Holographic conserved charges of rotating black holes

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Aspects of fluid/gravity correspondence

based on the work [arXiv:1406.7101], [arXiv:1410.1312]
with S. Hyun, S.-A. Park, S.-H. Yi

Motivations

Conserved charges in the gravity side

- We have consistent definitions for 'global' charges in general relativity.
  - ADM - asymptotically flat spacetime in Einstein gravity ['62 Arnowitt, Deser, Misner]
  - ADT - asymptotic conserved charges are obtained in a covariant manner useful for the asymptotically AdS space-time in Einstein as well as general higher derivative theories of gravity. ['82 Abbott, Deser]
  - covariant phase space method ['94 Iyer, Wald],...
- Conserved charges at the 'quasi-local' level are established.
  - In Einstein gravity ['01 Barnich, Brandt]
- In covariant theory of gravity ['13 Kim, Kulkarni, Yi]

Holographic conserved charges in the Asymptotic AdS

- Boundary stress tensor method to obtain holographic charges consistent with the dual field theory. ['99 Balasubramanian, Kraus]

Connection between traditional & holographic approach?

- Verified in asymptotically AdS geometry in Einstein gravity. ['05 Papadimitriou, Skenderis], ['05 Hollands, Ishibashi, Marolf]

Backgrounds

Abbott-Deser-Tekin (ADT) formalism ['82 Abbott, Deser]

- Asymptotic conserved charges are obtained in a covariant manner.
- Depends on the bulk Euler-Lagrange expression.
- Assuming the fast falloff behaviors of matter fields at the asymptotic infinity.

- Construct the covariant conserved quantity : on-shell ADT current

$$\mathcal{J}^{\mu} = \delta\mathcal{G}^{\mu\nu}\xi_{\nu}, \qquad \partial_{\mu}(\sqrt{-g}\mathcal{J}^{\mu}|_{on}) = 0$$

  where ξ : Killing vector, $\mathcal{G}^{\mu\nu}$: generalized Einstein tensor.
- Construct the ADT potential using Poincaré lemma : $\mathcal{J}^{\mu}|_{on} = \nabla_{\nu}Q^{\mu\nu}_{ADT}$
- Obtain the conserved (global) Killing charges

$$\delta Q(\xi) = \frac{1}{8\pi G}\int_{\mathcal{B}} d^{D-2}dx_{\mu\nu}\,\sqrt{-g}\,Q^{\mu\nu}_{ADT}$$

Backgrounds

On-shell ADT current $\mathcal{J}^{\mu}$: background dependent $\Rightarrow$ needs to be extended.
['13 Kim, Kulkarni, Yi]

- Off-shell ADT current $(\delta\xi^{\mu} = 0)$ ['07 Bouchareb, Clément], ['10 Nam, Park, Yi]

$$\mathcal{J}^{\mu}_{ADT} = \delta\mathcal{G}^{\mu\nu}\xi_{\nu} + \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\mathcal{G}^{\mu\nu}\xi_{\nu} + \mathcal{G}^{\mu\nu}\delta g_{\nu\rho}\xi^{\rho} - \frac{1}{2}\xi^{\mu}\mathcal{G}^{\alpha\beta}\delta g_{\alpha\beta}$$

which is identically conserved.

$$\partial_{\mu}(\sqrt{-g}\mathcal{J}^{\mu}_{ADT}) = 0$$

- ADT potential : $\mathcal{J}^{\mu}_{ADT} = \nabla_{\nu}Q^{\mu\nu}_{ADT}$
- Obtain the quasi-local conserved charges with one-parameter path in the solution space.

$$Q(\xi) = \frac{1}{8\pi G}\int_0^1 ds \int_{\mathcal{B}} d^{D-2}dx_{\mu\nu}\sqrt{-g}Q^{\mu\nu}_{ADT}(g;\xi|s)$$

$\Rightarrow$ Needs to be extended for the theory containing matter fields slow falling off.

Construction: bulk ADT charges

Off-shell ADT current for Killing vector ξ in the presence of matter fields ['14 Hyun, JJ, Park, Yi]

$$\sqrt{-g}\mathcal{J}^{\mu}_{ADT} = \sqrt{-g}\Big[\delta\mathbf{E}^{\mu\nu}\xi_{\nu} + \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\,\mathbf{E}^{\mu\nu}\xi_{\nu} + \mathbf{E}^{\mu\nu}\delta g_{\nu\rho}\,\xi^{\rho} + \frac{1}{2}\xi^{\mu}\mathcal{E}_{\Psi}\delta\Psi\Big]$$
$$= \delta\Big(\sqrt{-g}\,\mathbf{E}^{\mu\nu}\xi_{\nu}\Big) + \frac{1}{2}\sqrt{-g}\,\xi^{\mu}\mathcal{E}_{\Psi}\delta\Psi$$

- where $\mathcal{E}_{\Psi}\delta\Psi \equiv \mathcal{E}_{\alpha\beta}\delta g^{\alpha\beta} + \mathcal{E}_{\psi}\delta\psi$
- Identically conserved : $\partial_{\mu}(\sqrt{-g}\mathcal{J}^{\mu}_{ADT}) = 0$
- ADT potential : $\mathcal{J}^{\mu}_{ADT} = \nabla_{\nu}Q^{\mu\nu}_{ADT}$

Quasi-local conserved charges

$$Q(\xi) = \frac{1}{8\pi G}\int_0^1 ds\int_{\mathcal{B}} d^{D-2}dx_{\mu\nu}\sqrt{-g}Q^{\mu\nu}_{ADT}(g;\xi|s)$$

Construction: bulk ADT charges

Off-shell ADT current $\mathcal{J}^{\mu}_{ADT}$ in the presence of matter fields.

