- The anomalies A_{χ^α} must satisfy the Wess-Zumino condition

$$\delta_{\chi^\alpha} A_{\chi^\beta} - \delta_{\chi^\beta} A_{\chi^\alpha} = \left(\delta_{\chi^\alpha}\delta_{\chi^\beta} - \delta_{\chi^\beta}\delta_{\chi^\alpha}\right) W[\{g(x)\}]$$
$$= \delta_{[\chi^\alpha,\chi^\beta]} W[\{g(x)\}] = A_{[\chi^\alpha,\chi^\beta]}[\{g(x)\}]$$

- A trivial solution of this condition is one corresponding to A_{χ^α} being itself a gauge transformation of a local functional of the couplings, i.e.

$$A_{\chi^\alpha} = \delta_{\chi^\alpha} G[\{g(x)\}]$$

  Anomalies of this form are unphysical since they can be removed by adding $-G[\{g(x)\}]$ as a counterterm to the generating functional $W[\{g(x)\}]$.
- The space of possible physical anomalies is the space of all solutions to the WZ consistency condition modulo gauge transformations of local functionals.
- This problem can be formulated formally as a cohomology problem by introducing a Grassmann-valued BRST-like ghost.

- In the local RG formulation of QFT the local couplings act as sources of local operators, which can be defined as

$$\mathcal{O}_g(x) \sim \frac{\delta W[\{g\}]}{\delta g(x)}$$

- This definition of local operators leads to the anomalous Ward identities:

$$\delta_\chi W[\{g\}] = \sum_g \int \delta_\chi g(x) \mathcal{O}_g(x) = A_\chi$$

where $\delta_\chi g(x)$ denotes the transformation of the local couplings under the gauge transformation parameterized by χ.

Isotropic (relativistic) Weyl anomalies

- In order to gauge the Poincaré group in a relativistic QFT we introduce a metric $g_{(0)ij}(x)$ and define the theory covariantly on the resulting manifold so that the action is invariant under local diffeomorphisms:

$$\delta_\xi g_{(0)ij}(x) = D_{(0)i}\xi_j + D_{(0)j}\xi_i, \quad \delta_\sigma \varphi_{(0)}(x) = \mathcal{L}_\xi \varphi_{(0)}$$

- In a classically scale invariant QFT we also promote scaling transformations to local Weyl transformations:

$$\delta_\sigma g_{(0)ij}(x) = 2\delta\sigma(x) g_{(0)ij}(x), \quad \delta_\sigma \varphi_{(0)}(x) = -(d-\Delta)\delta\sigma(x)\varphi_{(0)}(x)$$

- Inserting these transformations of the local couplings in the above general Ward identity leads to the familiar diffeomorphism and trace Ward identities of a relativistic QFT:

$$\boxed{D_{(0)i}\mathcal{T}^i_j + \mathcal{O}\partial_j\varphi_{(0)} = 0, \quad \mathcal{T}^i_i + (d-\Delta)\varphi_{(0)}\mathcal{O} = \mathcal{A}}$$

where the stress tensor and the operator $\mathcal{O}(x)$ are defined via the relations

$$\mathcal{T}^{ij} := -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta g_{(0)ij}}, \quad \mathcal{O} := \frac{1}{\sqrt{g_{(0)}}}\frac{\delta W}{\delta \varphi_{(0)}}$$

- The possible contributions to the trace anomaly $\mathcal{A}$ are determined by the relative cohomology problem of the Weyl operator δ_σ with respect to diffeomorphisms, i.e. non-trivial cocycles in the cohomology of δ_σ built out of diffeomorphism invariants [Bonora, Pasti, Tonin '86].
- For relativistic theories it is known that:
  - for **odd** dimensions there are no Weyl anomalies (except for possible pure matter contributions)
  - for **even** dimensions there are two types of Weyl anomalies: the Euler density (type A) and the various Weyl invariants in the corresponding dimension (type B).
- This cohomological problem determines all possible terms that can appear in the conformal anomaly, but does not fix their coefficients, which depend on the particular theory.
- In the presence of matter sources writing down all possible diffeomorphism invariants of the right dilatation weight becomes more complicated.

Anisotropic (Lifshitz) Weyl anomalies

- The analogous problem for anisotropic Weyl transformations has been studied holographically in a number of recent works (e.g. [Baggio, de Boer, Holsheimer '11; Griffin, Hořava, Melby-Thompson '11]) and purely cohomologically by [Arav, Chapman, Oz '14].
- In Lifshitz theories time is singled out due to the anisotropic scaling symmetry governed by the dynamical exponent $z \neq 1$.
- A Lifshitz QFT is therefore naturally described on a manifold with a metric $g_{(0)ij}$ together with a codimension-1 foliation specified locally by a 1-form t_i that satisfies the Frobenius integrability condition

$$t \wedge dt = 0$$

- Below we will make extensive use of the normalized foliation 1-form

$$\boxed{\mathbb{n}_i \propto t_i, \quad \mathbb{n}_i \mathbb{n}^i = -1}$$

Note: a somewhat different covariantization of Lifshitz theories, corresponding to gauging the Schrödinger algebra, has been put forward in [Hartong, Kiritsis, Obers '14; Bergshoeff, Hartong, Rosseel '14; Hartong, Kiritsis, Obers '15]

