Entanglement and differential entropy for massive flavors

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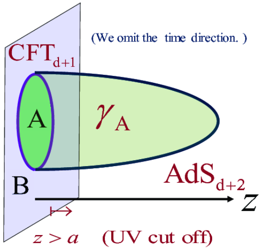
- There has been considerable interest recently in entanglement entropy.



(Takayanagi)

- $S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$.
- Holographic Ryu-Takayanagi (RT) prescription: area of co-dimension two minimal surface homologous to A

$$S_A = \frac{\mathcal{A}}{4G_N}$$

- Leading UV divergence: area of separating surface.

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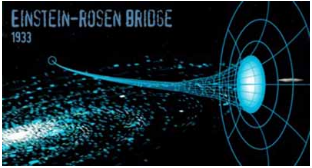
Significance of entanglement entropy

A useful computable, particularly in applied holography, but also



- Does entanglement entropy capture global structure in the dual spacetime?
- ER = EPR? (Maldacena and Susskind)

Other measures of entanglement

Many other measures of entanglement: holographic realisations?

- Consider density matrix ρ for theory with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.
- Define entanglement negativity $\mathcal{E} = \log \mathrm{Tr}(\rho^{T_2})$, with T_2 being a partial transpose over $\mathcal{H}_2$.
- Well studied in CFT (Calabrese et al) but replica trick requires a non-integral number of copies of the bulk!

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Key questions

1. Dependence of EE on shape and on field theory.
2. First law for EE.
3. Dependence of EE on the state in theory e.g. excited states.
4. Other bulk computables e.g. differential entropy and their roles in field theory.

(Balasubramanian, Hartman, Headrick, Hubeny, Liu, Mezei, Myers et al, Rangamani, Rosenhaus, Smolkin,Tayakanagi et al, . . .)

Massive flavors

- Brane systems such as D3/D7 are a natural framework in which to explore these questions.
- Well-understood dual field theory and phenomenologically interesting (Itzhaki et al '98, Karch and Katz '02).

Reference

- Peter Jones, Kostas Skenderis and Marika Taylor
  "Entanglement and differential entropy for massive flavors",
  to appear soon.
- Also phenomenological applications with Nick Evans.

Outline

- **The D3/D7 system**
- Entanglement entropy
- Field theory interpretation
- Differential entropy

The D3/D7 system

- Consider $\mathcal{N}_c$ D3-branes and $\mathcal{N}_f \ll \mathcal{N}_c$ parallel coincident D7-branes.
- In the decoupling limit the D7-branes wrap an $AdS_5 \times S^3$ submanifold of $AdS_5 \times S^5$

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx \cdot dx\right) + d\theta^2 + \sin^2\theta d\Omega_3^2 + \cos^2\theta d\phi^2$$

  i.e. $\theta = \pi/2$.
- The dual theory is a CFT, SYM coupled to $\mathcal{N} = 2$ massless hypermultiplets transforming in the bifundamental of $SU(\mathcal{N}_c) \times SU(\mathcal{N}_f)$.

Massive flavors

- Separating the D3 and D7 branes causes the hypers to become massive.
- From the brane probe perspective, the embedding is (Karch and Katz, '02)

$$\sin^2\theta = (1 - m^2 z^2),$$

i.e. the D7-branes extend to $z = 1/m$, with m the mass.

Deformation

- The corresponding deformation of the CFT is by a dimension three operator, the fermion mass,

$$I = I_{CFT} + m \int d^4 x \sqrt{-g} \mathcal{O}_3$$

where the holographic normalization of the operator (brane holographic renormalization (Karch et al, '05)) is

$$\langle \mathcal{O}_3(x) \mathcal{O}_3(0) \rangle = 16 T_7 \mathcal{R} \left(\frac{1}{x^6} \right)$$

with T_7 the D7-brane tension.

IR behaviour

- Integrating out the massive hypers leads to an effective IR theory

$$I = I_{SYM} + \frac{1}{m^2} \int d^4x \sqrt{-g} \mathcal{O}_6 + \cdots$$

  where $\mathcal{O}_6$ is a dimension six SYM operator, which breaks the R symmetry to $SO(4)$.
- Note that the finite extent of the probe D7-brane tallies with the field theory behaviour at energy scales far smaller than m.
- However, in the field theory there is no sharp transition at $\Lambda = m$.

