Numerical Holography: Isotropization & Quantum Quenches

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Close to equilibrium:

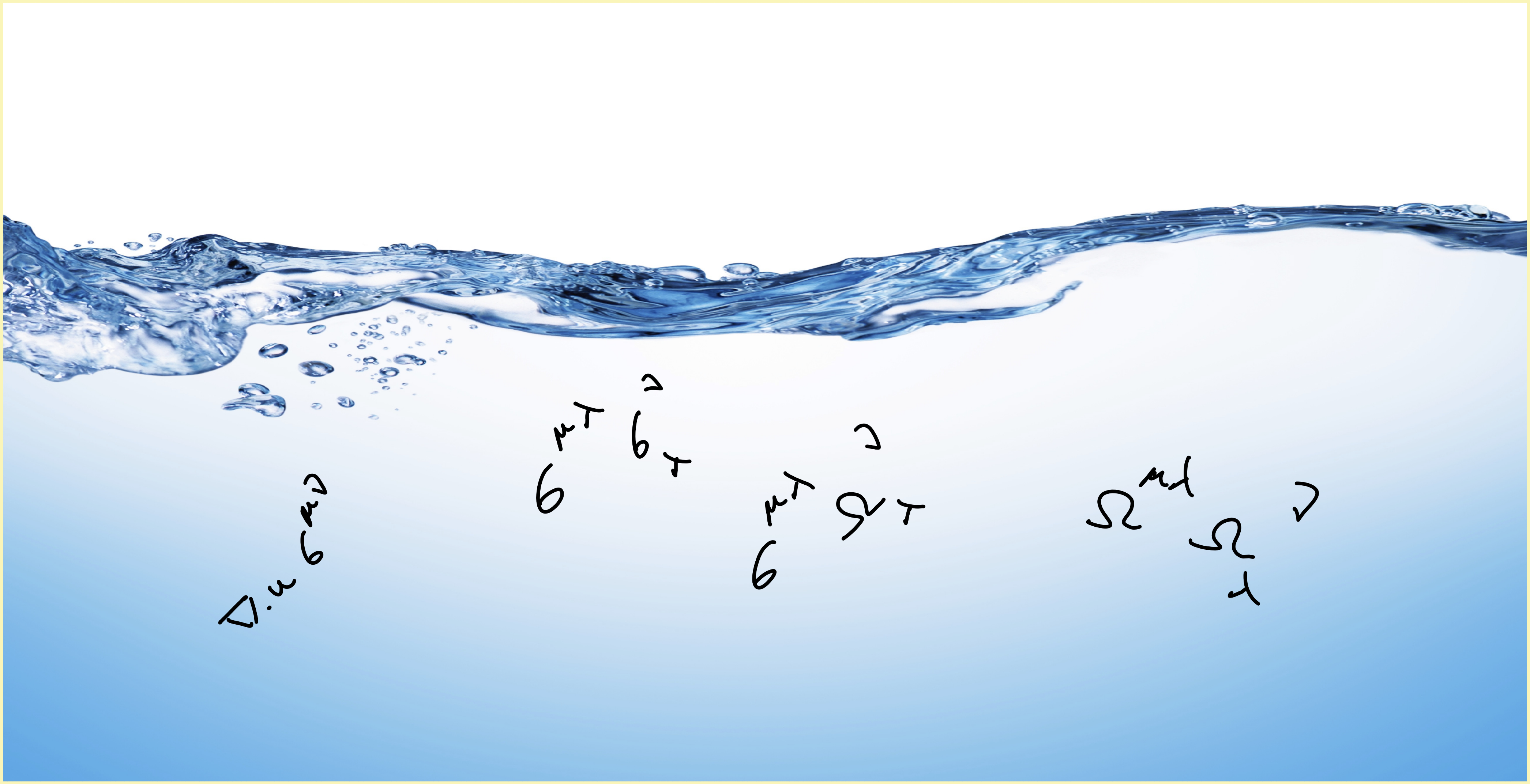
Starting from relativistic hydrodynamics,

$$\langle T^{\mu\nu} \rangle = T^{\mu\nu}_{eq.}(u^\mu, \mathcal{E}) + \Pi^{\mu\nu}$$

$$T^{\mu\nu}_{eq.} \equiv (\mathcal{E}+P)\, u^\mu u^\nu + P g^{\mu\nu}$$

$$\Pi^{\mu\nu} = -\eta\, \sigma^{\mu\nu} + \eta\, \tau_\pi \left(u\cdot\nabla \sigma^{\langle\mu\nu\rangle} + \frac{\nabla\cdot u}{3} \sigma^{\mu\nu} \right) + \cdots$$


∇.u
σ μν
Ω μν


$\nabla\cdot u\,\sigma^{\mu\nu}$
$\sigma^{\mu\tau}\sigma_{\tau}^{\nu}$
$\sigma^{\mu\tau}\Omega_{\tau}^{\nu}$
$\Omega^{\mu\lambda}\Omega_{\lambda}^{\nu}$

Expansion in background geometry: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \ll 1$

Considering a system where some nonuniformity, either in the initial conditions or in the spacetime geometry, forces $\sigma_{\mu\nu}$ etc. to be nonzero.