- The local symmetries under which the background fields $g_{(0)ij}$ and t_i transform are diffeomorphisms:

$$\delta_\xi g_{(0)ij}(x) = D_{(0)i}\xi_j + D_{(0)j}\xi_i, \quad \delta_\xi t_i = \mathcal{L}_\xi t_i = \xi^j D_{(0)j} t_i + D_{(0)i}\xi^j t_j$$

a subset of which are foliation preserving, i.e. $\mathcal{L}_\xi t_i \propto t_i$, and anisotropic Weyl rescalings:

$$\delta_\sigma t_i = 0, \quad \delta_\sigma \mathbb{n}_i = z\delta\sigma \mathbb{n}_i, \quad \delta_\sigma \mathbb{n}^i = -z\delta\sigma \mathbb{n}^i, \quad \delta_\sigma \varpi_{ij} = 2\delta\sigma\varpi_{ij}$$

where $\varpi_{ij} = g_{(0)ij} + \mathbb{n}_i\mathbb{n}_j$ is the projector to the tangent space of the foliation.

- These transformations lead respectively to the diffeomorphism and anisotropic Weyl Ward identities (in the notation of [Arav, Chapman, Oz '14])

$$\boxed{D_{(0)j}\mathcal{T}_i^j - 2\partial_{[i}\mathbb{n}_{j]}\mathcal{J}^j + \mathbb{n}_i D_{(0)j}\mathcal{J}^j = 0, \quad \varpi_{ij}\mathcal{T}^{ij} - z\mathbb{n}_i\mathbb{n}_j\mathcal{T}^{ij} = \mathcal{A}}$$

where

$$\mathcal{T}^{ij} \sim \frac{2}{\sqrt{-g_{(0)}}}\frac{\delta W}{\delta g_{(0)ij}}, \quad \mathcal{J}^i \sim \frac{1}{\sqrt{-g_{(0)}}}\frac{\delta W}{\delta t_i}$$

- The possible contributions to the anisotropic Weyl anomaly $\mathcal{A}$ can be determined by the relative cohomology of the anisotropic dilatation operator δ_σ with respect to foliation preserving diffeomorphisms.
- All possible contributions can be expressed in terms of the following geometric data [Chemissany, I.P. '14; Arav, Chapman, Oz '14]:

$$\sigma_{ij}, \quad \mathbb{q}_i = \mathbb{n}^j D_j \mathbb{n}_i, \quad \mathbb{K}_{ij} = \frac{1}{2}\mathcal{L}_{\mathbb{n}}\sigma_{ij}, \quad \mathbb{R}_{ijkl}[\sigma]$$

as well as time, $\mathcal{L}_{\mathbb{n}}$, and spatial $\mathbb{D}_i$, derivatives of any of these tensors and any additional matter fields.
- The cohomological problem involves writing down the all possible diffeomorphism invariant combinations of the correct dilatation weight and does not determine the coefficients of the allowed terms.

Outline

Holographic computation of Weyl anomalies

[Henningson, Skenderis ’98]

- Local sources arise as integration functions in the asymptotic solutions of the equations of motion.
- The gauge transformations of these sources correspond to the subset of asymptotic bulk diffeomorphims leaving the asymptotic expansions form-invariant, i.e. Penrose-Brown-Henneaux (PBH) diffeomorphisms.
- Conformal anomalies arise due to local counterterms that explicitly depend on the UV cutoff.

- The Lifshitz (Lif) metric is [Kachru, Liu, Mulligan '08]

$$ds^2_{d+2} = \ell^2 u^{-2} \left(du^2 - u^{-2(z-1)} dt^2 + dx^a dx^a \right)$$

with dynamical exponent $z \neq 1$

- This metric is invariant under the scaling transformation

$$x^a \to \lambda x^a \ , \ t \to \lambda^z t \ , \ u \to \lambda u$$

- The conformal boundary is located at $u = 0$
- The null energy condition

$$T_{\mu\nu} k^\mu k^\nu \geq 0, \quad k^\mu k_\mu = 0$$

requires

$$z \geq 1$$

Einstein-Proca-Dilaton (EPD) model

$$S_\xi = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}e^{d\xi\phi}\left(R[g]-\alpha_\xi\partial_\mu\phi\partial^\mu\phi - Z_\xi(\phi)F^2 - W_\xi(\phi)B^2 - V_\xi(\phi)\right)$$

- The Stückelberg ω renders the vector field

$$B_\mu = A_\mu - \partial_\mu\omega$$

  invariant under the $U(1)$ transformation

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \quad \omega \to \omega + \Lambda$$

- We work in the "dual frame" so that asymptotically locally Lifshitz boundary conditions generically correspond to asymptotically hyperscaling violating Lifshitz boundary conditions in the Einstein frame.