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Entanglement entropy

- To compute EE for D3/D7 we should find the full backreacted metric, asymptoting to $AdS_5 \times S^5$, extract the effective 5d Einstein metric and apply the RT formula.
- Backreacted metric depends on $(z, \theta, \phi) \rightarrow$ cohomogeneity three problem.
- Smearing over the sphere simplifies problem (Mas, Nunez, Ramallo et al) but is obscure in field theory.
- Extracting 5d Einstein metric from an inhomogeneous 10d metric is also not simple.

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Probe approximation

As usual we work in the quenched approximation $\mathcal{N}_f \ll \mathcal{N}_c$.

- Effectively we therefore consider solving

$$I = I_{\rm sugra} - T_7 \int d^8 \sqrt{-\gamma} + \cdots$$

  with γ_{ab} the induced brane metric, iteratively around $AdS_5 \times S^5$.

- EE is sensitive to the 5d Einstein metric so we cannot work just with D-brane action, even at order T_7.
- Backreaction at linear order in T_7 is complex, as embedding breaks symmetry to $E^{3,1} \times SO(4)$.

Shortcuts?

(Karch et al, 2013/2014) have proposed various shortcut methods:

1. Exploit CHM map for spherical entangling regions i.e. solve a BH perturbation problem.
   - Only works for spherical regions.
   - CHM map is very complicated for D7-brane embedding (intractable for finite mass, even at zero density).
2. Use results of (Graham/Karch) for cohomogeneity two brane embeddings.
   - Change in EE depends only on a subset of the metric perturbation in ten dimensions.
   - How would this method work at higher order in T_7 or for other brane embeddings?

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A systematic perturbative approach

The second method is actually equivalent to using Kaluza-Klein holography, (Skenderis, M.T. '06).

- Consider any background which is a perturbation of $AdS_5 \times S^5$, i.e.

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx \cdot dx\right) + d\Omega_5^2$$
$$+\delta g_{mn}(z, x_\mu, \theta_i).$$

  The pertubations can be decomposed in terms of spherical harmonics.

- Kaluza-Klein holography gives an algorithmic approach to extracting the 5d Einstein metric (and all other 5d fields) from the perturbation harmonics.

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Perturbation in 5d Einstein metric

- In the case at hand, the change in the 5d Einstein metric is particularly simple: for $z \leq 1/m$

$$\delta(ds^2) = \frac{1}{z^2}\left(f(z)dz^2 + h(z)dx \cdot dx\right)$$

with

$$\tilde{f}(z) = (f(z) + zh'(z)) = \frac{t_0}{12}(1 - m^2 z^2)^2$$

where $t_0 = \mathcal{V}_{S^3} T_7 = 2\pi^2 T_7$.

- The gauge invariant combination is $\tilde{f}(z)$; require also $h(1/m) = h'(1/m) = 0$ for continuity of metric and first derivative at $z = 1/m$.

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Change in EE

We consider three types of domains:

1. Slab in y, z plane, width $l = \Delta x$.
2. Half plane $x > 0$; $\Delta x \to \infty$ limit of slab.
3. Spherical region, of radius l; Casini-Huerta-Myers (CHM) case.

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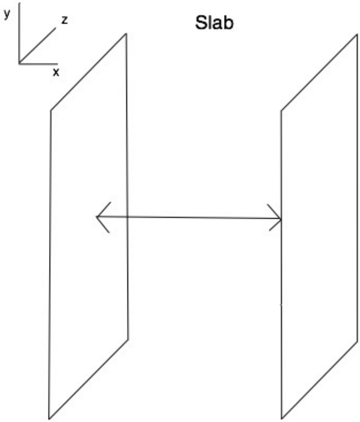
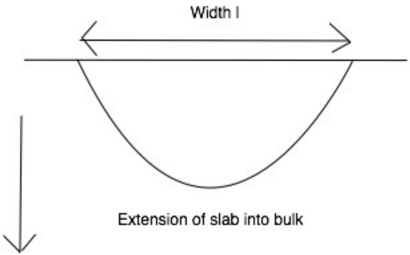
Entangling surfaces



Entangling surfaces

Suppressing the y, z directions:



Results for EE

For the slab:

- AdS_5 result:

$$S = \frac{L^2}{2G_N}\left(\frac{1}{2\epsilon^2} + \frac{\sqrt{\pi}\Gamma(-\frac{1}{3})}{6\Gamma(\frac{1}{6})z^{*2}}\right)$$

  with L^2 the regulated area of the y, z directions, ϵ the UV cutoff and z^* the turning point of the bulk entangling surface.