It was particularly convenient to consider an initially uniform, equilibrium system in flat space but to introduce perturbatively weak and slowly varying spacetime nonuniformity which causes the fluid to experience shear and vorticity.

Consider the expectation value of the Stress-tensor for a system initially (time $t_0 \ll 0$) in equilibrium at temperature T, subject to a spacetime dependent metric perturbation $h_{\alpha\beta}(x)$, with $h_{\mu\nu}(t \leq t_0) = 0$. The stress tensor was determined

$$\langle T^{\mu\nu}(0) \rangle = \mathrm{Tr}\, e^{-\beta H}\, \tilde{T} e^{i\int_{t_0}^{0} dt' H(t')}\, T^{\mu\nu}\, T e^{-i\int_{t_0}^{0} dt'' H(t'')}$$

$$\langle T^{\mu\nu} \rangle_h = G_r^{\mu\nu}(0) - \frac{1}{2}\int G_{ra}^{\mu\nu,\alpha\beta}(0,x)\, h_{\alpha\beta}(x) + \frac{1}{8}\iint G_{raa}^{\mu\nu,\alpha\beta,\gamma\delta}(0,x,y)\, h_{\alpha\beta}(x)\, h_{\gamma\delta}(y)$$

Far from equilibrium dynamics:

Use gauge/gravity duality to study, far from equilibrium strongly interacting dynamics.

We hope to solve for dynamics of interesting initial states.

Heavy-ion collisions: homogeneous isotropization, boost invariant flow, colliding planar shocks, colliding "nuclei", turbulence.

Dynamical quenches.

Computationally one of the simplest cases is the homogeneous isotropization.

Physically this corresponds to central rapidity region in heavy-ion collisions, where the initial distribution of partons is anisotropic.

$$\langle T_{\mu\nu} \rangle = \frac{N_c^2}{2\pi^2} \, \mathrm{diag}\left[\mathcal{E} \,,\, P_L(t) \,,\, P_T(t) \,,\, P_T(t) \right]$$

$$P_L(t) = \frac{1}{3}\mathcal{E} - \frac{2}{3}\Delta P(t) \qquad P_T(t) = \frac{1}{3}\mathcal{E} + \frac{1}{3}\Delta P(t)$$

$$\Delta P = P_T - P_L$$

Holographic setup: In the generalized Eddington-Finkelstein

$$ds^2 = 2dtdr - A dt^2 + \Sigma^2 e^{-2B} dx_L^2 + \Sigma^2 e^{B} dx_T^2$$

where A, and B are all functions of time t and the

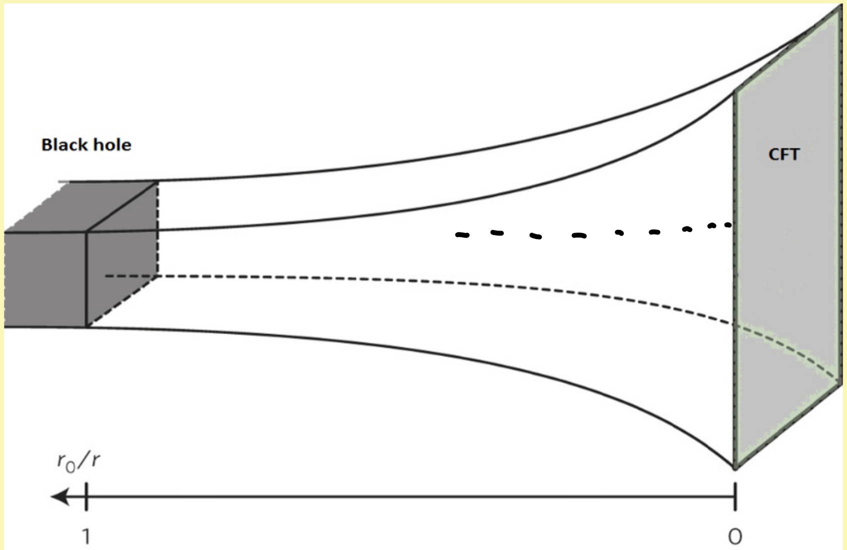
$$A = r^2 + \frac{a_4}{r} - \frac{2b_4^2}{7r^6} + \cdots, \quad B = \frac{b_4}{r^4} + \frac{b_4'}{r^5} + \cdots, \quad \Sigma = r - \frac{b_4^2}{7r^7} + \cdots$$