- This model admit Lif or hvLf solutions at least asymptotically provided the potentials are of the form

$$V_\xi = V_o e^{2(\rho+\xi)\phi}, \qquad Z_\xi = Z_o e^{-2(\xi+\nu)\phi}, \qquad W_\xi = W_o e^{2\sigma\phi}$$

- The parameters are related to the parameters of the Lif solutions

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2, \quad A = \frac{\mathcal{Q}}{\epsilon Z_o}e^{\epsilon r}dt, \quad \phi = \mu r, \quad \omega = const.$$

as

$$\rho = -\xi, \quad \nu = -\xi + \frac{\epsilon - z}{\mu}, \quad \sigma = \frac{z-\epsilon}{\mu},$$

$$\epsilon = \frac{(\alpha_\xi + d^2\xi^2)\mu^2 - d\mu\xi + z(z-1)}{z-1}, \quad \mathcal{Q}^2 = \frac{1}{2}Z_o(z-1)\epsilon,$$

$$W_o = 2Z_o\epsilon(d+z+d\mu\xi-\epsilon), \quad V_o = -d(1+\mu\xi)(d+z+d\mu\xi) - (z-1)\epsilon.$$

- ADM decomposition

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

- Radial ADM Lagrangian:

$$\begin{aligned} L \quad = \quad & \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-\gamma}Ne^{d\xi\phi}\Big\{R[\gamma] + K^2 - K^{ij}K_{ij} + \frac{2d\xi}{N}K(\dot{\phi} - N^i\partial_i\phi) \\ & -\frac{\alpha_\xi}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - \alpha_\xi\gamma^{ij}\partial_i\phi\partial_j\phi \\ & -Z_\xi(\phi)\left(\frac{2}{N^2}\gamma^{ij}(F_{ri} - N^k F_{ki})(F_{rj} - N^l F_{lj}) + \gamma^{ij}\gamma^{kl}F_{ik}F_{jl}\right) \\ & -W_\xi(\phi)\left(\frac{1}{N^2}\left(A_r - N^i A_i - \dot{\omega} + N^i\partial_i\omega\right)^2 + \gamma^{ij}B_iB_j\right) - V_\xi(\phi)\Big\} \end{aligned}$$

- Hamiltonian:

$$
\begin{aligned}
H &= \int d^{d+1}x \left(\dot{\gamma}_{ij}\pi^{ij} + \dot{A}_i\pi^i + \dot{\phi}\pi_\phi + \dot{\omega}\pi_\omega\right) - L \\
&= \int d^{d+1}x \left(N\mathcal{H} + N_i\mathcal{H}^i + A_r\mathcal{F}\right)
\end{aligned}
$$

- where

$$
\begin{aligned}
\mathcal{H} = -\frac{\kappa^2}{\sqrt{-\gamma}}e^{-d\xi\phi}\left(2\pi^{ij}\pi_{ij} - \frac{2}{d}\pi^2 + \frac{1}{2\alpha}\left(\pi_\phi - 2\xi\pi\right)^2 + \frac{1}{4}Z_\xi^{-1}\pi^i\pi_i + \frac{1}{2}W_\xi^{-1}\pi_\omega^2\right) \\
+ \frac{\sqrt{-\gamma}}{2\kappa^2}e^{d\xi\phi}\left(-R[\gamma] + \alpha_\xi\partial^i\phi\partial_i\phi + Z_\xi(\phi)F^{ij}F_{ij} + W_\xi(\phi)B^iB_i + V_\xi(\phi)\right)
\end{aligned}
$$

$$
\mathcal{H}^i = -2D_j\pi^{ji} + F^i{}_j\pi^j + \pi_\phi\partial^i\phi - B^i\pi_\omega
$$

$$
\mathcal{F} = -D_i\pi^i + \pi_\omega
$$

Canonical momenta

- From off-shell Lagrangian:

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(K\gamma^{ij} - K^{ij} + \frac{d\xi}{N}\gamma^{ij}\left(\dot{\phi} - N^k\partial_k\phi\right)\right),$$

$$\pi^{i} = \frac{\delta L}{\delta \dot{A}_{i}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}Z_\xi(\phi)\frac{4}{N}\gamma^{ij}\left(F_{rj} - N^k F_{kj}\right),$$

$$\pi_\phi = \frac{\delta L}{\delta \dot{\phi}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(2d\xi K - \frac{2\alpha_\xi}{N}(\dot{\phi} - N^i\partial_i\phi)\right),$$

$$\pi_\omega = \frac{\delta L}{\delta \dot{\omega}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}W_\xi(\phi)\frac{2}{N}\left(\dot{\omega} - N^i\partial_i\omega - A_r + N^i A_i\right)$$

- From on-shell action:

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}, \quad \pi^{i} = \frac{\delta \mathcal{S}}{\delta A_{i}}, \quad \pi_\phi = \frac{\delta \mathcal{S}}{\delta \phi}, \quad \pi_\omega = \frac{\delta \mathcal{S}}{\delta \omega}$$

Flow equations

- Combining the two expressions for the momenta:

$$\dot{\gamma}_{ij} = -\frac{4\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\left(\gamma_{ik}\gamma_{jl} - \frac{\alpha_\xi + d^2\xi^2}{d\alpha} \gamma_{ij}\gamma_{kl} \right) \frac{\delta}{\delta\gamma_{kl}} - \frac{\xi}{2\alpha}\gamma_{ij}\frac{\delta}{\delta\phi} \right) \mathcal{S},$$

$$\dot{A}_i = -\frac{\kappa^2}{2} \frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} Z_\xi^{-1}(\phi)\gamma_{ij}\frac{\delta}{\delta A_j}\mathcal{S},$$

$$\dot{\phi} = -\frac{\kappa^2}{\alpha}\frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\frac{\delta}{\delta\phi} - 2\xi\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} \right) \mathcal{S},$$

$$\dot{\omega} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} W_\xi^{-1}(\phi)\frac{\delta}{\delta\omega}\mathcal{S}$$