- Consider a theory of gravity with arbitrary matter fields $\psi = (\phi^I, A_\mu, \cdots)$

$$I[g,\psi] = \frac{1}{16\pi G}\int d^D x\,\sqrt{-g}\,\mathcal{L}(g,\psi)$$

- for the generic variation

$$\delta I[g,\psi] = \frac{1}{16\pi G}\int d^D x\left[\sqrt{-g}\big(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}_\psi \delta\psi\big) + \partial_\mu \Theta^\mu(\delta g, \delta\psi)\right]$$

- for the diffeomorphism variation

$$\begin{aligned}
\delta_\zeta(\sqrt{-g}\mathcal{L}) &= \sqrt{-g}\big(-\mathcal{E}^{\mu\nu}\delta_\zeta g_{\mu\nu} + \mathcal{E}_\psi\delta_\zeta\psi\big) + \partial_\mu\Theta^\mu(\delta g,\delta\psi) \\
&= \sqrt{-g}\big(2\zeta_\nu\nabla_\mu\mathcal{E}^{\mu\nu} + \mathcal{E}_\psi\pounds_\zeta\psi\big) + \partial_\mu\big(\Theta^\mu - 2\sqrt{-g}\mathcal{E}^{\mu\nu}\zeta_\nu\big) \\
&= \partial_\mu\big(\Theta^\mu - 2\sqrt{-g}\mathcal{E}^{\mu\nu}\zeta_\nu + \sqrt{-g}\mathcal{Z}^{\mu\nu}\zeta_\nu\big) \\
&= \partial_\mu\big(\Theta^\mu - 2\sqrt{-g}\mathbf{E}^{\mu\nu}\zeta_\nu\big) \qquad \text{with } \mathbf{E}^{\mu\nu} \equiv \mathcal{E}^{\mu\nu} - \frac{1}{2}\mathcal{Z}^{\mu\nu} \\
&= \partial_\mu\big(\zeta^\mu\sqrt{-g}\mathcal{L}\big)
\end{aligned}$$

Comparison: covariant phase space approach

Connection between the off-shell ADT formalism and the covariant phase space method?

- Off-shell Noether current & potential

$$J^\mu(\zeta) = 2\sqrt{-g}\mathbf{E}^{\mu\nu}\zeta_\nu + \zeta^\mu\sqrt{-g}\,\mathcal{L} - \Theta^\mu(\pounds_\zeta g, \pounds_\zeta \psi) \equiv \partial_\nu K^{\mu\nu}$$

- Symplectic current in the covariant phase space formalism. ['90 Lee, Wald]

$$\omega^\mu(\delta_1\Psi\,,\,\delta_2\Psi) \equiv \delta_2\Theta^\mu(\delta_1\Psi) - \delta_1\Theta^\mu(\delta_2\Psi)$$

- Using above, finally comes

$$2\sqrt{-g}\mathcal{J}^\mu_{ADT}(\zeta,\delta\Psi) = \partial_\nu\Big(\delta K^{\mu\nu}(\zeta) - 2\zeta^{[\mu}\Theta^{\nu]}(\delta\Psi)\Big) - \omega^\mu(\pounds_\zeta\Psi\,,\,\delta\Psi)$$

And for Killing vector ξ

$$2\sqrt{-g}\,Q^{\mu\nu}_{ADT}(\xi,\delta\Psi) = \delta K^{\mu\nu}(\xi) - 2\xi^{[\mu}\Theta^{\nu]}(\delta\Psi) \equiv W^{\mu\nu}(\xi,\delta\Psi)$$

where $W^{\mu\nu}$: potential in the covariant phase space approach.

Construction: boundary conserved charges

Boundary current in the asymptotic AdS space

- Consider the renomalized action

$$I_r[g,\psi] = I[g,\psi] + I_{GH}[\gamma] + I_{ct}[\gamma,\psi]$$

with radial decomposed metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2 dr^2 + \gamma_{ij}(r,x)(dx^i + N^i dr)(dx^j + N^j dr)$$

- for the generic variation

$$\delta I_r^{on}[\gamma,\psi] = \frac{1}{16\pi G}\int_{\mathcal{B}} d^d x\,\sqrt{-\gamma}\left[T_B^{ij}\delta\gamma_{ij} + \Pi_\psi \delta\psi\right]$$

- for the boundary diffeomorphism variation

$$\begin{aligned}\sqrt{-\gamma}\left[T_B^{ij}\delta_\zeta\gamma_{ij} + \Pi_\psi\delta_\zeta\psi\right] &= \sqrt{-\gamma}\left[2\zeta_j\nabla_i T_B^{ij} + \Pi_\psi \pounds_\zeta\psi\right] + \partial_i\left(2\sqrt{-\gamma}\,T_B^{ij}\zeta_j\right)\\ &= \partial_i\left(2\sqrt{-\gamma}\,T_B^{ij}\zeta_j + \sqrt{-\gamma}\,\mathcal{Z}_B^{ij}\zeta_j\right)\\ &= \partial_i\left(2\sqrt{-\gamma}\,\mathbf{T}_B^{ij}\zeta_j\right) \qquad \text{with } \mathbf{T}_B^{ij} \equiv T_B^{ij} + \frac{1}{2}\mathcal{Z}_B^{ij}\end{aligned}$$

Construction: boundary conserved charges

Analogous to the bulk case, construct the identically conserved 'boundary' current $\mathcal{J}_B^i$.