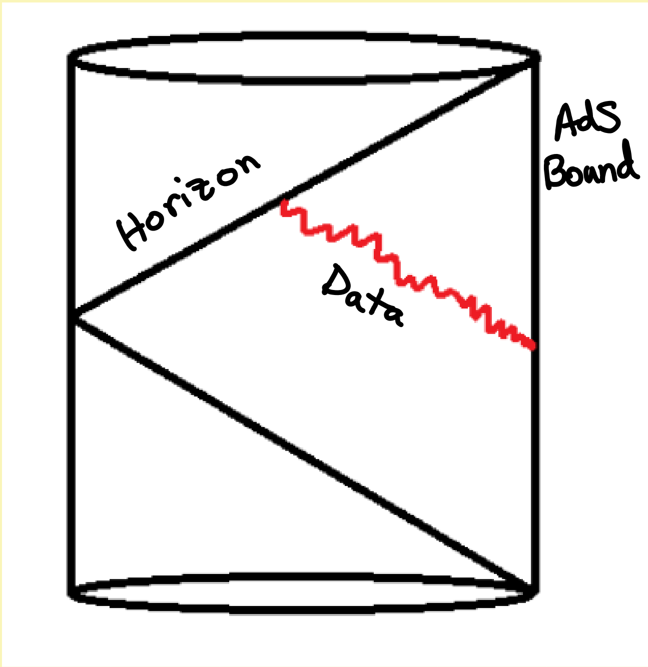
The normalizable modes are determind by the boundary

$$\mathcal{E} = -\frac{3a_4}{4}, \qquad \Delta P(t) = 3b_4$$

To be compared with the static black brane solution dual to a plasma

$$A = r^2\left(1 - \frac{r_h^4}{r^4}\right), \quad \Sigma = r, \quad B = 0$$





$$\Sigma(\dot{\Sigma})' + 2\Sigma'\dot{\Sigma} - 2\Sigma^2 = 0$$

$$\Sigma(\dot{B})' + \frac{3}{2}(\Sigma'\dot{B} + B'\dot{\Sigma}) = 0$$

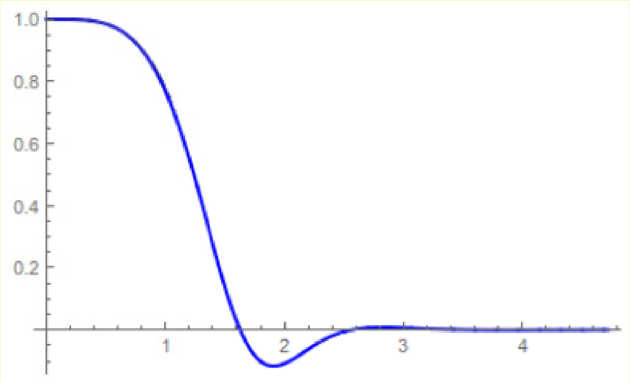
$$A'' + 3B'\dot{B} - 12\frac{\Sigma'\dot{\Sigma}}{\Sigma^2} + 4 = 0$$

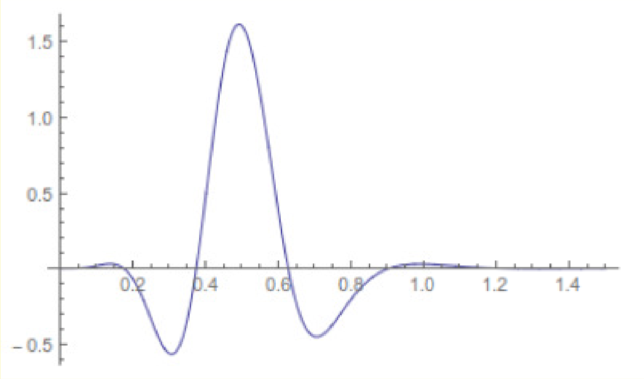
$$\ddot{\Sigma} + \frac{1}{2}(\dot{B}^2\Sigma - A'\dot{\Sigma}) = 0$$