Zero derivative solution of the HJ equation

- The zero order solution of the HJ equation contains no transverse derivatives:

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2}\int d^{d+1}x\sqrt{-\gamma}U(\phi, A_i A^i)$$

- Inserting this ansatz into the Hamiltonian constraint yields a PDE for $U(X,Y)$, where $X := \phi$, $Y := B_i B^i = A_i A^i$ (cf. superpotential equation)

$$\frac{1}{2\alpha}\left(U_X - \xi(d+1)U + 2\xi Y U_Y\right)^2 + Z_\xi^{-1}(X) Y U_Y^2$$
$$-\frac{1}{2d}\left((d+1)U + 2(d-1)YU_Y\right)\left(U - 2YU_Y\right) = \frac{1}{2}e^{2d\xi X}\left(W_\xi(X)Y + V_\xi(X)\right)$$

- This equation for the 'superpotential' $U(X,Y)$ determines the zero derivative solution of the Hamilton-Jacobi equation: It can be used to holographically renormalize any homogeneous background of the equations of motion and any exact solution of this PDE leads to exact solutions of the equations of motion via the flow equations.

Asymptotically locally Lifshtiz backgrounds

- Imposing Lifshitz boundary conditions requires that asymptotically the gauge invariant vector field behaves as

$$\boxed{B_i \sim B_{oi} = \sqrt{\frac{z-1}{2\epsilon}} Z_\xi^{-1/2}(\phi)\mathbb{n}_i}$$

where $\mathbb{n}_i$ is the canonically normalized time foliation 1-form.

- This in turn implies that the superpotential $U(X,Y)$ must satisfy

$$U(X, Y_o(X)) \sim e^{d\xi X}\left(d(1+\mu\xi)+z-1\right)$$
$$U_Y(X, Y_o(X)) \sim -\epsilon e^{d\xi X} Z_\xi(X)$$
$$U_X(X, Y_o(X)) \sim e^{d\xi X}\left(-\mu\alpha_\xi + d\xi(d+z)\right)$$

- Hence, the asymptotic form of the zero order solution of the HJ equation is

$$\mathcal{S}_{(0)} \sim \frac{1}{\kappa^2}\int_{\Sigma_r} d^{d+1}x\sqrt{-\gamma}e^{d\xi\phi}\left(d(1+\mu\xi)+\frac{1}{2}(z-1)-\epsilon Z_\xi(\phi)B_iB^i\right)$$

Taylor expansion of the superpotential

- Since Lifshitz boundary conditions require that $B_i \sim B_{oi}$ asymptotically, the solution of the HJ equation can be expressed as a Taylor series in $B_i - B_{oi}$
- The zero derivative solution $\mathcal{S}_{(0)}$ can be Taylor expanded in

$$Y - Y_o = 2B_o^i(B_i - B_{oi}) + \mathcal{O}(B - B_o)^2$$

where $Y_o \equiv B_o^i B_{oi}$, as

$$U = e^{(d+1)\xi\phi}\left(u_0(\phi) + Y_o^{-1}u_1(\phi)(Y - Y_o(\phi)) + Y_o^{-2}u_2(\phi)(Y - Y_o(\phi))^2 + \cdots\right)$$

- Inserting this expansion in the superpotential equation for $U(X, Y)$ leads to a tower of equations for the functions $u_n(\phi)$

- An additional relation between the functions $u_0(\phi)$ and $u_1(\phi)$ is imposed by the consistency of the Taylor expansion, i.e. requiring that

$$\dot{Y} - \dot{Y}_o = \mathcal{O}(Y - Y_o)$$

- In a bottom up approach these equations can be used to *define* the potentials $V(\phi)$, $Z(\phi)$ and $W(\phi)$ in terms of $u_0(\phi)$ and $u_1(\phi)$, with all $u_n(\phi)$ for $n \geq 2$ being determined in terms of these functions.
- Lifshitz boundary conditions require

$$u_0(\phi) \sim (z - 1 + d(1 + \mu\xi))\, e^{-\xi\phi}$$
$$u_1(\phi) \sim \frac{1}{2}(z-1)e^{-\xi\phi}$$

- The function $u_2(\phi)$ satisfies a quadratic (Riccati) equation and determines the scaling behavior of the independent mode $Y - Y_o$, while $u_n(\phi)$ with $n \geq 3$ satisfy linear equations.

Recursive solution of the HJ equation

- To summarize the above analysis, we have shown that the most general zero derivative solution of the HJ equation takes the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int_{\Sigma_r} d^{d+1}x \sqrt{-\gamma} U(\phi, B^2)$$

where for Lifshitz boundary conditions the superpotential $U(X, Y)$ admits a Taylor expansion in $Y - Y_o$. Moreover, this zero derivative solution is the asymptotically leading one, with derivative terms entering only in asymptotically subleading orders.