- The turning point is linearly related to the slab width

$$l = c_0 z^*.$$

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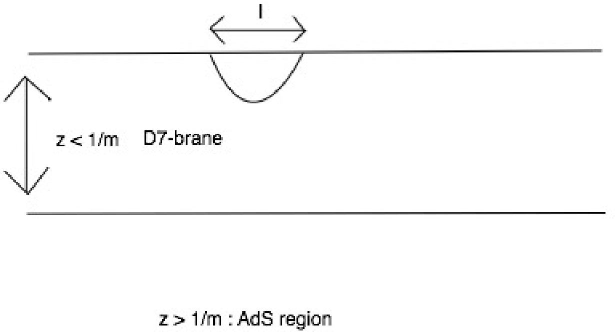
Entangling surfaces

Since the D7-branes extend only to $z = 1/m$, the entanglement depends on whether the turning point of the entangling surface is as $z^* < 1/m$ or $z^* \geq 1/m$.



Results for EE

- For $mz^* \leq 1$:

$$\delta S = \frac{t_0 L^2}{48 G_N}\left(\frac{1}{2\epsilon^2} + \frac{2}{3}m^2 + \frac{\sqrt{\pi}}{12 z^{*2}}\frac{\Gamma(-1/3)}{\Gamma(7/6)} + m^4 z^{*2}\frac{\sqrt{\pi}}{12}\frac{\Gamma(1/3)}{\Gamma(11/6)} + \frac{2}{3}m^2 \log(\epsilon^3/2z^{*3})\right) + \delta S_{\rm gauge}(m,\epsilon).$$

- For $mz^* \gg 1$:

$$\delta S = \frac{t_0 L^2}{48 G_N}\left(\frac{1}{2\epsilon^2} + 2m^2 \log(m\epsilon) - \frac{1}{48 m^4 z^{*6}} + \cdots\right) + \delta S_{\rm gauge}(m,\epsilon)$$

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Comments

- The entangement entropy has a fourth order phase transition at $mz^* = 1$.
- The gauge dependent terms depend on our choice of $h(z)$, i.e. the gauge choice for the metric.
- The relation between the slab width l and the turning point z^* is corrected perturbatively:

$$l = (c_0 + t_0 c_1(z^*) + \cdots)z^*$$

Other entangling regions

- The half space is obtained as the $l \to \infty$ limit at fixed m:

$$\delta S = \frac{t_0 L^2}{96 G_N} \left(\frac{1}{2\epsilon^2} + 2m^2 \log(m\epsilon) \right) + \delta S_{\rm gauge}(m, \epsilon)$$

- There are analogous results for the spherical region; fifth order transition at $mz^* = 1$.

- Recall that the (gauge invariant) metric perturbation is

$$\tilde{f}(z) = \frac{t_0}{12}(1 - m^2 z^2)^2 \qquad z \leq \frac{1}{m}$$

  and hence second derivatives of the metric are discontinuous at $mz = 1$.
- The EE for the slab inherits a discontinuity only at fourth order i.e. in

$$\left(\frac{\partial^4 S}{\partial l^4} \right)_m$$

  (due to symmetry, the shift in the turning point etc).

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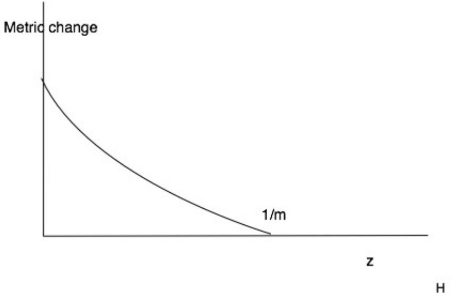
Phase transitions

- In the fully backreacted solution we would not expect sharp discontinuities at $z = 1/m$.
- The phase transitions at $mz^* = 1$ would therefore seem to be relics of the probe approximation.