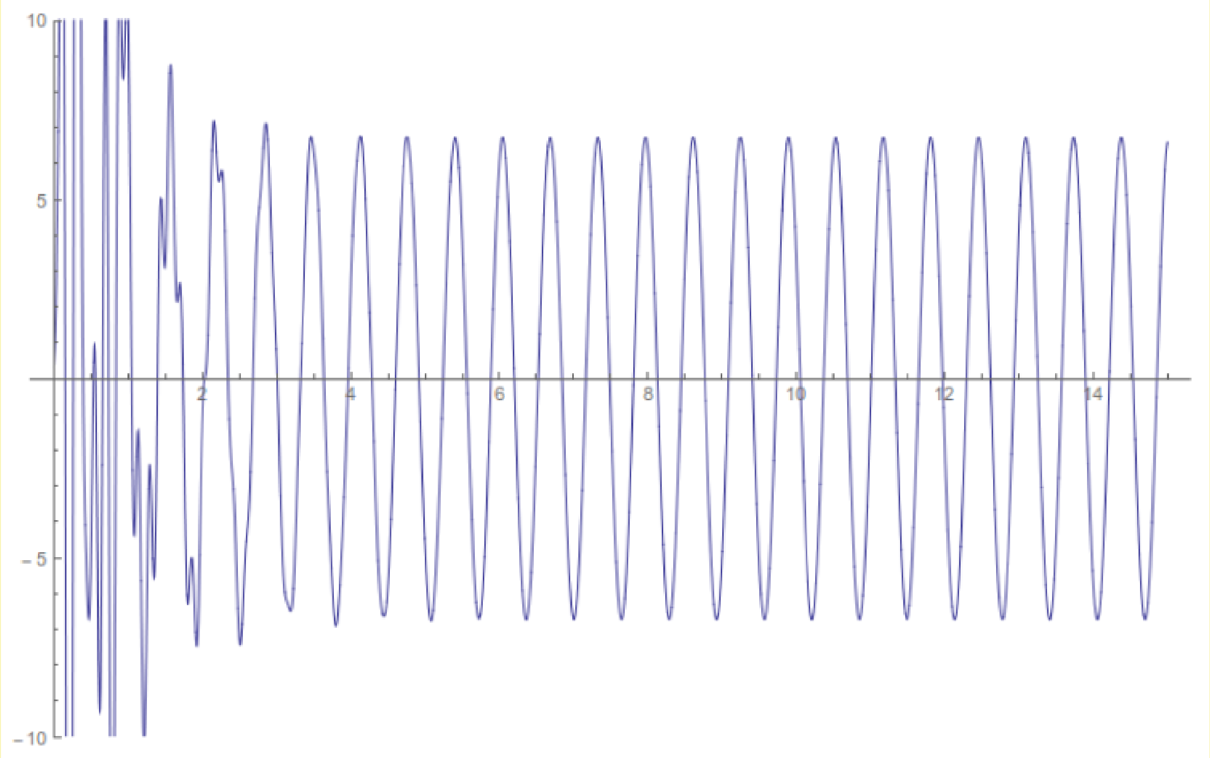
$$\Sigma'' + \frac{1}{2}B'^2\Sigma = 0$$

$$h' \equiv \partial_r h$$

$$\dot{h} \equiv \partial_t h + \frac{1}{2}A\partial_r h$$







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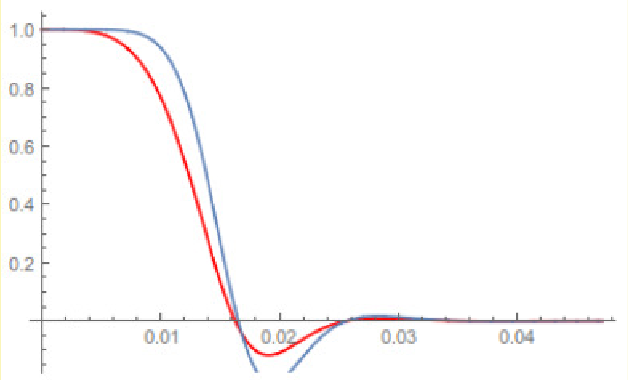
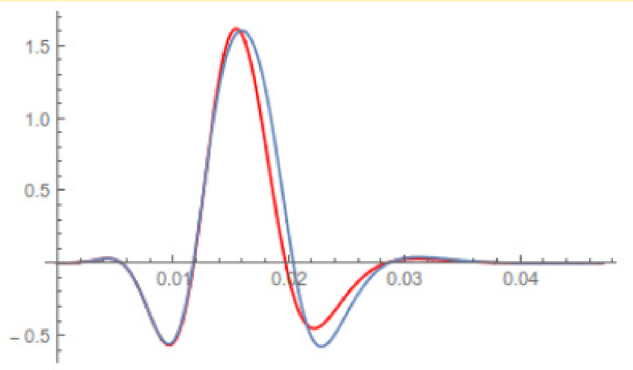
What can we understand from a strongly coupled non-abelian plasma out of equilibrium?