- Boundary ADT-like current for a boundary Killing vector ξ_B

$$\sqrt{-\gamma}\mathcal{J}_B^i(\xi_B) \equiv -\delta\Big(\sqrt{-\gamma}\,\mathbf{T}_B^{ij}\xi_j^B\Big) + \frac{1}{2}\sqrt{-\gamma}\,\xi_B^i\Big(T_B^{kl}\delta\gamma_{kl} + \Pi_\psi\delta\psi\Big)$$

- Boundary conserved charges

$$\begin{aligned} Q_B(\xi_B) &= \frac{1}{8\pi G}\int_{\partial\mathcal{B}} d^{d-1}x_i \int ds\ \sqrt{-\gamma}\,\mathcal{J}_B^i(\xi_B) \\ &= -\frac{1}{8\pi G}\int_{\partial\mathcal{B}} d^{d-1}x_i\,\sqrt{-\gamma}\,\Big(\mathbf{T}_B^{ij} + \frac{1}{2}\Delta\mathcal{A}\Big)\xi_j^B \end{aligned}$$

where $\Delta\mathcal{A} \equiv \mathcal{A} - \mathcal{A}_{vac} = 0$.

Equivalent with the boundary stress tensor method!

Equivalence: bulk ADT potential & boundary off-shell current

Relation between bulk & boundary formalisms?

- Fefferman-Graham coordinates for an asymptotically AdS space

$$ds^2 = d\eta^2 + \gamma_{ij} dx^i dx^j$$

- Modified surface term from renormalized action

$$\tilde{\Theta}^\eta(\delta\Psi) = \Theta^\eta(\delta\Psi) + \delta(2\sqrt{-\gamma}L_{GH}) + \delta(\sqrt{-\gamma}L_{ct}) = \sqrt{-\gamma}\Big(T_B^{ij}\delta\gamma_{ij} + \Pi_\psi \delta\psi\Big)$$

- For the diffeomorphism variation $\pounds_\xi \Psi$

$$\tilde{\Theta}^\eta(\pounds_\xi \Psi) = \sqrt{-\gamma}\Big(2T_B^{ij}\nabla_i \zeta_j + \Pi_\psi \pounds_\zeta \psi\Big) = \partial_i\Big(2\sqrt{-\gamma}\,\mathbf{T}_B^{ij}\,\zeta_j\Big)$$

Equivalence: bulk ADT potential & boundary off-shell current

Relation between bulk & boundary formalisms?

- Modified Noether current from renormalized action

$$\tilde{J}^\eta = \partial_i \tilde{K}^{\eta i}(\zeta) = \zeta^\eta \sqrt{-\gamma}\mathcal{L}_r^{on} - \tilde{\Theta}^\eta(\pounds_\zeta \Psi)$$

- The asymptotic behavior of general diffeomorphism parameter ζ

$$\zeta^\eta \sim \mathcal{O}(e^{-d\eta})\,, \qquad \zeta^i \sim \mathcal{O}(1)$$

  which is preserving the asymptotic gauge choice and the renormalized action.

This asymptotic behavior in the diffeomorphism parameter ζ allows us to discard $\zeta^\eta\sqrt{-\gamma}\mathcal{L}_r^{on}$ term when we approach the boundary.

Equivalence: bulk ADT potential & boundary off-shell current

By using this result, one can see that the Noether potential $\tilde{K}^{\eta i}$ becomes

$$\tilde{K}^{\eta i} = 2\sqrt{-\gamma}\,\mathbf{T}^{ij}_{B}\,\zeta_j + \partial_j(\sqrt{-\gamma}\,\mathcal{U}^{ij}_{B})\,,$$

where $\mathcal{U}^{ij}_{B}$ is an arbitrary anti-symmetric second rank tensor irrelevant to obtaining conserved charges. As a result, the relation between the ADT and Noether potentials for a Killing vector ξ becomes

$$\sqrt{-g}Q^{\eta i}_{ADT}|_{\eta\to\infty} = -\delta\Big(\sqrt{-\gamma}\,\mathbf{T}^{ij}_{B}\,\xi^{B}_{j}\Big)+\frac{1}{2}\sqrt{-\gamma}\,\xi^{i}_{B}\Big(T^{kl}_{B}\delta\gamma_{kl}+\Pi_{\psi}\delta\psi\Big) \equiv \sqrt{-\gamma}\mathcal{J}^{i}_{B}$$

$$Q(\xi) = \frac{1}{8\pi G}\int_{\mathcal{B}} d^{D-2}x_{\eta i}\int ds\sqrt{-g}Q^{\eta i}_{ADT}$$

$$||$$

$$Q_B(\xi_B) = \frac{1}{8\pi G}\int_{\partial\mathcal{B}} d^{d-1}x_i\,\int ds\,\sqrt{-\gamma}\,\mathcal{J}^{i}_{B}$$

$\therefore$ The bulk potential and the boundary current lead to the same conserved charges.

