Conformal Field Theory in Momentum space

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Aspects of fluid/gravity correspondence
Thessaloniki, 16-20 February 2015

Introduction

- Conformal invariance imposes strong constraints on correlation functions.
- It determines two- and three-point functions of scalars, conserved vectors and the stress-energy tensor [Polyakov (1970)] ... [Osborn, Petkou (1993)]. For example,

$$\langle \mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\mathcal{O}_3(\boldsymbol{x}_3)\rangle$$
$$= \frac{c_{123}}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\boldsymbol{x}_2-\boldsymbol{x}_3|^{\Delta_2+\Delta_3-\Delta_1}|\boldsymbol{x}_3-\boldsymbol{x}_1|^{\Delta_3+\Delta_1-\Delta_2}}.$$

- It determines the form of higher point functions up to functions of cross-ratios.

Introduction

- These results (and many others) were obtained in position space.
- This is in stark contrast with general QFT were Feymnan diagrams are typically computed in momentum space.
- While position space methods are powerful, typically they
  - provide results that hold only at separated points ("bare" correlators).
  - are hard to extend beyond CFTs
- The purpose of this work is to provide a first principles analysis of CFTs in momentum space.

Introduction

- Momentum space results were needed in several recent applications:
  - Holographic cosmology [McFadden, KS](2010)(2011) [Bzowski, McFadden, KS (2011)(2012)] [Pimentel, Maldacena (2011)][Mata, Raju,Trivedi (2012)] [Kundu, Shukla,Trivedi (2014)].
  - Studies of 3d critical phenomena [Sachdev et al (2012)(2013)]

References

- Adam Bzowski, Paul McFadden, KS
  Implications of conformal invariance in momentum space
  1304.7760
- Adam Bzowski, Paul McFadden, KS
  Renormalized scalar 3-point functions
  15xx.xxxx
- Adam Bzowski, Paul McFadden, KS
  Renormalized tensor 3-point functions
  15xx.xxxx

Conformal invariance

- Conformal transformations consists of dilatations and special conformal transformations.
- Dilatations $\delta x^\mu = \lambda x^\mu$, are linear transformations, so their implications are easy to work out.
- Special conformal transforms, $\delta x^\mu = b^\mu x^2 - 2x^\mu b \cdot x$, are non-linear, which makes them difficult to analyse (and also more powerful).
- The corresponding Ward identities are partial differential equations which are difficult to solve.

Conformal invariance

- In position space one overcomes the problem by using the fact that special conformal transformations can be obtained by combining inversions with translations and then analyzing the implications of inversions.
- In momentum space we will see that one can actually directly solve the special conformal Ward identities.

Conformal Ward identities

- These are derived using the conformal transformation properties of conformal operators. For scalar operators:

$$\langle \mathcal{O}_1(\boldsymbol{x}_1)\cdots\mathcal{O}_n(\boldsymbol{x}_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d}\langle \mathcal{O}_1(\boldsymbol{x}'_1)\cdots\mathcal{O}_n(\boldsymbol{x}'_n)\rangle$$

- For (infinitesimal) dilatations this yields

$$0 = \left[\sum_{j=1}^{n}\Delta_j + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right]\langle \mathcal{O}_1(\boldsymbol{x}_1)\ldots\mathcal{O}_n(\boldsymbol{x}_n)\rangle.$$

- In momentum space this becomes

$$0 = \left[\sum_{j=1}^{n}\Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right]\langle \mathcal{O}_1(\boldsymbol{p}_1)\ldots\mathcal{O}_n(\boldsymbol{p}_n)\rangle,$$

Special conformal Ward identity

- For (infinitesimal) special conformal transformations this yields

$$0 = \left[\sum_{j=1}^{n} \left(2\Delta_j x_j^\kappa + 2 x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_{j\kappa}} \right) \right] \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle$$

- In momentum space this becomes

$$0 = \mathcal{K}^\mu \langle \mathcal{O}_1(\boldsymbol{p}_1) \dots \mathcal{O}_n(\boldsymbol{p}_n) \rangle,$$

$$\mathcal{K}^\mu = \left[\sum_{j=1}^{n-1} \left(2(\Delta_j - d) \frac{\partial}{\partial p_j^\kappa} - 2 p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_j^\kappa} + (p_j)_\kappa \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_{j\alpha}} \right) \right]$$

Special conformal Ward identities

- To extract the content of the special conformal Ward identity we expand $\mathcal{K}^\mu$ is a basis of linear independent vectors, the $(n-1)$ independent momenta,

$$\mathcal{K}^\kappa = p_1^\kappa \mathcal{K}_1 + \ldots + p_{n-1}^\kappa \mathcal{K}_{n-1}.$$

- Special conformal Ward identities constitute $(n-1)$ differential equations.