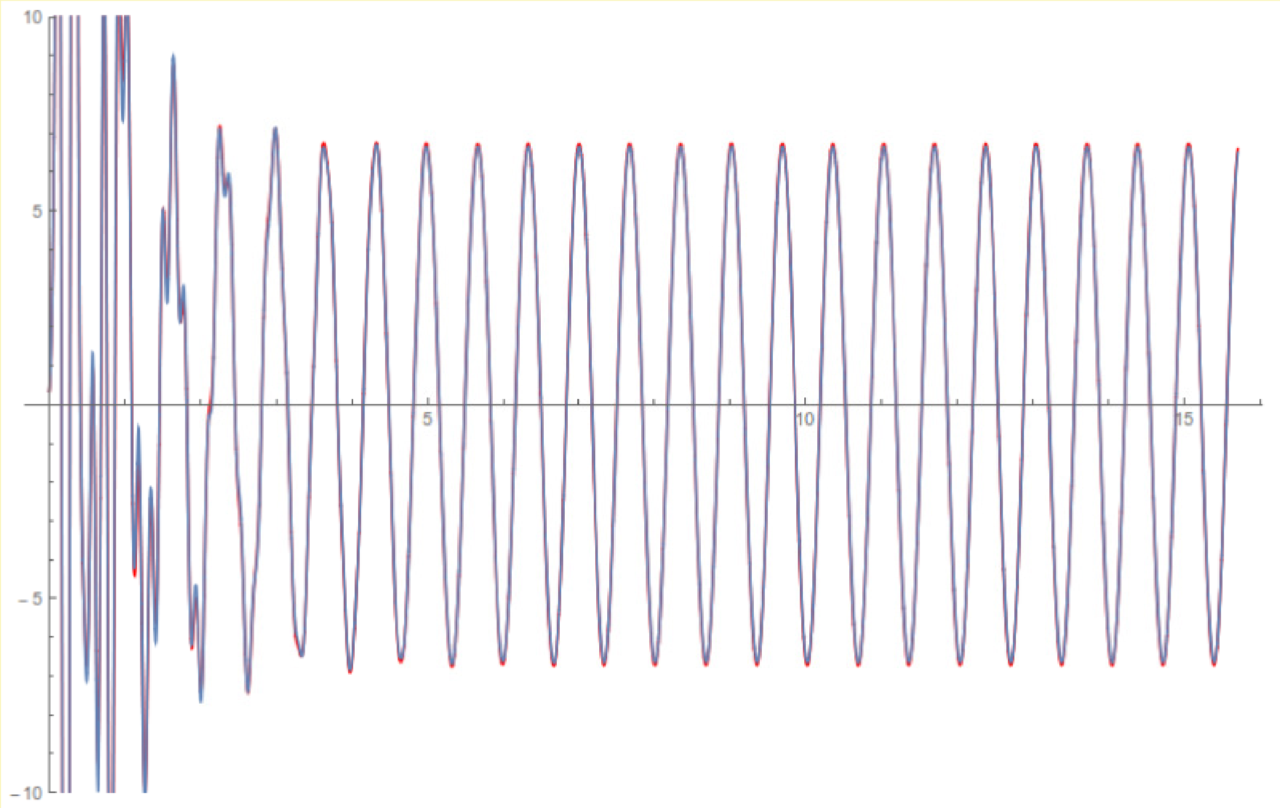
Let's slightly break the symmetry between z and x,y axis.

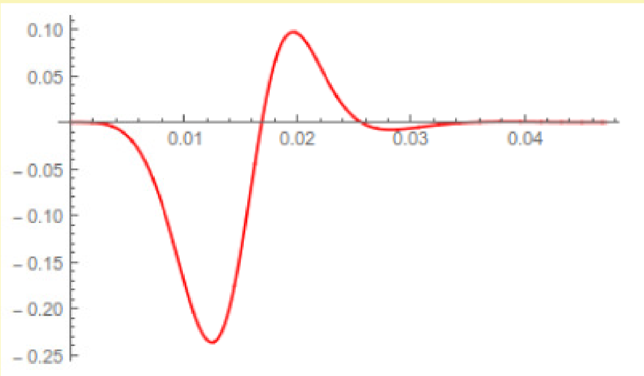
$$g_{xy} = \Sigma^2 H(t,r)$$

Intuitionally, we can compare it with a picture that we have from the isotropization of the expanding universe.


1.0
0.8
0.6
0.4
0.2
0.01
0.02
0.03
0.04


1.5
1.0
0.5
−0.5
0.01
0.02
0.03
0.04


10
5
−5
−10
5
10
15


0.10
0.05
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0.02
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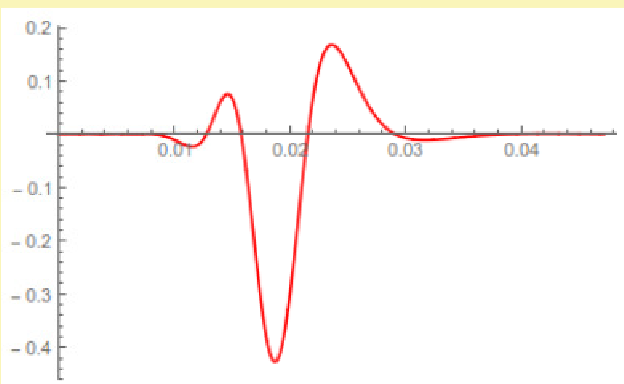
− 0.05
− 0.10
− 0.15
− 0.20
− 0.25