- In order to systematically determine these asymptotically subleading derivative terms of the solution of the HJ equation, we expand $\mathcal{S}$ in a covariant expansion in eigenfunctions of a suitable operator.
- For backgrounds with asymptotic scaling invariance one can use the dilatation operator [I. P. & Skenderis 2004] but in the presence of an asymptotically running dilaton, meaning that asymptotic scale invariance is broken, this is not sufficient.
- Instead we need an operator such that $\mathcal{S}_{(0)}$ is an eigenfunction for any superpotential $U(\phi, B^2)$.

- In fact there are two mutually commuting such operators:

$$\widehat{\delta} := \int d^{d+1}x \left(2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right), \quad \delta_B := \int d^{d+1}x \left(2Y^{-1} B_i B_j \frac{\delta}{\delta\gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right)$$

which satisfy

$$\widehat{\delta}\mathcal{S}_{(0)} = (d+1)\mathcal{S}_{(0)}, \quad \delta_B \mathcal{S}_{(0)} = \mathcal{S}_{(0)}, \quad [\widehat{\delta}, \delta_B] = 0$$

- This allows us to seek a solution of the HJ equation in the form of a *graded* covariant expansion in simultaneous eigenfunctions of both $\widehat{\delta}$ and δ_B:

$$\mathcal{S} = \sum_{k=0}^{\infty} \mathcal{S}_{(2k)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \mathcal{S}_{(2k,2\ell)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \int d^{d+1}x \mathcal{L}_{(2k,2\ell)}$$

where

$$\widehat{\delta}\mathcal{S}_{(2k,2\ell)} = (d+1-2k)\mathcal{S}_{(2k,2\ell)}, \quad \delta_B \mathcal{S}_{(2k,2\ell)} = (1-2\ell)\mathcal{S}_{(2k,2\ell)}, \quad 0 \leq \ell \leq k$$

- The operator $\widehat{\delta}$ counts derivatives
- The operator δ_B annihilates the projection operator $\sigma^i_j := \delta^i_j - Y^{-1} B^i B_j$ and counts derivatives contracted with B_i, which asymptotically become time derivatives since $B_i \sim B_{oi} \propto \mathbb{n}_i$

- Inserting the covariant expansion of $\mathcal{S}$ in simultaneous eigenfunctions of $\widehat{\delta}$ and δ_B in the Hamilton-Jacobi equation (Hamiltonian constraint) results in a system of recursive first order functional *linear* equations for the higher derivative terms:

$$\frac{1}{\alpha}\left(U_X-(d+1)\xi U+2\xi Y U_Y\right)\frac{\delta}{\delta\phi}\int\mathcal{L}_{(2k,2\ell)}+$$
$$\left((2Y+Z_\xi^{-1})U_Y+\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)\right)B_i\frac{\delta}{\delta B_i}\int\mathcal{L}_{(2k,2\ell)}-$$
$$\left(\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)(d+1-2k)+2YU_Y(1-2\ell)\right)\mathcal{L}_{(2k,2\ell)}=$$
$$e^{d\xi\phi}\mathcal{R}_{(2k,2\ell)}$$

- The inhomogeneous term $\mathcal{R}_{(2k,2\ell)}$ involves derivatives of lower order terms as well as the 2-derivative sources from the Hamiltonian constraint

- The covariant expansion of $\mathcal{S}$ in simultaneous eigenfunctions of $\widehat{\delta}$ and δ_B, and hence the above recursion relations, is independent of the specific choice of boundary conditions
- In order to impose Lifshitz boundary conditions we must additionally expand $\mathcal{S}_{(2k,2\ell)}$ in $B_i - B_{oi}$ at each order of the covariant expansion as

$$\begin{aligned}\mathcal{L}_{(2k,2\ell)} &= \mathcal{L}^0_{(2k,2\ell)}[\gamma(x),\phi(x)] \\ &+ \int d^{d+1}x'(B_i(x')-B_{oi}(x'))\mathcal{L}^{1i}_{(2k,2\ell)}[\gamma(x),\phi(x);x'] + \mathcal{O}\left(B-B_o\right)^2\end{aligned}$$

- Inserting this Taylor expansion in the above recursion relations eliminates the derivative with respect to B_i, resulting in first order linear functional differential equations in ϕ only. Such functional differential equations appear in the relativistic case as well, e.g. for non-conformal branes or Improved Holographic QCD, and they can be solved systematically [I.P. '11].

Solution of the recursion relations up to $\mathcal{O}(B-B_o)$

- The inhomogeneous solution of these linear functional differential equations takes the form

$$\mathcal{L}^0_{(2k,2\ell)} = e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)}\int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi})e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})}\mathcal{R}^0_{(2k,2\ell)},$$

$$\varpi^i_j\mathcal{L}^{1j}_{(2k,2\ell)} = Z_\xi^{\frac{1}{2}}e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)}\int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi})e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})}Z_\xi^{-\frac{1}{2}}\varpi^i_j\mathcal{R}^{1j}_{(2k,2\ell)},$$

$$B_{oj}(x)\mathcal{L}^{1j}_{(2k,2\ell)} = \Omega^{-1}e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)}\int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi})e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})}\Omega B_{oj}\widehat{\mathcal{R}}^{1j}_{(2k,2\ell)}$$

where $\mathcal{C}_{k,\ell} := d+1-2k+(z-1)(1-2\ell)$,

$$\mathcal{K}(\phi) := \frac{\alpha}{e^{\xi\phi}\left(u_0'+\frac{Z'}{Z}u_1\right)} \sim -\frac{1}{\mu}, \quad e^{\mathcal{A}(\phi)} = Z_\xi^{-\frac{1}{2(\epsilon-z)}} \sim e^{\phi/\mu}$$

and the $\Omega(\phi)$ can be expressed in terms of u_0, u_1 and u_2.