Conformal Ward identities

- ➢ Poincaré invariant n-point function in $d \geq n$ spacetime dimensions depends on $n(n-1)/2$ kinematic variables.
- ➢ Thus, after imposing $(n-1)+1$ conformal Ward identities we are left with

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

undetermined degrees of freedom.

- ➢ This number equals the number of conformal ratios in n variables in $d \geq n$ dimensions.
- ➠ It is not known however what do the cross ratios become in momentum space.

Outline

Scalar 2-point function

- The dilatation Ward identity reads

$$0 = \left[d - \Delta_1 - \Delta_2 + p\frac{\partial}{\partial p}\right] \langle O_1(\boldsymbol{p}) O_2(-\boldsymbol{p})\rangle$$

- The 2-point function is a homogeneous function of degree $(\Delta_1 + \Delta_2 - d)$:

$$\langle O_1(\boldsymbol{p}) O_2(-\boldsymbol{p})\rangle = c_{12} p^{\Delta_1+\Delta_2-d}.$$

where c_{12} is an integration constant.

Scalar 2-point function

- The special conformal Ward identity reads

$$0 = \mathcal{K}\langle O_1(\boldsymbol{p}) O_2(-\boldsymbol{p})\rangle, \qquad \mathcal{K} = \frac{d^2}{dp^2} - \frac{2\Delta_1 - d - 1}{p}\frac{d}{dp}$$

- Inserting the solution of the dilatation Ward identity we find that we need

$$\Delta_1 = \Delta_2$$

Scalar 2-point function

The general solution of the conformal Ward identities is:

$$\langle O_\Delta(\boldsymbol{p}) O_\Delta(-\boldsymbol{p})\rangle = c_{12} p^{2\Delta-d}.$$

- This solution is trivial when

$$\Delta = \frac{d}{2} + k, \qquad k = 0, 1, 2, \ldots$$

  because then correlator is local,

$$\langle O(\boldsymbol{p}) O(-\boldsymbol{p})\rangle = c p^{2k} \to \langle O(\boldsymbol{x}_1) O(\boldsymbol{x}_2)\rangle \sim \Box^k \delta(x_1 - x_2)$$

- Let ϕ_0 the source of O. It has dimension $d - \Delta = d/2 - k$. The term

$$\phi_0 \Box^k \phi_0$$

  has dimension d and can act as a local counterterm.

Position space [Petkou, KS (1999)]

- In position space, it seems that none of these are an issue:

$$\langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle = \frac{C}{x^{2\Delta}}$$

- This expression however is valid only at separated points, $x^2 \neq 0$.
- Correlation functions should be well-defined distributions, and in particular they should have well-defined Fourier transforms.
- Fourier transforming we find:

$$\int d^d\boldsymbol{x}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{x^{2\Delta}} = \frac{\pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)} p^{2\Delta-d},$$

- This is well-behaved, except when $\Delta = d/2 + k$, where k is a positive integer.

Regularization

- We use dimensional regularisation to regulate the theory

$$d \mapsto d + 2u\epsilon, \qquad \Delta_j \mapsto \Delta_j + (u+v)\epsilon$$

- In the regulated theory, the solution of the Ward identities is the same as before but the integration constant may depend on the regulator,

$$\langle O(\boldsymbol{p}) O(-\boldsymbol{p}) \rangle_{\text{reg}} = c(\epsilon, u, v) p^{2\Delta - d + 2v\epsilon}.$$

Regularization and Renormalization

$$\langle O(\boldsymbol{p})O(-\boldsymbol{p})\rangle_{\text{reg}} = c(\epsilon, u, v)p^{2\Delta-d+2v\epsilon}.$$

- Now, in local CFTs:

$$c(\epsilon, u, v) = \frac{c^{(-1)}(u, v)}{\epsilon} + c^{(0)}(u, v) + O(\epsilon)$$

- This leads to

$$\langle O(\boldsymbol{p})O(-\boldsymbol{p})\rangle_{\text{reg}} = p^{2k}\left[\frac{c^{(-1)}}{\epsilon} + c^{(-1)}v\log p^2 + c^{(0)} + O(\epsilon)\right].$$

- We need to renormalise ....