0.2
0.1
0.01
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− 0.2
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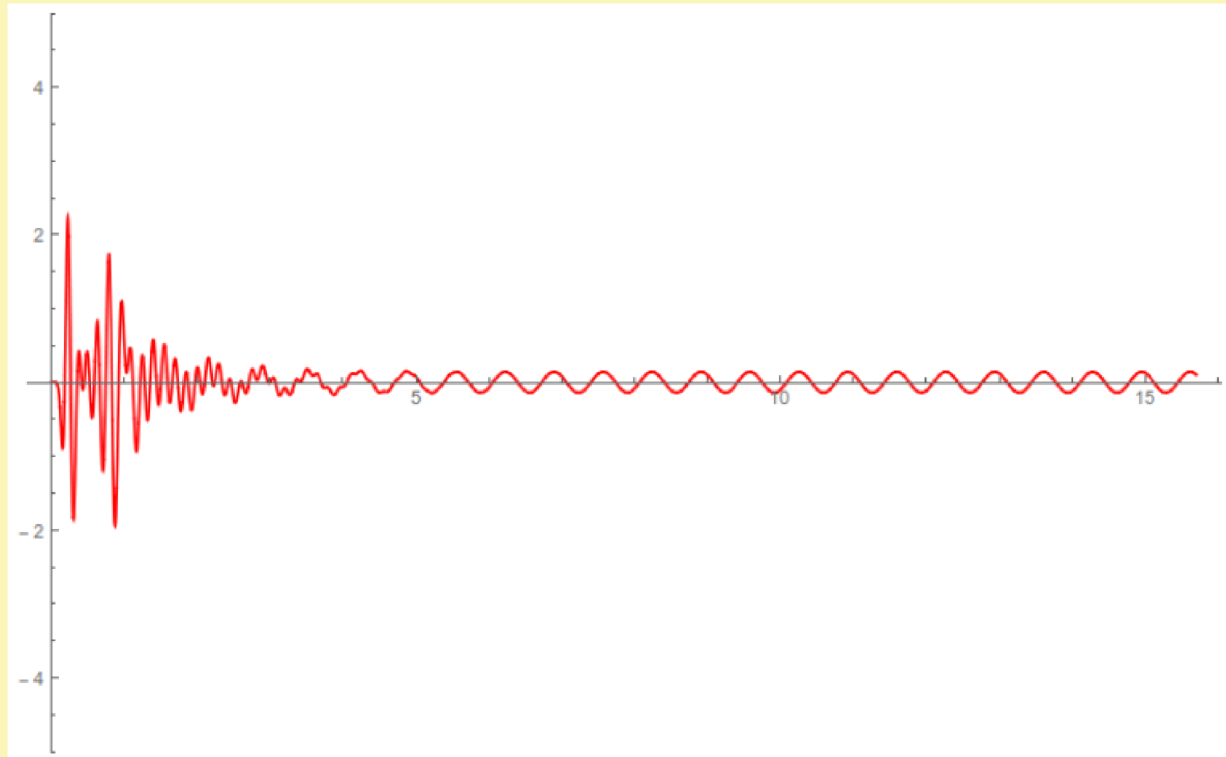
− 0.4


4
2
5
10
15
− 2
− 4

Our small perturbation could mimic the nonlinear dynamics.

Full consideration of the problem requires solving the whole nonlinear equations.

Modification of the boundary is required and It's known to give rise to turbulance!

$$\Sigma(\dot{\Sigma})' + 2\Sigma'\dot{\Sigma} - 2\Sigma^2 = 0$$

ϕ

$$\Sigma(\dot{B})' + \frac{3}{2}\left(\Sigma'\dot{B} + B'\dot{\Sigma}\right) = 0$$

$\dot{\phi}$, ϕ'

$$A'' + 3B'\dot{B} - 12\frac{\Sigma'\dot{\Sigma}}{\Sigma^2} + 4 = 0$$

$$\ddot{\Sigma} + \frac{1}{2}\left(\dot{B}^2\Sigma - A'\dot{\Sigma}\right) = 0$$

$$\Sigma'' + \frac{1}{2}B'^2\Sigma = 0$$

$$\Sigma(\dot{\Sigma})' + 2\Sigma'\dot{\Sigma} - 2\Sigma^2 + \frac{1}{12}m^2\phi^2\Sigma^2 = 0$$

$$\Sigma(\dot{\phi})' + \frac{3}{2}(\Sigma'\dot{\phi} + \phi'\dot{\Sigma}) - \frac{1}{2}\Sigma m^2\phi = 0$$

$$A'' + \dot{\phi}\phi' - 12\frac{\Sigma'\dot{\Sigma}}{\Sigma^2} + 4 - \frac{1}{6}m^2\phi^2 = 0$$

$$\ddot{\Sigma} + \frac{1}{2}(\dot{B}^2\Sigma - A'\dot{\Sigma}) = 0$$

$$\Sigma'' + \frac{1}{2}B'^2\Sigma = 0$$

This brings us to the next topic:

The system we are interested to study is that of a scalar field on an AdS-black brane spacetime.

The equations of motion follow by varying the 5-dimentional Einstein-Hilbert action:

$$S_5 = \frac{1}{16\pi G_5} \int d\xi^5 \sqrt{-g} \left(R + 12 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 \right)$$

Due to physical applications, we look at $m^2=-3$. According to the dictionary, this scalar field is dual to a fermionic mass operator with $\Delta=3$ in a thermal $\mathcal{N}=2^*$. $m^2=(\Delta-d)=-3$ with $d=4$.

Our background again will be the infalling Eddington-Finkelstein metric with follwing dependency for the scalar field:

$$ds_5^2 = -A(t,r)\,dt^2 + \Sigma^2(t,r)\,d\vec{x}^2 + 2\,dr\,dt$$

Also: $\phi(t,r)$

What is a quantum quench?