Applications: Model

Consider the model

$$I[g,\phi] = \frac{1}{16\pi G}\int d^D x\ \sqrt{-g}\left[R - 2\Lambda - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]$$

- For the generic variation of the Lagrangian

$$\delta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}\Big(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}_\phi\delta\phi\Big) + \partial_\mu\Theta^\mu$$

- EOM are $\mathcal{E}^{\mu\nu} = 0$, $\mathcal{E}_\phi = 0$, where

$$\begin{aligned}\mathcal{E}_{\mu\nu} &\equiv G^\Lambda_{\mu\nu} - T_{\mu\nu} \\ &= \Big[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda\, g_{\mu\nu}\Big] - \Big[\frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}\big(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\big)\Big] \\ \mathcal{E}_\phi &\equiv \nabla^2\phi - \frac{\partial V}{\partial\phi}\end{aligned}$$

- The surface terms

$$\Theta^\mu(\delta g,\delta\phi) = \Theta^\mu_g(\delta g) + \Theta^\mu_\phi(\delta\phi) = \sqrt{-g}\left[2g^{\alpha[\mu}\nabla^{\beta]}\delta g_{\alpha\beta} - \delta\phi\partial^\mu\phi\right]$$

ADT potential

In this model, the total off-shell ADT potential is given by the sum of the metric and scalar contributions as

$$Q^{\mu\nu}_{ADT}(\xi\,;\,\delta g,\delta\phi) = Q^{\mu\nu}_{ADT}(\xi\,;\,\delta g) + Q^{\mu\nu}_{ADT}(\xi\,;\,\delta\phi)$$

$$\begin{aligned} Q^{\mu\nu}_{ADT}(\xi\,;\,\delta g) = &-\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}\nabla^{[\mu}\xi^{\nu]} + \xi^{[\mu}\nabla_\alpha\delta g^{\nu]\alpha} - \xi_\alpha\nabla^{[\mu}\delta g^{\nu]\alpha} \\ &- g_{\alpha\beta}\xi^{[\mu}\nabla^{\nu]}\delta g^{\alpha\beta} + \delta g^{\alpha[\mu}\nabla_\alpha\xi^{\nu]} \end{aligned}$$

$$Q^{\mu\nu}_{ADT}(\xi\,;\,\delta\phi) = \delta\phi\,\xi^{[\mu}\partial^{\nu]}\phi$$

Counter terms and boundary stress tensor

- The generic forms of the GH and counter terms in this model.

The GH term for the Einstein gravity is given by extrinsic curvature scalar at the boundary $K(\gamma)$.

$$L_{GH} = K(\gamma)$$

The counter terms $L_{ct}(\gamma, \phi)$ consist of two parts for the pure gravity and for the scalar field.

$$L_{ct} = 2K_{ct}(\gamma) + \Phi_{ct}(\phi)$$

$$K_{ct}(\gamma) = -(d-1) - \frac{1}{2(d-2)} R_B - \frac{1}{2(d-4)(d-2)^2} \left(R^B_{ij} R^{ij}_B - \frac{d}{4(d-1)} R_B^2 \right) + \cdots$$

$$\Phi_{ct}(\phi) = \alpha_1 \phi^2 + \alpha_2 \phi^4 + \cdots$$

where R^B_{ij} and R_B are intrinsic Ricci tensor and scalar at the boundary. And α_k are determined to cancel the divergences in the renormalized action at the boundary.

Counter terms and boundary stress tensor

- The resultant form of the boundary stress tensor.

$$
\begin{aligned}
T_B^{ij} &= T_g^{ij} + T_\phi^{ij} \\
T_g^{ij} &= K\gamma^{ij} - K^{ij} - (d-1)\gamma^{ij} + \frac{1}{2(d-2)}\left(R_B^{ij} - \frac{1}{2}R_B\gamma^{ij}\right) + \cdots \\
T_\phi^{ij} &= \frac{\gamma^{ij}}{2}\left(\alpha_1\,\phi^2 + \alpha_2\,\phi^4 + \cdots\right)
\end{aligned}
$$

One may note that in this case $\mathbf{T}_B^{ij} = T_B^{ij}$, since we are considering a scalar field only.

- The renormalized momentum of the scalar field.

$$\sqrt{-\gamma}\,\Pi_\phi = \sqrt{-\gamma}\Big[-\partial_\eta\phi + 2\alpha_1\phi + 4\alpha_2\phi^3 + \cdots\Big]$$

The radial expansion

In the Fefferman-Graham coordinates for an asymptotically AdS space

$$ds^2 = d\eta^2 + \gamma_{ij} dx^i dx^j$$

Assumptions

- boundary metric taken to be flat as $\gamma_{ij}^{(0)} = \eta_{ij}$
- scalar field depends only on the radial coord. η

The leading order of scalar field is given by

$$\phi \sim e^{-(d-\Delta_\pm)\eta} \phi_\pm$$

where

- ϕ_+ : leading order term of the non-normalizable mode
- ϕ_- : leading order term of the normalizable mode
- $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$

The mass of the scalar field has the Breitenlohner-Freedman(BF) bound: $m^2 = m_{BF}^2 = -\frac{d^2}{4}$.