Renormalization

- Let ϕ_0 the source that couples to O,

$$S[\phi_0] = S_0 + \int d^{d+2u\epsilon}\boldsymbol{x}\, \phi_0 O.$$

- The divergence in the 2-point function can be removed by the addition of the counterterm action

$$S_{\text{ct}} = a_{\text{ct}}(\epsilon, u, v) \int d^{d+2u\epsilon}\boldsymbol{x}\, \phi_0 \Box^k \phi_0 \mu^{2v\epsilon},$$

- Removing the cut-off we obtain the renormalised correlator:

$$\langle O(\boldsymbol{p}) O(-\boldsymbol{p}) \rangle_{ren} = p^{2k} \left[C \log \frac{p^2}{\mu^2} + C_1 \right]$$

Anomalies

- The counter term breaks scale invariance and as result the theory has a conformal anomaly.
- The 2-point function depend on the scale [Petkou, KS (1999)]

$$\mathcal{A}_2 = \mu \frac{\partial}{\partial \mu} \langle O(\boldsymbol{p}) O(-\boldsymbol{p}) \rangle = c p^{2\Delta - d},$$

- The integrated anomaly is Weyl invariant

$$A = \int d^d \boldsymbol{x} \, \phi_0 \Box^k \phi_0$$

On a curved background, $\Box^k$ is replaced by the "k-th power of the conformal Laplacian", P^k.

Outline

1 2-point functions

2 3-point functions

3 Conclusions

Scalar 3-point functions

We would now like to understand 3-point functions at the same level:

- What is the general solution of the conformal Ward identities?
- What is the analogue of the condition

$$\Delta = \frac{d}{2} + k, \qquad k = 0, 1, 2, ...$$

- Are there new conformal anomalies associated with 3-point functions and if yes what is their structure?

Conformal Ward identities

- Dilatation Ward identity

$$0 = \left[2d - \Delta_t + \sum_{j=1}^{3} p_j \frac{\partial}{\partial p_j} \right] \langle O_1(\boldsymbol{p}_1) O_2(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle$$

$\Delta_t = \Delta_1 + \Delta_2 + \Delta_3$

- ➠ The correlation is a homogenous function of degree $(2d - \Delta_t)$.
- The special conformal Ward identities give rise to two scalar 2nd order PDEs.

Special conformal Ward identities

- Special conformal WI

$$0 = \mathrm{K}_{12} \langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_2(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle = \mathrm{K}_{23} \langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_2(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle,$$

where

$$\mathrm{K}_{ij} = \mathrm{K}_i - \mathrm{K}_j,$$
$$\mathrm{K}_j = \frac{\partial^2}{\partial p_j^2} + \frac{d + 1 - 2\Delta_j}{p_j} \frac{\partial}{\partial p_j}, \ (i, j = 1, 2, 3).$$

- This system of differential equations is precisely that defining Appell's F_4 generalised hypergeometric function of two variables. [Coriano, Rose, Mottola, Serino][Bzowski, McFadden, KS] (2013).

Scalar 3-point functions

- There are four linearly independent solutions of these equations.
- Three of them have unphysical singularities at certain values of the momenta leaving one physically acceptable solution.
- We thus recover the well-known fact that scalar 3-point functions are determined up to a constant.

Scalar 3-pt functions and triple-K integrals

- The physically acceptable solution has the following *triple-K integral* representation:

$$\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle = C_{123} p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \int_0^\infty dx\, x^{\frac{d}{2}-1} K_{\Delta_1-\frac{d}{2}}(p_1 x) K_{\Delta_2-\frac{d}{2}}(p_2 x) K_{\Delta_3-\frac{d}{2}}(p_3 x),$$

where $K_\nu(p)$ is a Bessel function and C_{123} is an constant.