- EE is UV divergent.
- In a field theory it is natural to define a renormalized EE:

$$S_{\text{ren}} = S_{\text{bare}} + S_{\text{ct}}$$

- For the RT surface, the counterterms arise from volume renormalisation (Witten, Graham), i.e.

$$S_{\text{ren}} = \frac{1}{4G_N}\int_\gamma d^{d-1}x\sqrt{g} - \frac{1}{4(d-2)G_N}\int_{\partial\gamma} d^{d-2}x\sqrt{H} + \cdots$$

with H the induced boundary metric.

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Massless flavors

For massless flavors we find

$$S_{\text{ren}} = \frac{1}{4G_N}\int_{\gamma} d^3x\sqrt{g} - \frac{1}{8G_N}(1+\frac{t_0}{24})\int_{\partial\gamma} d^2x\sqrt{H} + \cdots$$

where ellipses denote terms which vanish for flat boundary regions.

Scheme dependence and finite quantities

- The gauge dependence found earlier cancels in the renormalized quantity i.e. no dependence on metric perturbation $h(\epsilon)$.
- The counterterm contributions also do not depend on Δx for the slab.
- There are additional logarithmic counterterms in the massive case.
- For the spherical region there are counterterms depending on the curvature of the boundary of the entangling surface .

Finite quantities by differentiation

Various proposals exist to isolate finite terms in EE:

- Finite mass in $d = 4$ (Hertzberg, Wilczek)

$$S_{HW} = m^4 \frac{\partial^2 S}{\partial (m^2)^2}$$

- Finite slab width (Cardy et al)

$$S_l = l \frac{\partial S}{\partial l}$$

- Spherical regions in UV conformal theories (Liu and Mezei)

$$S_{LM} = l \frac{\partial}{\partial l} \left(\frac{\partial S}{\partial l} - 2S \right).$$

for $d = 4$.

Finite quantities by differentiation

- One can use the renormalized entanglement entropy to show why each of these is indeed finite in a mass deformed CFT.
- With finite m and l these differentiated quantities are of limited use, as each throws away terms with physical interpretations:
- E.g. for a slab with $ml \gg 1$:

$$\delta S_l = \frac{t_0 L^2}{48 G_N} \left(\frac{1}{8 m^4 z^{*6}} \right)$$

and

$$\delta S_m = \frac{t_0 L^2}{48 G_N} \left(m^2 - \frac{1}{8 m^4 z^{*6}} \right)$$

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- **Field theory interpretation**
- Differential entropy

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Deformations of the CFT

- Consider first the half space:

$$\delta S = \frac{t_0 L^2}{48 G_N}\left(\frac{1}{2\epsilon^2} + 2m^2 \log(m\epsilon)\right) + \delta S_{\rm gauge}(m,\epsilon)$$

- The $m \to 0$ limit follows from conformal invariance and agrees with the result for free massless hypermultiplets.

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Deformations of the CFT

- At finite mass the CFT is deformed as

$$I = I_{\rm CFT} + m \int d^4x \sqrt{-g} \mathcal{O}_3 .$$

- The change in the entanglement entropy under a relevant perturbation of dimension $\Delta = (d+2)/2$ has been argued to contain universal log divergences (Rosenhaus, Smolkin):

$$\delta S = \mathcal{N} m^2 \frac{(d-2)}{4(d-1)} \frac{\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})} \mathcal{A} \log \left(\frac{\epsilon_{UV}}{\epsilon_{IR}} \right) ,$$

  with $\mathcal{N}$ the operator normalisation and $\mathcal{A}$ the area of the slab.
- Note that the change at order m vanishes, due to conformal invariance.

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Deformations of the CFT

- Using the known operator normalisation we indeed obtain

$$\delta S = \frac{2\pi t_0}{3} m^2 \mathcal{A} \log \left(\frac{\epsilon_{UV}}{\epsilon_{IR}} \right)$$

  in agreement with our result, setting $\epsilon_{IR} = 1/m$.
- Moreover, the result agrees with the results for free massive hypers, i.e. there is a non-renormalisation theorem (which was not obvious given $\mathcal{N} = 2$ susy).

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Finite slab width

The modular Hamiltonian is not known for a finite width slab.
But:

- The massless case:

$$S = \frac{L^2}{2G_N}\left(\frac{1}{2\epsilon^2} + \frac{\sqrt{\pi}\Gamma(-\frac{1}{3})}{6\Gamma(\frac{1}{6})z^{*2}}\right) + \frac{t_0 L^2}{48 G_N}\left(\frac{1}{2\epsilon^2} + \frac{\sqrt{\pi}}{2}\frac{\Gamma(-\frac{1}{3})}{\Gamma(\frac{1}{6})z^{*2}}\right)$$

can be understood in terms of free (conformal) fields (Hertzberg, Wilczek etc)..