From QM we know that if we perturb the Hamiltonian by some parameter adiabatically,

$$H_\lambda = H_0 + \lambda \, \delta H$$

the system will follow the changes in $\lambda(t)$.

But sudden changes of the coupling forces the system into a superposition of the states of the new Hamiltonian. This is known as quantum quenches.

It's interesting to investigate:

$$\mathcal{L}_{SYM} + \lambda_\Delta(t)\, \mathcal{O}_\Delta$$

with

$$\lambda_\Delta = m_\Delta^2$$

$$\lambda_\Delta(t) = \frac{1}{2}\lambda^0 \left(1 \pm \tanh\frac{t}{\tau}\right)$$

$\tau \to \infty$ Adiabatic

$\tau \to 0$ Abrubt

Boundary:

$$\phi = \lambda u + \lambda' u^2 + u^3 \left(\rho_2 + \ln u \left(\frac{1}{2} \lambda'' + \frac{1}{6} \lambda^3 \right) \right) + \cdots$$

$$A = \frac{1}{u^2} - \frac{1}{6} \lambda^2 + u^2 \left(a_2 + \ln u \left(\frac{1}{6} \lambda'^2 - \frac{1}{6} \lambda \lambda'' - \frac{1}{36} \lambda^4 \right) \right) + \cdots$$

$$\Sigma = \frac{1}{u} - \frac{1}{12} u \lambda^2 - \frac{1}{9} u^2 \lambda \lambda' + \cdots$$

The curious case is when the scalar field is small and can be linearized.

Application of the Chebyshev-spectral method:

Let d/dz be a derivative operator applied to a known function u (z), with z defined over the interval [-1,1] and we assume the interval is partitioned into N subintervals by a grid of (N+1) points, taking N to be equal to 2.

$$D = \frac{1}{2}\begin{bmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{bmatrix}$$

z_0	z_1	z_2
•	•	•
u_0	u_1	u_2

$$u(z) = u_0 \frac{(z-z_1)(z-z_2)}{(z_0-z_1)(z_0-z_2)} + u_1 \frac{(z-z_2)(z-z_0)}{(z_1-z_2)(z_1-z_0)} + u_2 \frac{(z-z_0)(z-z_1)}{(z_2-z_0)(z_2-z_1)}$$

To get a general representation of the differentian matrices, we divide the interval [-1,1] into N subintervals. The Chebyshev Gauss-Lobatto points are

$$z_p = -\cos\frac{p\pi}{N} \quad p=0,\ldots,N$$

$$u(z) = \sum_{p=0}^{N} u_p \, l_p(z) \quad \text{with} \quad l_p(z) = \prod_{j=0,\, j\neq p}^{N} \frac{z - z_j}{z_p - z_j}$$

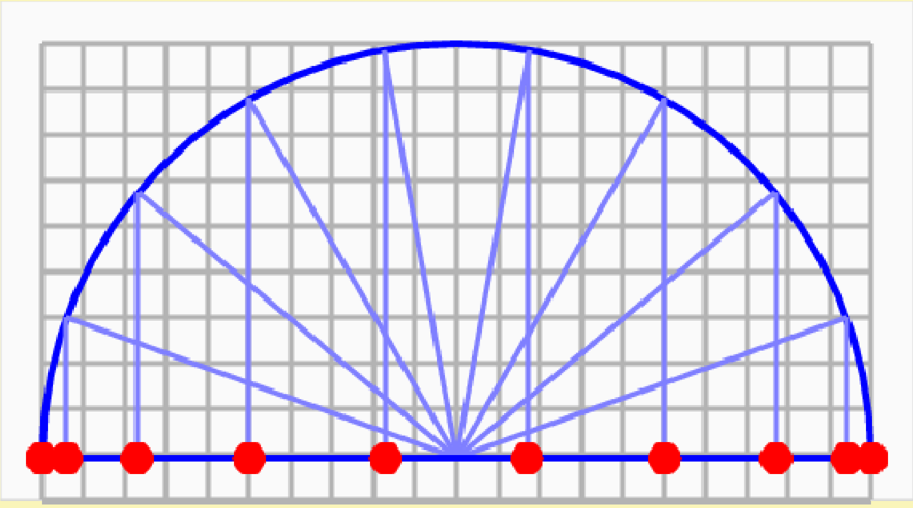
$$D = \begin{bmatrix} -\frac{2N^2+1}{6} & & & -\frac{(-1)^N}{2} \\ & \ddots & \star & \\ & & -\frac{z_j}{2(1-z_j^2)} & \\ & \star & & \ddots \\ \frac{(-1)^N}{2} & & & \frac{2N^2+1}{6} \end{bmatrix}$$

$$\star \equiv \frac{\bar{c}_i}{\bar{c}_j} \frac{(-1)^{i+j}}{z_i - z_j}$$

$$\bar{c}_n = \begin{cases} 2 & n=0 \text{ or } n=N \\ 1 & 1 \leq n \leq N-1 \end{cases}$$

Why do we use non-uniform grids?