- This is the general solution of the conformal Ward identities.

Triple K-integrals

- Triple-K integrals,

$$I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_1,p_2,p_3) = \int_0^\infty dx\, x^\alpha \prod_{j=1}^{3} p_j^{\beta_j} K_{\beta_j}(p_j x),$$

  are the building blocks of all 3-point functions.
- The integral converges provided

$$\alpha > \sum_{j=1}^{3} |\beta_j| - 1$$

- The integral can be defined by analytic continuation provided

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 \neq -2k,$$

  where k is any non-negative integer.

Renormalization and anomalies

- If the equality holds,

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2k,$$

  the integral cannot be defined by analytic continuation.
- Non-trivial subtractions and renormalization may be required and this may result in conformal anomalies.
- Physically when this equality holds, there are new terms of dimension d that one can add to the action (counterterms) and/or new terms that can appear in T^μ_μ (conformal anomalies).

Scalar 3-pt function

- For the triple-K integral that appears in the 3-pt function of scalar operators the condition becomes

$$\frac{d}{2} \pm (\Delta_1 - \frac{d}{2}) \pm (\Delta_2 - \frac{d}{2}) \pm (\Delta_3 - \frac{d}{2}) = -2k$$

- There are four cases to consider, according to the signs needed to satisfy this equation. We will refer to the 4 cases as the $(- - -)$, $(- - +)$, $(- + +)$ and $(+ + +)$ cases.
- Given Δ_1, Δ_2 and Δ_3 these relations may be satisfied with more than one choice of signs and k.

Procedure

- To analyse the problem we will proceed by using dimensional regularisation

$$d \mapsto d + 2u\epsilon, \qquad \Delta_j \mapsto \Delta_j + (u+v)\epsilon$$

- In the regulated theory the solution of the conformal Ward identity is given in terms of the triple-K integral but now the integration constant C_{123} in general will depend on the regulator ϵ, u, v.
- We need to understand the singularity structure of the triple-K integrals and then renormalise the correlators.
- We will discuss each case in turn.

The $(- - -)$ case

$$\Delta_1 + \Delta_2 + \Delta_3 = 2d + 2k$$

- This the analogue of the $\Delta = d/2 + k$ case in 2-point functions.
- There are possible counterterms

$$S_{\text{ct}} = a_{\text{ct}}(\epsilon, u, v) \int d^d\boldsymbol{x}\, \Box^{k_1}\phi_1 \Box^{k_2}\phi_2 \Box^{k_3}\phi_3$$

  where $k_1 + k_2 + k_3 = k$. The same terms may appear in T^μ_μ as new conformal anomalies.
- After adding the countertrems one may remove the regulator to obtain the renormalised correlates.

Example: $\Delta_1 = \Delta_2 = \Delta_3 = 2$, $d = 3$

- The source ϕ for an operator of dimension 2 has dimension 1, so ϕ^3 has dimension 3.
- Regularizing:

$$\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = C_{123}\left(\frac{\pi}{2}\right)^{3/2}\int_0^\infty dx\, x^{-1+\epsilon} e^{-x(p_1+p_2+p_3)}$$

$$= C_{123}\left(\frac{\pi}{2}\right)^{3/2}\left[\frac{1}{\epsilon} - (\gamma_E + \log(p_1+p_2+p_3)) + O(\epsilon)\right].$$

Renormalization and anomalies

- We add the counterterm

$$S_{ct} = -\frac{C_{123}}{3!\epsilon}\left(\frac{\pi}{2}\right)^{3/2}\int d^{3+2\epsilon}\boldsymbol{x}\ \phi^3\mu^{-\epsilon}$$

- This leads to the renormalized correlator,

$$\langle\mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = -C_{123}\left(\frac{\pi}{2}\right)^{3/2}\log\frac{p_1+p_2+p_3}{\mu}$$

- The renormalized correlator is not scale invariant

$$\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = C_{123}\left(\frac{\pi}{2}\right)^{3/2}$$