- If $\mu=0$ (e.g. for Einstein-Proca theory) the corresponding solutions can be expressed *algebraically* in terms of the source terms.

Structure of the HJ solution

- The general asymptotic solution of the HJ equation obtained via the above algorithm takes the form

$$\boxed{\mathcal{S}=\sum_{k,\ell,m\ |\ \mathcal{C}_{k,\ell}+\theta-m\Delta_-\geq 0}\int\cdots\int(B-B_o)^m\mathcal{S}^m_{(2k,2\ell)}+\widehat{\mathcal{S}}_{ren}+\cdots}$$

  where $\Delta_+=d+z-\theta-\Delta_-$ is the scaling dimension of the scalar operator dual to the mode

$$\psi:=Y_o^{-1}B_o^j(B_j-B_{oj})$$

  and $(B-B_o)^m\mathcal{S}^m_{(2k,2\ell)}$ has dilatation weight $\mathcal{C}_{k,\ell}+\theta-m\Delta_-$, while $\widehat{\mathcal{S}}_{ren}$ has dilatation weight 0.
- All terms $(B-B_o)^m\mathcal{S}^m_{(2k,2\ell)}$ with $\mathcal{C}_{k,\ell}+\theta-m\Delta_-\geq 0$ are determined by the recursion algorithm.
- For $\mathcal{C}_{k,\ell}+\theta-m\Delta_-<0$ these terms are powerlike divergent in the UV, while terms with $\mathcal{C}_{k,\ell}+\theta-m\Delta_-=0$ have a pole which via dimensional regularization leads to a logarithmic divergence. Such logarithmically divergent terms give rise to the conformal anomaly when $\mu=0$, but they can be absorbed in the dilaton when $\mu\neq 0$.

- The covariant local counterterms that render the on-shell action finite and the variational problem with Lifshitz boundary conditions well posed are

$$\mathcal{S}_{ct} := -\sum_{k,\ell,m \mid \mathcal{C}_{k,\ell}+d\mu\xi-m\Delta_-\geq 0} \int\cdots\int (B-B_o)^m \mathcal{S}^m_{(2k,2\ell)}$$

- The renormalized part of the on-shell action is therefore given by the UV-finite term $\widehat{\mathcal{S}}_{ren}$, which corresponds to an independent contribution to the HJ solution and can be parameterized as

$$\widehat{\mathcal{S}}_{ren} = \int d^{d+1}x \left(\gamma_{ij}\widehat{\pi}^{ij} + B_i\widehat{\pi}^i + \phi\widehat{\pi}_\phi\right)$$

where $\widehat{\pi}^{ij}$, $\widehat{\pi}^i$ and $\widehat{\pi}_\phi$ are undetermined integration functions of the HJ equation.

- Inserting this general asymptotic solution of the HJ equation, including the undetermined term $\widehat{\mathcal{S}}_{ren}$, in the first order flow equations one can systematically derive the generalized asymptotic Fefferman-Graham expansions for the bulk fields, including the sources and 1-point functions of the dual operators.
- The sources generically correspond to integration constants of the flow equations, while the 1-point functions are related to the integration constants of the HJ solution in $\widehat{\mathcal{S}}_{ren}$.
- Decomposing the induced fields as

$$\gamma_{ij}dx^i dx^j = -(n^2 - n_a n^a)dt^2 + 2n_a dt dx^a + \sigma_{ab}dx^a dx^b, \quad A_i dx^i = a dt + A_a dx^a,$$

where the indices a, b run from 1 to d, and introducing the linear combinations

$$\widehat{\mathcal{T}}^{ij} := -\frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\left(2\widehat{\pi}^{ij} + Y_o^{-1}B_o^i B_o^j B_{ok}\widehat{\pi}^k\right),$$

$$\widehat{\mathcal{O}}_\phi := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\left(\widehat{\pi}_\phi + (\nu+\xi)B_{oi}\widehat{\pi}^i\right),$$

$$\widehat{\mathcal{O}}_\psi := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}B_{oi}\widehat{\pi}^i, \quad \widehat{\mathcal{E}}^i := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\sqrt{-Y_o}\sigma^i_j\widehat{\pi}^j,$$

the full set of sources and VEVs is (cf. energy-momentum complex [Ross '09]):