The radial expansion

Class I : BF-bound saturated ($m^2 = m^2_{BF} = -\frac{d^2}{4}$)

The radial expansion of the scalar field

$$\phi = e^{-\frac{d}{2}\eta}\Big(\phi_{(0)} + \cdots\Big)$$

Linearized analysis : to see the back reaction of the metric to the scalar field.

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \cdots$$

The linearized EOM of our specific model become

$$h''_{ij} + (d-4)h'_{ij} + (4-2d)h_{ij} - e^{2\eta}\eta_{ij}\left(h'' + dh'\right) = 0\,,$$
$$(d-1)h' - \frac{d^2}{4}e^{-d\eta}\phi^2_{(0)} = 0\,, \qquad h \equiv e^{-2\eta}\eta^{ij}h_{ij}$$
$$\varphi'' + d\varphi' - m^2\varphi = 0\,,$$

where primes denote derivatives with respect to η and $\gamma_{ij} \equiv e^{2\eta}\eta_{ij} + h_{ij}$ and $\phi \equiv e^{-\frac{d}{2}\eta}\phi_{(0)} + \varphi$.

Since the leading order contribution of the scalar field to the metric starts from the order $e^{-d\eta}$, the linear analysis is sufficient to compute conserved charges.

The radial expansion

the leading order coefficient $\gamma_{ij}^{(d)}$ in metric satisfies the trace relation,

$$\eta^{ij}\gamma_{ij}^{(d)} = -\frac{d}{4(d-1)}\phi_{(0)}^2 \, .$$

The form of the coefficients $\gamma_{ij}^{(d)}$ would be further specified by the metric ansatz of the solution. As in the case of pure Einstein gravity, these coefficients can be used to determine the conserved charges.

The radial expansion

Class II : BF-bounded $m^2 > m_{BF}^2 = -\frac{d^2}{4}$

The radial expansion of the scalar field ($\Delta_\phi = \Delta_+$)

$$\phi = e^{-(d-\Delta_\phi)\eta}\left(\phi_{(0)} + e^{-2(d-\Delta_\phi)\eta}\phi_{(2)} + e^{-4(d-\Delta_\phi)\eta}\phi_{(4)} + \cdots\right)$$

for the even scalar potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 + \cdots$$

We restrict to the case $\Delta_\phi < d$ so that this non-normalizable mode does not change the asymptotic AdS structure.

The radial expansion of the metric solution

$$\gamma_{ij} = e^{2\eta}\left[\eta_{ij} + e^{-2(d-\Delta_\phi)\,\eta}\gamma_{ij}^{(2d-2\Delta_\phi)} + \cdots + e^{-d\eta}\gamma_{ij}^{(d)} + \cdots\right]$$

where the leading order term in the expansion is

$$\gamma_{ij}^{(2d-2\Delta_\phi)} = -\frac{\phi_{(0)}^2}{4(d-1)}\eta_{ij}\,.$$

3D Black hole solutions: *Class I*

- Consider the three-dimensional AdS black hole space with scalar hair.

Class I : $m^2 = m_{BF}^2 = -1$

The most general solution obtained by solving the linearized EOM

$$\gamma_{ij}^{(2)} = \begin{pmatrix} C_1 + \frac{1}{4}\phi_{(0)}^2 & -C_2 \\ -C_2 & C_1 - \frac{1}{4}\phi_{(0)}^2 \end{pmatrix}$$

where C_1 and C_2 are arbitrary parameters which turn out to be proportional to the mass and the angular momentum, respectively, of AdS black holes with scalar hair.

3D Black hole solutions: *Class I*

1. Mass : $M_{ADT} \equiv M^g_{ADT} + M^\phi_{ADT} = \frac{1}{4G}\left(C_1 - \frac{1}{4}\phi^2_{(0)}\right) + \frac{1}{16G}\phi^2_{(0)} = \frac{1}{4G}C_1$

   Computed by the ADT potential for the time-like Killing vector $\xi^i_T = (1,0)$

$$\sqrt{-g}\,Q^{\eta i}_{ADT}(\xi_T\,;\,\delta g)\big|_{\eta\to\infty} = \left(\,\delta C_1 - \frac{1}{2}\phi_{(0)}\delta\phi_{(0)}\,,\ \delta C_2\,\right)$$
$$\sqrt{-g}\,Q^{\eta i}_{ADT}(\xi_T\,;\,\delta \phi)\big|_{\eta\to\infty} = \left(\,\frac{1}{2}\phi_{(0)}\,\delta\phi_{(0)}\,,\ 0\,\right)$$

2. Angular momentum : $J_{ADT} \equiv J^g_{ADT} + J^\phi_{ADT} = \frac{1}{4G}C_2$

   Computed by the ADT potential for the rotational Killing vector $\xi^i_R = (0,1)$

$$\sqrt{-g}Q^{\eta i}_{ADT}(\xi_R\,;\,\delta g)\big|_{\eta\to\infty} = \left(-\,\delta C_2\,,\ -\delta C_1 - \frac{1}{2}\phi_{(0)}\delta\phi_{(0)}\right)$$
$$\sqrt{-g}Q^{\eta i}_{ADT}(\xi_R\,;\,\delta g)\big|_{\eta\to\infty} = \left(\ 0\ ,\ 0\ \right)$$