- There is a new conformal anomaly:

$$\langle T\rangle = -\phi\langle\mathcal{O}\rangle + \frac{1}{3!}C_{123}\left(\frac{\pi}{2}\right)^{3/2}\phi^3.$$

The $(- - +)$ case

$$\Delta_1 + \Delta_2 - \Delta_3 = d + 2k$$

- In this case the new local term one can add to the action is

$$S_{\text{ct}} = a_{\text{ct}} \int d^d x \Box^{k_1} \phi_1 \Box^{k_2} \phi_2 O_3$$

  where $k_1 + k_2 = k$.
- In this case we have renormalization of sources,

$$\phi_3 \to \phi_3 + a_{\text{ct}} \Box^{k_1} \phi_1 \Box^{k_2} \phi_2$$

- The renormalised correlator will satisfy a Callan-Symanzik equation with beta function terms.

Callan-Symanzik equation

- The quantum effective action $\mathcal{W}$ (i.e., the generating functional of renormalised connected correlators) obeys the equation

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_i \int d^d\vec{x}\,\beta_i \frac{\delta}{\delta\phi_i(\vec{x})}\right)\mathcal{W} = \int d^d\vec{x}\,\mathcal{A},$$

- This implies that for 3-point functions we have

$$\mu\frac{\partial}{\partial\mu}\langle O_i(p_1)O_j(p_2)O_j(p_3)\rangle =$$
$$\beta_{j,ji}\big(\langle O_j(p_2)O_j(-p_2)\rangle + \langle O_j(p_3)O_j(-p_3)\rangle\big) + \mathcal{A}^{(3)}_{ijj},$$

$$\beta_{i,jk} = \frac{\delta^2\beta_i}{\delta\phi_j\delta\phi_k}\Big|_{\{\phi_l\}=0}, \quad \mathcal{A}^{(3)}_{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3) = -\frac{\delta^3}{\delta\phi_i(\vec{x}_1)\delta\phi_j(\vec{x}_2)\delta\phi_k(\vec{x}_3)}\int d^d\vec{x}\,\mathcal{A}(\{\phi_l(\vec{x})\})$$

Example: $\Delta_1 = 4, \Delta_2 = \Delta_3 = 3$ in $d = 4$

- $\Delta_1 + \Delta_2 + \Delta_3 = 10 = 2d + 2k$, which satisfies the $(- - -)$-condition with $k = 1$.
- There is an anomaly

$$\int d^d x \phi_0 \phi_1 \Box \phi_1$$

- $\Delta_1 + \Delta_2 - \Delta_3 = 4 = d + 2k$, which satisfies the $(- - +)$-condition with $k = 0$. There is the counterterm

$$\int d^4 x \phi_0 \phi_1 O_3$$

- There is a beta function for ϕ_1.

$\langle O_4 O_3 O_3 \rangle$

$$\langle O_4(\boldsymbol{p}_1) O_3(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle = \alpha \left(2 - p_1 \frac{\partial}{\partial p_1} \right) I^{\text{(non-local)}}$$
$$+ \ \frac{\alpha}{8} \left[(p_2^2 - p_3^2) \log \frac{p_1^2}{\mu^2} \left(\log \frac{p_3^2}{\mu^2} - \log \frac{p_2^2}{\mu^2} \right) \right.$$
$$- (p_2^2 + p_3^2) \log \frac{p_2^2}{\mu^2} \log \frac{p_3^2}{\mu^2}$$
$$\left. (p_1^2 - p_2^2) \log \frac{p_3^2}{\mu^2} + (p_1^2 - p_3^2) \log \frac{p_2^2}{\mu^2} + p_1^2 \right]$$

$\langle O_4 O_3 O_3 \rangle$

$$
\begin{aligned}
I^{(\text{non-local})} = -\frac{1}{8}\sqrt{-J^2}\Bigg[\frac{\pi^2}{6} - 2\log\frac{p_1}{p_3}\log\frac{p_2}{p_3} \\
+ \log\left(-X\frac{p_2}{p_3}\right)\log\left(-Y\frac{p_1}{p_3}\right) \\
- Li_2\left(-X\frac{p_2}{p_3}\right) - Li_2\left(-Y\frac{p_1}{p_3}\right)\Bigg],
\end{aligned}
$$

$$
J^2 = (p_1+p_2-p_3)(p_1-p_2+p_3)(-p_1+p_2+p_3)(p_1+p_2+p_3),
$$

$$
X = \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{-J^2}}{2p_2p_3}, \qquad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{-J^2}}{2p_1p_3}.
$$