	1-point function	source
spatial stress tensor	$\widehat{\Pi}^i_j := \mathbb{\sigma}^i_k \mathbb{\sigma}_{jl} \mathcal{T}^{kl} \sim e^{-(d+z-\theta)r} \Pi^i_j(x)$	$\sigma_{(0)ab}$
momentum density	$\widehat{\mathcal{P}}^i := -\mathbb{\sigma}^i_k \mathbb{n}_l \mathcal{T}^{kl} \sim e^{-(d+2-\theta)r} \mathcal{P}^i(x)$	$n_{(0)a}$
energy density	$\widehat{\mathcal{E}} := -\mathbb{n}_k \mathbb{n}_l \mathcal{T}^{kl} \sim e^{-(d+z-\theta)r} \mathcal{E}(x)$	$n_{(0)}$
energy flux	$\widehat{\mathcal{E}}^i \sim e^{-(d+2z-\theta)r} \mathcal{E}^i(x)$	0
dilaton	$\widehat{\mathcal{O}}_\phi \sim e^{-(d+z+d\mu\xi)r} \mathcal{O}_\phi(x)$	$\phi_{(0)}$
composite scalar	$\widehat{\mathcal{O}}_\psi \sim e^{-\Delta_+ r} \mathcal{O}_\psi(x)$	ψ_-

- The momentum constraint of the radial Hamiltonian formalism leads to the diffeomorphism Ward identities

$$\mathbb{D}_j\widehat{\Pi}^i_i + \mathbb{q}_j\widehat{\Pi}^j_i + \mathbb{n}^j D_j\widehat{\mathcal{P}}_i + \mathbb{K}\widehat{\mathcal{P}}_i + \mathbb{K}^i_i\widehat{\mathcal{P}}_j + \mathbb{n}_i\mathbb{q}_j\widehat{\mathcal{P}}^j - \widehat{\mathcal{E}}\mathbb{q}_i + \widehat{\mathcal{O}}_\phi\mathbb{D}_i\phi + \widehat{\mathcal{O}}_\psi\mathbb{D}_i\psi = 0,$$
$$\mathbb{n}^i D_i\widehat{\mathcal{E}} + \mathbb{K}\widehat{\mathcal{E}} - \mathbb{K}^i_j\widehat{\Pi}^j_i + \mathbb{D}_i\widehat{\mathcal{E}}^i + \widehat{\mathcal{O}}_\phi\mathbb{n}^i D_i\phi = 0, \tag{1}$$

  where $\mathbb{D}_i$ is the covariant derivative w.r.t. σ_{ij}, $\mathbb{K}_{ij} = \mathbb{D}_i\mathbb{n}_j$ is the extrinsic curvature of the constant time slices, and $\mathbb{q}_i = \mathbb{n}^k D_k\mathbb{n}_i$.

- The transformation of the renormalized action under *local* anisotropic boundary Weyl transformations leads to the trace Ward identity

$$z\widehat{\mathcal{E}} + \widehat{\Pi}^i_i + \Delta_-\psi\widehat{\mathcal{O}}_\psi - \mu\widehat{\mathcal{O}}_\phi = 0, \qquad \mu \neq 0,$$
$$z\widehat{\mathcal{E}} + \widehat{\Pi}^i_i + \Delta_-\psi\widehat{\mathcal{O}}_\psi = \mathcal{A}, \qquad \mu = 0,$$

  where $\mathcal{A}$ is the conformal anomaly, corresponding to all terms satisfying $\mathcal{C}_{k,\ell} + \theta - m\Delta_- = 0$.

Outline

- The recursion relations are algebraic in this case and lead to the asymptotic solution

$$\mathcal{S} = \int d^2x dt \left(\mathcal{L}^0_{(0)} + \mathcal{L}^0_{(2,0)} + r_o \mathcal{L}^0_{(2,2)} + r_o \mathcal{L}^0_{(4,0)} \right)$$

where r_o is the radial cutoff and

$$\begin{aligned}
\mathcal{L}^0_{(0)} &= \frac{\sqrt{-\gamma}}{2\kappa^2} 6, \\
\mathcal{L}^0_{(2,0)} &= \frac{\sqrt{-\gamma}}{2\kappa^2} \frac{1}{2} \left(\mathbb{R} - 2D_k \mathbb{q}^k + \frac{1}{2} \mathbb{q}^k \mathbb{q}_k \right) \simeq \frac{\sqrt{-\gamma}}{2\kappa^2} \frac{1}{2} \left(\mathbb{R} + \frac{1}{2} \mathbb{q}^k \mathbb{q}_k \right), \\
\mathcal{L}^0_{(2,2)} &= \frac{\sqrt{-\gamma}}{2\kappa^2} \left(\mathbb{K}^{kl} \mathbb{K}_{kl} + 2\mathbb{n}^k D_k \mathbb{K} + \frac{3}{2} \mathbb{K}^2 \right) \simeq \frac{\sqrt{-\gamma}}{2\kappa^2} \left(\mathbb{K}^{kl} \mathbb{K}_{kl} - \frac{1}{2} \mathbb{K}^2 \right), \\
\mathcal{L}^0_{(4,0)} &= \frac{\sqrt{-\gamma}}{2\kappa^2} \frac{1}{4} \left\{ \left(\mathbb{D}_i \mathbb{q}_j + \frac{1}{2} \mathbb{q}_i \mathbb{q}_j - \frac{1}{2} \sigma_{ij} \left(\mathbb{D}_k \mathbb{q}^k + \frac{1}{2} \mathbb{q}_k \mathbb{q}^k \right) \right)^2 \right. \\
&\qquad \left. - \frac{1}{2} \left(D_k \mathbb{q}^k - \frac{1}{2} \mathbb{q}_k \mathbb{q}^k \right)^2 + \frac{1}{2} \mathbb{R} \mathbb{q}_k \mathbb{q}^k \right\} \simeq 0
\end{aligned}$$