3D Black hole solutions: *Class I*

In this class, the boundary stress tensor is

$$(\mathbf{T}_g)^i{}_j = \begin{pmatrix} -C_1 + \frac{1}{4}\phi_{(0)}^2 & -C_2 \\ -C_2 & C_1 + \frac{1}{4}\phi_{(0)}^2 \end{pmatrix} \quad (\mathbf{T}_\phi)^i{}_j = \begin{pmatrix} -\frac{1}{4}\phi_{(0)}^2 & 0 \\ 0 & -\frac{1}{4}\phi_{(0)}^2 \end{pmatrix}$$

- It is straightforward to confirm the equivalence of the bulk and boundary conserved charges for Killing vectors ξ_T and ξ_R.
- The equivalence relation holds for the metric and matter part separately.

3D Black hole solutions: *Class I*

Some known black hole solutions belong to this class

- BTZ black hole solutions ['92 Banados, Teitelboim, Zanelli]

$$ds^2 = -\frac{(r^2 - r_-^2)(r^2 - r_+^2)}{r^2}dt^2 + \frac{r^2}{(r^2 - r_-^2)(r^2 - r_+^2)}dr^2 + r^2\left(d\theta - \frac{r_- r_+}{r^2}dt\right)^2$$

These are solutions in pure gravity with a cosmological constant or solutions without scalar hair, $\phi_{(0)} = 0$. After transforming to FG coordinates,

$$C_1 = \frac{r_-^2 + r_+^2}{2}\,, \qquad C_2 = r_- r_+\,,$$

which reproduce the well-known expressions of the total mass and angular momentum of BTZ black holes

$$M = \frac{r_-^2 + r_+^2}{8G}\,, \qquad J = \frac{r_- r_+}{4G}\,.$$

3D Black hole solutions: *Class I*

- The extremal rotating black holes with scalar hair ['12 Hyun, Jeong, Yi]

$$ds^2 = r^2\left[-1+\frac{\mu_0}{r^2}+\mathcal{O}(\frac{1}{r^3})\right]dt^2+\frac{1}{r^2}\left[1+\frac{\mu_0-\frac{1}{2}\phi_{(0)}^2}{r^2}+\mathcal{O}(\frac{1}{r^3})\right]dr^2$$
$$+r^2\left[d\theta-\left(\frac{\mu_0}{2r^2}+\mathcal{O}(\frac{1}{r^3})\right)dt\right]^2$$
$$\phi(r)=\frac{\phi_{(0)}}{r}+\mathcal{O}(\frac{1}{r^2}),$$

These are solutions corresponding to the case $C_1 = C_2 = \frac{\mu_0}{2}$.

The total mass and angular momentum of these black holes are

$$M = J = \frac{\mu_0}{8G},$$

which satisfy the extremality condition.

3D Black hole solutions: *Class II*

Class II : $-1 < m^2 < 0$

In this class, we apply our formalism to those solutions given in ['06 Henneaux, Martinez, Troncoso, Zanelli]

The scalar potential with a cosmological constant

$$V(\phi) - 2 = -2\left[\cosh^6(\frac{\phi}{4}) + \nu\sinh^6(\frac{\phi}{4})\right]$$

The radial expansion of the scalar field in FG coordinates

$$\phi = e^{-\frac{1}{2}\eta}\left(\phi_{(0)} + \frac{1}{48}\phi_{(0)}^3 e^{-\eta} + \cdots\right)$$

The radial expansion of the metric solution up to the $e^{-2\eta}$ order

$$\gamma_{ij}^{(1)} = -\frac{1}{4}\phi_{(0)}^2\,\eta_{ij}\,, \qquad \gamma_{ij}^{(2)} = \frac{3}{128}\phi_{(0)}^4\left[\eta_{ij} + \frac{(1+\nu)}{4}\delta_{ij}\right]$$

3D Black hole solutions: *Class II*

1. Mass : $M_{ADT} \equiv M^g_{ADT} + M^\phi_{ADT} = \frac{1}{4G}\frac{3(1+\nu)}{512}\phi^4_{(0)}$

   The ADT potentials for the time-like Killing vector ξ^i_T

$$\sqrt{-g}\,Q^{\eta i}_{ADT}(\xi_T\,;\,\delta g)\Big|_{\eta\to\infty} = \Big[-\frac{1}{4}e^\eta\phi_{(0)}\delta\phi_{(0)} + \frac{1}{32}\phi^3_{(0)}\delta\phi_{(0)} + \frac{3(1+\nu)}{128}\phi^3_{(0)}\delta\phi_{(0)}\Big]\xi^i_T$$

$$\sqrt{-g}\,Q^{\eta i}_{ADT}(\xi_T\,;\,\delta\phi)\Big|_{\eta\to\infty} = \Big[\frac{1}{4}e^\eta\phi_{(0)}\delta\phi_{(0)} - \frac{1}{32}\phi^3_{(0)}\delta\phi_{(0)}\Big]\xi^i_T$$