Callan-Symanzik equation

- It satisfies

$$\mu\frac{\partial}{\partial\mu}\langle O_4(\boldsymbol{p}_1)O_3(\boldsymbol{p}_2)O_3(\boldsymbol{p}_3)\rangle =$$
$$\frac{\alpha}{2}\left(p_2^2\log\frac{p_2^2}{\mu^2}+p_3^2\log\frac{p_3^2}{\mu^2}-p_1^2+\frac{1}{2}(p_2^2+p_3^2)\right).$$

- This is indeed the correct Callan-Symanzik equation.
  (Recall that $\langle O_3(\boldsymbol{p})O_3(\boldsymbol{p})\rangle = p^2\log\frac{p^2}{\mu^2}$)

The $(+++)$ and $(-++)$ cases

- In these cases it is the representation of the correlator in terms of the triple-K integral that is singular, not the correlator itself,

$$\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle = C_{123} p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \times I_{d/2-1,\{\Delta_1-d/2,\Delta_3-d/2,\Delta_3-d/2\}}$$

  Taking the integration constant $C_{123} \sim \epsilon^m$ for appropriate m and sending $\epsilon \to 0$ results in an expression that satisfies the original (non-anomalous) Ward identity.

- In other words, the Ward identities admit a solution that is finite.

The $(+++)$ case

$$\Delta_1 + \Delta_2 + \Delta_3 = d - 2k$$

- For example, for $k = 0$ the finite solution to the Ward identities is

$$\langle O_1(\boldsymbol{p}_1) O_2(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle = c p_1^{(\Delta_1 - \Delta_2 - \Delta_3)} p_2^{(\Delta_2 - \Delta_1 - \Delta_3)} p_3^{(\Delta_3 - \Delta_1 - \Delta_2)}$$

- When the operators have these dimensions there are "multi-trace" operators which are classically marginal

$$\mathcal{O} = \Box^{k_1} O_1 \Box^{k_2} O_2 \Box^{k_3} O_3$$

where $k_1 + k_2 + k_3 = k$.

The $(-++)$ case

$$\Delta_1 - \Delta_2 - \Delta_3 = 2k$$

- For $k = 0$, there are "extremal correlators" . In position, the 3-point function is a product of 2-point functions

$$\langle O_1(\boldsymbol{x}_1) O_2(\boldsymbol{x}_2) O_3(\boldsymbol{x}_3) \rangle = \frac{c_{123}}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|^{2\Delta_2} |\boldsymbol{x}_3 - \boldsymbol{x}_1|^{2\Delta_3}}$$

- In momentum space, the finite solution to the Ward identities is

$$\langle O_1(\boldsymbol{p}_1) O_2(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle = c_{123} p_2^{(2\Delta_2 - d)} p_3^{(2\Delta_3 - d)}$$

- When the operators have these dimensions there are "multi-trace" operators of dimension Δ_1

$$\mathcal{O} = \Box^{k_2} O_2 \Box^{k_3} O_3$$

where $k_2 + k_3 = k$.

Tensorial correlators

- New issues arise for tensorial correlation functions, such as those involving stress-energy tensors and conserved currents
- Lorentz invariance implies that the tensor structure will be carried by tensors constructed from the momenta p^{μ} and the metric $\delta_{\mu\nu}$.
- After an appropriate parametrisation, the analysis becomes very similar to the one we discussed here.
- In particular, these correlator are also given in terms of triple-K integrals.

Outline

Conclusions/Outlook

- We obtained the implications of conformal invariance for three-point functions working in momentum space.
- We discussed renormalization and anomalies.
- The presence of "beta function" terms in the Callan - Symanzik equation for CFT correlators is new.
- It would be interesting to understand how to extend the analysis to higher point functions. What is the momentum space analogue of cross-ratios?
- Bootstrap in momentum space?