- The anisotropic Weyl anomaly therefore is (see also [Baggio, de Boer, Holsheimer '11; Griffin, Hořava, Melby-Thompson '11])

$$\boxed{\mathcal{A} = \frac{\sqrt{-\gamma}}{2\kappa^2}\left(\mathbb{K}^{kl}\mathbb{K}_{kl} - \frac{1}{2}\mathbb{K}^2\right)}$$

- This is one of the two possible non-trivial cocycles that can appear for $d = z = 2$ with this field content, the other one being the invariant

$$\sqrt{-\gamma}\left(\mathbb{R} + \mathbb{D}_k \mathfrak{q}^k\right)^2$$

involving only spatial derivatives.

- The algorithm gives the logarithmic contributions to $\mathcal{L}$ in terms of Weyl cocycles plus total derivative terms. However, evaluating the trace Ward identity expresses the anomaly directly in terms of Weyl cocycles.

$d = z = 2$, $\mu = 0$ EPD model

- Keeping the an asymptotically constant dilaton (i.e. $\mu = 0$) leads to a solution of the form

$$\mathcal{S} = \int d^2x dt e^{\phi} \left(\mathcal{L}^0_{(0)} + \mathcal{L}^0_{(2,0)} + r_o \mathcal{L}^0_{(2,2)} + r_o \mathcal{L}^0_{(4,0)} \right)$$

where $\mathcal{L}^0_{(0)}$, $\mathcal{L}^0_{(2,0)}$ and $\mathcal{L}^0_{(2,2)}$ are as for EP model, while now

$$\begin{aligned}
\mathcal{L}^0_{(4,0)} \quad = \quad & \frac{\sqrt{-\gamma}}{2\kappa^2} \frac{1}{4} \left\{ \left(\mathbb{R}^{ij} - \mathbb{D}^{(i} \mathfrak{q}^{j)} - \frac{1}{2} \mathfrak{q}^i \mathfrak{q}^j \right)^2 + \frac{1}{2} \left(\mathbb{R} - 2 D_k \mathfrak{q}^k + \frac{1}{2} \mathfrak{q}^k \mathfrak{q}_k \right)^2 \right. \\
& + \frac{1}{4} \left(\mathbb{R} + \frac{1}{2} (\mathfrak{q}^k \mathfrak{q}_k - 2 D_k \mathfrak{q}^k) \right)^2 \\
& - \left(\mathbb{R} - 2 D_k \mathfrak{q}^k + \frac{1}{2} \mathfrak{q}^k \mathfrak{q}_k \right) \left(\mathbb{R} - D_k \mathfrak{q}^k + \frac{1}{2} \mathfrak{q}^k \mathfrak{q}_k \right) \\
& - \frac{1}{8} \left(\mathbb{R} - 3 D_k \mathfrak{q}^k + \frac{1}{2} \mathfrak{q}^k \mathfrak{q}_k \right)^2 \\
& \left. + \frac{1}{24} \left(\mathbb{R} + \frac{1}{2} \mathfrak{q}^k \mathfrak{q}_k + D_k \mathfrak{q}^k \right)^2 + \frac{1}{12} \left(\mathbb{R} - \mathfrak{q}^k \mathfrak{q}_k + D_k \mathfrak{q}^k \right)^2 \right\} \\
\simeq \quad & -\frac{\sqrt{-\gamma}}{2\kappa^2} \frac{1}{16} \left(\mathbb{R} + \mathbb{D}_k \mathfrak{q}^k \right)^2
\end{aligned}$$

- The anisotropic Weyl anomaly in this case therefore is

$$\boxed{\mathcal{A} = \frac{\sqrt{-\gamma}}{2\kappa^2} e^{\phi} \left[\left(\mathbb{K}^{kl}\mathbb{K}_{kl} - \frac{1}{2}\mathbb{K}^2 \right) - \frac{1}{16} \left(\mathbb{R} + \mathbb{D}_k \mathbb{q}^k \right)^2 \right]}$$

  which includes both possible cocycles for $d = z = 2$.
- To my knowledge this is the only known model where the potential term actually contributes to the anomaly.

Outline

1. Anomalies from cohomology
2. Holographic Lifshitz theories from EPD gravity
3. Examples
4. Concluding remarks

Concluding remarks

- Lifshitz QFT can be defined covariantly in terms of a background metric and a 1-form that describes the time foliation.
- Lifshitz scale anomalies correspond to the relative cohomology of the anisotropic Weyl operator with respect to foliation preserving diffeomorphims.
- As for relativistic theories, Lifshitz scale anomalies in theories with a weakly coupled holographic dual correspond to the logarithmically divergent counterterms that explicitly depend on the radial cutoff.
- Such terms can be computed systematically using a general recursive procedure for solving the radial Hamilton-Jacobi equation.