2. Angular momentum : $J_{ADT} \equiv J^g_{ADT} + J^\phi_{ADT} = 0$

   The ADT potentials for the rotational Killing vector ξ^i_R

$$\sqrt{-g}\,Q^{\eta i}_{ADT}(\xi_R\,;\,\delta g)\Big|_{\eta\to\infty} = \Big[-\frac{1}{4}e^\eta\phi_{(0)}\delta\phi_{(0)} + \frac{1}{32}\phi^3_{(0)}\delta\phi_{(0)} - \frac{3(1+\nu)}{128}\phi^3_{(0)}\delta\phi_{(0)}\Big]\xi^i_R$$

$$\sqrt{-g}\,Q^{\eta i}_{ADT}(\xi_R\,;\,\delta\phi)\Big|_{\eta\to\infty} = \Big[\frac{1}{4}e^\eta\phi_{(0)}\delta\phi_{(0)} - \frac{1}{32}\phi^3_{(0)}\delta\phi_{(0)}\Big]\xi^i_R$$

3D Black hole solutions: *Class II*

Now, we turn to the boundary formalism. In this case, we choose counter terms of the scalar field as

$$\Phi_{ct} = -\frac{1}{4}\phi^2 - \frac{1}{96}\phi^4$$

By using this form of counter terms, one can see that

$$\begin{aligned}
\sqrt{-\gamma}\,(T_G)^i{}_j =& \left[\frac{1}{8}e^{\eta}\phi_{(0)}^2 - \frac{3}{128}\phi_{(0)}^4 + \frac{3(1+\nu)}{512}\phi_{(0)}^4\right]\delta^i_j - \frac{3(1+\nu)}{256}\phi_{(0)}^4\delta^{it}\delta_{jt} \\
\sqrt{-\gamma}\,(T_\phi)^i{}_j =& -\left[\frac{1}{8}e^{\eta}\phi_{(0)}^2 - \frac{3}{128}\phi_{(0)}^4\right]\delta^i_j \\
\sqrt{-\gamma}\,\Pi_\phi =& \, 0
\end{aligned}$$

Once again, it is straightforward to confirm the equivalence relation for Killing vectors ξ_T and ξ_R. As a result, the identical expression for the mass and angular momentum can be obtained through the boundary stress tensor method as well.

5 dim. AdS Kerr black hole

- B.H. mass (6.49), (6.50) in [hep-th/0505190]

$$M = M_{\text{Casimir}} + \frac{2\pi^2 m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)}{\kappa^2 \Xi_a^2 \Xi_b^2}, \quad M_{\text{Casimir}} \equiv \frac{3\pi^2 l^2}{4\kappa^2}\left(1 + \frac{(\Xi_a - \Xi_b)^2}{9\Xi_a\Xi_b}\right)$$

: Casimir energy depends on the rotation parameters of the AdS Kerr black hole.

- B.H. mass (14) in [hep-th/0507034]

$$E_{\text{tot}} = \frac{3\pi^2 l^2}{4\kappa^2} + \frac{2\pi^2 m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b)}{\kappa^2 \Xi_a^2 \Xi_b^2}$$

: Casimir energies that are necessarily independent of the black hole rotation parameters.

$\Rightarrow$ We would like to compensate this with modification of the holographic charges.

Backgrounds: bulk ADT charges

Off-shell ADT bulk current

$$\sqrt{-g}\mathcal{J}^{\mu}_{ADT} = \delta\Big(\sqrt{-g}\,\mathbf{E}^{\mu\nu}\xi_{\nu}\Big) - \sqrt{-g}\,\mathbf{E}^{\mu}{}_{\nu}\,\delta\xi^{\nu} + \frac{1}{2}\sqrt{-g}\,\xi^{\mu}\mathcal{E}_{\Psi}\delta\Psi$$

- Identically conserved : $\partial_{\mu}(\sqrt{-g}\mathcal{J}^{\mu}_{ADT}) = 0$
- ADT potential : $\mathcal{J}^{\mu}_{ADT} = \nabla_{\nu}Q^{\mu\nu}_{ADT}$
- ADT potential in terms of Noether potential

$$2\sqrt{-g}Q^{\mu\nu}_{ADT}(\xi, \delta\Psi\,;\,\Psi) = \delta K^{\mu\nu}(\xi\,;\,\Psi) - K^{\mu\nu}(\delta\xi\,;\,\Psi) - 2\xi^{[\mu}\Theta^{\nu]}(\delta\Psi\,;\,\Psi)$$

Quasi-local conserved charges

$$Q(\xi) = \frac{1}{8\pi G}\int_0^1 ds\int_{\mathcal{B}} d^{D-2}dx_{\mu\nu}\sqrt{-g}Q^{\mu\nu}_{ADT}(g;\xi|s)$$

$\Rightarrow$ with same procedure, identically conserved current at boundary.