Only approximately uniform in the central part. Points tend to get closer when approaching the extremities at z=+1 or z=-1. The mesh size scales as 1/N²



In a uniform grid error peaks out near the extremities and is larger than the non-uniform grid. Infact we get exponential improvement!

$$\Sigma(\dot{\Sigma})' + 2\Sigma'\dot{\Sigma} - 2\Sigma^2 + \frac{1}{12}m^2\phi^2\Sigma^2 = 0$$

$$\Sigma(\dot{\phi})' + \frac{3}{2}(\Sigma'\dot{\phi} + \phi'\dot{\Sigma}) - \frac{1}{2}\Sigma m^2\phi = 0$$

$$A'' + \dot{\phi}\phi' - 12\frac{\Sigma'\dot{\Sigma}}{\Sigma^2} + 4 - \frac{1}{6}m^2\phi^2 = 0$$

$$\ddot{\Sigma} + \frac{1}{2}(\dot{B}^2\Sigma - A'\dot{\Sigma}) = 0$$

$$\Sigma'' + \frac{1}{2}B'^2\Sigma = 0$$

With the redefinition of: $\hat{\phi}(t,u) = \phi_{log}(t,u) + \phi_c(t,u)$

One can write both sides of the EOM for the scalar field in terms of the truncated sum over Chebyshev polynomials

$$\partial_t \phi_c = \pi_c + \frac{1-\tau^4 u^4}{2}\partial_u \phi_c + K_{log}$$

$$\partial_u \pi_c - \frac{1}{2u}\pi_c = -\left(J_\pi + \partial_u \pi_{log} - \frac{1}{2u}\pi_{log} \right)$$

$$J_\pi = \frac{1}{4u}\partial_u \hat{\phi} - \frac{1}{4}\tau^4 u^3 \partial_u \hat{\phi} + \frac{1}{2}\tau^4 u^2 \hat{\phi}$$

$$\pi \equiv \partial_t \hat{\phi} + \frac{\tau^4 u^4 - 1}{2}\partial_u \hat{\phi}$$

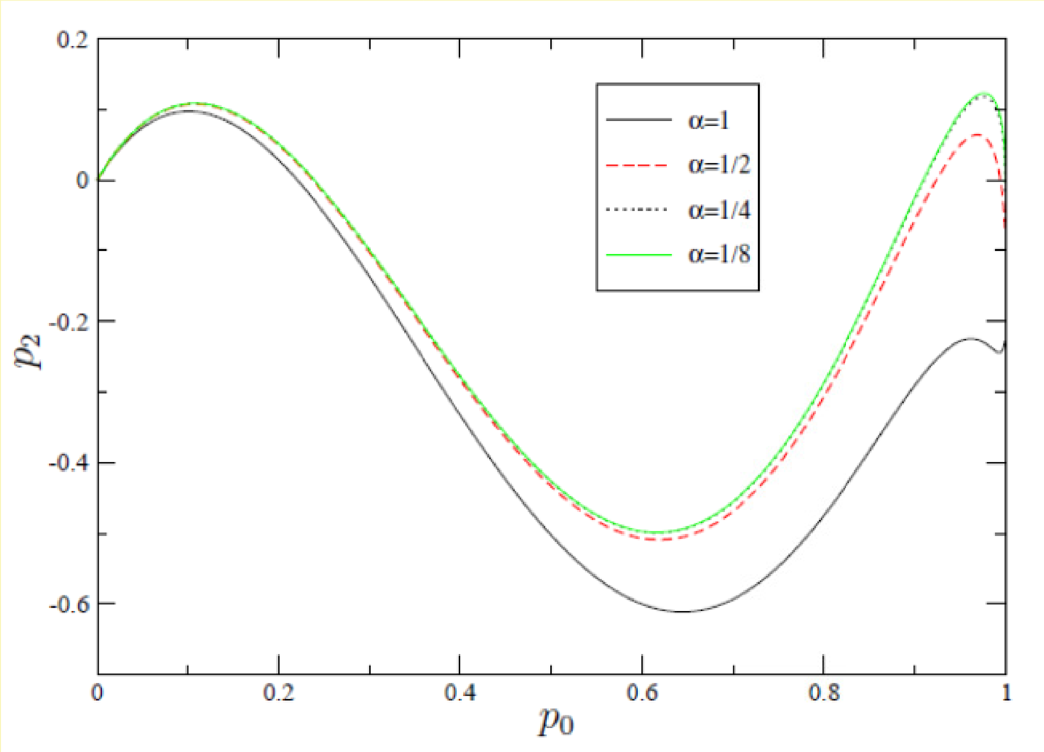
One main Challenge:

Look for the universal behaviors!

Logarithms have to be subtracted.

They are a source of numerical instabilities.

Work in progress!



A. Buchel, R. Myers, A. van Niekerk