Finite slab width

We can also understand the $ml \gg 1$ limit:

- The leading finite contribution is

$$\delta S = \frac{t_0 L^2}{48 G_N} \left(- \frac{1}{48 m^4 z^{*6}} \right).$$

- Integrating out the massive flavors results in

$$I = I_{SYM} + \frac{1}{m^2} \int d^4 x \sqrt{-g} \mathcal{O}_6$$

with $\mathcal{O}_6$ an R-charged operator.

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Finite slab width

- Symmetry implies that the leading contribution to the entanglement entropy is at order $1/m^4$.
- By translational invariance along the slab the EE scales as L^2.
- Hence

$$\delta S \sim \frac{L^2}{m^4 l^6}$$

on dimensional grounds, since there is no other scale in the theory.

We may also be able to match the coefficient (?)

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Differential entropy

- The differential entropy is defined as

$$E = \sum_{k=1}^{\infty} [S(I_k) - S(I_k \cap I_{k+1})]$$

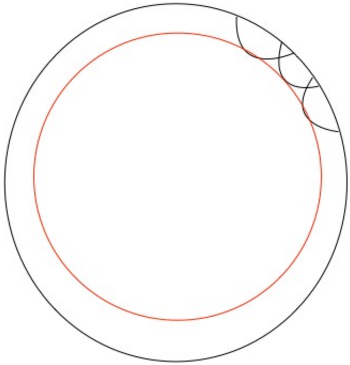
  where $\{I_k\}$ is a set of intervals partitioning the boundary.
- We will take $\{I_k\}$ to be slabs of width Δx, with intersection of width $(\Delta x - L_x/n)$, and take $n \to \infty$.

Holes and differential entropy

- In AdS_5 the differential entropy computes the area of a hole of radius z^*, the turning point of the entangling surface associated with each slab.
- This equivalence can be proved geometrically (Balasubramanian et al; Myers et al; Headrick et al).

Differential entropy

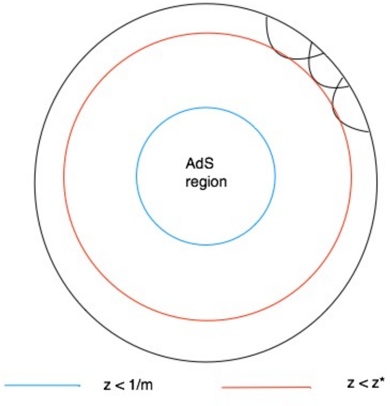
Witten diagram showing differential entropy: differential entropy computes area of red hole.



Differential entropy for massive flavor system

- The massive flavor system is asymptotic to $AdS_5 \times S^5$ but the symmetry is broken to $E^{3,1} \times SO(4)$.
- What does the differential entropy actually compute?

Differential entropy for flavor system



Differential entropy for massive flavor system

It still computes the area of a hole in the 5d Einstein metric.

- For $ml \gg 1$ the metric is just AdS_5, yet the differential entropy is changed:

$$E = \frac{V}{4G_N}\left(\frac{c_0^3}{(\Delta x)^3} + \frac{t_0 c_0^6}{384 m^4 (\Delta x)^7}\right)$$

  with c_0 the number such that $\Delta x = c_0 z^* + \cdots$.

- The metric is unchanged, but the relation between Δx and the turning points of the entangling surface z^* is changed.

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Differential entropy for massive flavor system

- The change is consistent with the viewpoint of the IR theory as an irrelevant deformation of SYM.
- Differential entropy however seems to tell us only about the 5d metric, not the 10d spacetime.

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Conclusions

- We have developed a systematic method for computing EE for probe brane systems, which is more widely applicable to other 10d spacetimes.
- Finite terms in the EE may be obtained using volume renormalization for the minimal surfaces.
- Exact coefficients in the EE can be matched.
- Differential entropy computes the area in the 5d Einstein metric, not the "physical" 10d metric.

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Outlook

- Phenomenology: finite temperature, finite density, phase transitions?
- Interpretations of differential entropy in the field theory?
- General results for shape and field theory dependence (including irrelevant deformations)?