Construction: modified holographic charges

Construct the boundary conserved current

$$\sqrt{-\gamma}\mathcal{J}_B^i(\xi_B) \equiv -\,\delta\Big(\sqrt{-\gamma}\,\mathbf{T}_B^{ij}\xi_j^B\Big) + \sqrt{-\gamma}\,\mathbf{T}_B{}^i{}_j\,\delta\xi_B^j + \frac{1}{2}\sqrt{-\gamma}\,\xi_B^i\Big(T_B^{kl}\delta\gamma_{kl} + \Pi_\psi\delta\psi\Big)$$

1. $\sqrt{-\gamma}\,\mathbf{T}_B^{ij}\xi_j^B$: the conserved currents in conventional holographic charges
2. Identically conserved : $\partial_i\left(\sqrt{-\gamma}\mathcal{J}_B^i\right) = 0$
3. Equivalent with bulk expression.

$$\sqrt{-g}Q_{ADT}^{\eta i}|_{\eta\to\infty} = \sqrt{-\gamma}\mathcal{J}_B^i\,.$$

Modified holographic conserved charges

$$Q_B(\xi_B) = \frac{1}{8\pi G}\int_{\partial\mathcal{B}} d^{d-1}x_i \int ds\,\sqrt{-\gamma}\,\mathcal{J}_B^i(\xi_B)$$

Example: 5 dim. AdS Kerr B.H.

AdS Kerr black hole solutions in Boyer-Lindquist coordinates

$$
\begin{aligned}
ds^2 = &-\frac{\Delta_r}{\rho^2}\Big(dt - a\Delta_\phi d\phi - b\Delta_\psi d\psi\Big)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\
&+ \frac{\Delta_\theta \sin^2\theta}{\rho^2}\Big(adt - \frac{r^2+a^2}{1-a^2}d\phi\Big)^2 + \frac{\Delta_\theta \cos^2\theta}{\rho^2}\Big(bdt - \frac{r^2+b^2}{1-b^2}d\psi\Big)^2 \\
&+ \frac{1+1/r^2}{\rho^2}\Big(abdt - b(r^2+a^2)\Delta_\phi d\phi - a(r^2+b^2)\Delta_\psi d\psi\Big)^2 ,
\end{aligned}
$$

where $\rho^2 \equiv r^2 + a^2\cos^2\theta + b^2\sin^2\theta$,

$$
\Delta_r \equiv (r^2+a^2)(r^2+b^2)\Big(1+\frac{1}{r^2}\Big) - 2m ,
$$

$$
\Delta_\theta \equiv 1 - a^2\cos^2\theta - b^2\sin^2\theta , \quad \Delta_\phi \equiv \frac{\sin^2\theta}{1-a^2} , \quad \Delta_\psi \equiv \frac{\cos^2\theta}{1-b^2} .
$$

Example: 5 dim. AdS Kerr B.H.

$$\int d^3x_i\sqrt{-\gamma}\Big[\mathbf{T}^i_{B\;j}\delta\xi^j_T+\frac{1}{2}\,\xi^i_T\big(T^{kl}_B\delta\gamma_{kl}+\Pi_\psi\delta\psi\big)\Big] = -\frac{\pi^2(a^2-b^2)(2-a^2-b^2)}{6(1-a^2)(1-b^2)}\left[\frac{a\delta a}{1-a^2}-\frac{b\delta b}{1-b^2}\right],$$

In Fefferman-Graham coordinates,

$$ds^2 = d\eta^2 + \gamma_{ij}dx^i dx^j\,, \quad \gamma_{ij} = \sum_{n=0} e^{-2(n-1)\eta}\gamma^{(n)}_{ij}\,,$$

with time-like Killing vector $\xi^i_T\partial_i = \partial_t - a\partial_\phi - b\partial_\psi$, the finite mass expression is given by

$$\delta Q_B(\xi_T) = \delta\left(\frac{\pi m(3-a^2-b^2-a^2b^2)}{4G(1-a^2)^2(1-b^2)^2}\right) = \delta\left(\frac{2\pi^2 m(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)}{\kappa^2\Xi_a^2\Xi_b^2}\right)$$

Example: 5 dim. AdS Kerr B.H.

In asymptotically canonical AdS coordinates,

$$
\begin{aligned}
ds^2 =& -(1+y^2)dt^2 + \frac{dy^2}{1+y^2-\frac{2m}{\Delta_{\hat{\theta}}^2 y^2}} + y^2 d\hat{\Omega}_3^2 \\
&+ \frac{2m}{\Delta_{\hat{\theta}}^3 y^2}(dt - a\sin^2\hat{\theta}d\hat{\phi} - b\cos^2\hat{\theta}d\hat{\psi})^2 + \cdots \,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{\hat{\theta}} &\equiv 1 - a^2\sin^2\hat{\theta} - b^2\cos^2\hat{\theta}\,, \\
d\hat{\Omega}_3^2 &\equiv d\hat{\theta}^2 + \sin^2\hat{\theta}d\hat{\phi} + \cos^2\hat{\theta}d\hat{\psi}\,.
\end{aligned}
$$

one can check explicitly that mass and angular momentums in these non-rotating coordinates are given by the same expressions as in the rotating ones. $\Rightarrow$ frame independence confirmed!

Conclusion

- We have constructed a quasi-local formalism for conserved charges in a general theory of gravity with diffeomorphism symmetry in the presence of arbitrary matter fields.
- As an application of our formalism, we have considered some examples in order to show some details in our formalism concretely.
- We propose the modified form of the conventional holographic conserved charges which provides us the frame-independent expressions for charges.
- As an explicit example, we consider 5-dimensional AdS Kerr black holes and show that our form of holographic conserved charges gives us the identical expressions in the rotating and non-rotating frames.

$$M = M_{casimir}(m = a = b = 0) + \frac{\pi m(3 - a^2 - b^2 - a^2 b^2)}{4G(1 - a^2)^2(1 - b^2)^2} \, ,$$