Toda equation, $\mathcal{N} = 2$ SCFTs and Euclidean holography

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based on works with
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Synopsis

Continual Toda equation as a toolkit for studying:

Supergravity solutions:

1. 11d SUGRA solutions with $SO(2,4) \times SO(3) \times U(1)_R$ isometry.
2. $\nexists$ additional $U(1)$ symmetry – No smearing.
3. Explore the 11d landscape of qualitatively different solutions – qualitatively different SCFTs.

Euclidean holography:

1. 4d WSD & Einstein metrics with a symmetry, i.e. Przanowski–Tod & Calderbank–Pedersen.
2. Fefferman–Graham expansion and boundary data.

Plan of the talk

Gravity duals of $\mathcal{N} = 2$ SCFTs

General M-theory solution which preserves four dimensional $\mathcal{N} = 2$ superconformal symmetry, with $SO(2,4) \times SO(3) \times U(1)_R$ isometry, was constructed in Lin–Lunin–Maldacena (2004):

$$
\begin{aligned}
&ds_{11}^2 = \kappa_{11}^{\frac{2}{3}} e^{2\lambda} \left(4\, ds^2_{\mathrm{AdS}_5} + z^2 e^{-6\lambda} d\Omega_2^2 + \frac{4}{1 - z\,\partial_z\Psi}\,(d\varphi + \omega)^2 - \frac{\partial_z\Psi}{z}\,\gamma_{ij} dx^i dx^j\right),\\
&\omega = \omega_x dx + \omega_y dy\,, \quad \omega_x = \frac{1}{2}\partial_y\Psi\,, \quad \omega_y = -\frac{1}{2}\partial_x\Psi\,,\\
&\gamma_{ij} dx^i dx^j = dz^2 + e^{\Psi}(dx^2 + dy^2)\,, \quad e^{-6\lambda} = -\frac{\partial_z\Psi}{z(1 - z\,\partial_z\Psi)}\,, \quad G_4 = dC_3 = \kappa_{11}\, F_2 \wedge d\Omega_2\,,\\
&F_2 = 2(d\varphi + \omega)\wedge d\left(z^3 e^{-6\lambda}\right) + 2z\left(1 - z^2 e^{-6\lambda}\right)d\omega - \partial_z e^{\Psi} dx \wedge dy\,, \quad \kappa_{11} = \frac{\pi \ell_P^3}{2}\,,
\end{aligned}
$$

where $\Psi(x, y, z)$ satisfies the continual Toda equation:

$$\left(\partial_x^2 + \partial_y^2\right)\Psi + \partial_z^2\, e^{\Psi} = 0\,,$$

where $z \in [0, z_c]$ and z_c : $e^{\Psi} \sim z_c - z\,, \quad \partial_z\Psi \longrightarrow \infty$.
Regularity of the metric when S^2 shrinks to zero size, requires

$$z = 0: \quad e^{\Psi} = \text{finite} \neq 0\,, \quad \partial_z\Psi = 0\,, \quad \partial_z\Psi/z = \text{finite} \neq 0.$$

PLAN OF THE TALK

The continual Toda equation

The molecule of Toda

$$H = \frac{1}{2}\,p_z^2 + \sum_{i=1}^{z-1}\left(\frac{1}{2}\,p_i^2 + e^{q_i - q_{i+1}}\right) \Longrightarrow \ddot{\Psi}_i + \sum_{j=1}^{z-1} K_{i|j}\, e^{\Psi_j} = 0\,,$$

$$\Psi_i(T) = q_i - q_{i+1}\,, \qquad K_{i|i} = 2\,, \qquad K_{i|i+1} = K_{i+1|i} = -1\,, \quad i = 1, 2, \ldots, z-1\,, \quad z \in \mathbb{N}\,.$$

where $K_{i|j}$ is the Cartan matrix of the classical Lie algebra $A_{z-1} : SU(z)$.

The Toda field theory

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{z-1} K_{i|j}^{-1}\,\partial\Psi_i\bar{\partial}\Psi_j + \sum_{i=1}^{z-1} e^{\Psi_i} \Longrightarrow \partial\bar{\partial}\Psi_i = \sum_{j=1}^{z-1} K_{i|j} e^{\Psi_j}\,, \quad q = \frac{1}{2}\,(x + i\,y)\,, \quad \partial = \frac{\partial}{\partial q} = \partial_x - i\partial_y\,,$$

The infinite limit – Saveliev (1990)

$$z \to \infty\,, \quad K(z, z') = -\delta''(z - z') \quad \Longrightarrow \quad \boxed{\partial\bar{\partial}\Psi + \partial_z^2\, e^{\Psi} = 0}$$

Methods for solving Toda

Solutions for special cases

1. Separability:

$$e^{\Psi} = c_3 \frac{|\partial f|^2}{(1 - c_3|f|^2)^2} \left(-z^2 + c_1 z + c_2\right) , \qquad f = f(q) ,$$

Maldacena–Núñez (2000):
$$e^{\Psi} = 4 \frac{N^2 - z^2}{(1 - r^2)^2} , \qquad r^2 = x^2 + y^2 ,$$
$$0 \leqslant z \leqslant N , \qquad 0 \leqslant r \leqslant 1 .$$

2. Extra $U(1)$ symmetry:

$$\mathrm{AdS}_7 \times S^4: \qquad e^{\Psi} = \coth^2 \zeta , \qquad r = \sinh^2 \zeta \sin^2 \vartheta , \qquad z = \cosh^2 \zeta \cos^2 \vartheta ,$$

and the Maldacena–Núñez solution.

EXTRA $U(1)$ – ELECTROSTATICS

Extra $U(1)$ symmetry – Ward's transformation: Ward (1990)

$$(r, z, \Psi) \mapsto (\rho, \eta, \Phi) : \qquad \ln r = \partial_\eta \Phi\,, \quad z = \rho \partial_\rho \Phi\,, \quad \rho = r\, e^{\Psi(r,z)/2}\,,$$

where the Toda equation is "replaced" by a Poisson equation

$$\frac{1}{r}\partial_r\left(r\partial_r\Psi\right) + \partial_z^2 e^\Psi = \delta(M_5) \Longrightarrow \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\Phi\right) + \partial_\eta^2\Phi = \frac{\lambda(\eta)\delta(\rho)}{\rho}\,.$$

Boundary condition at $z = 0$: Infinite conducting plane with a charge density $\lambda(\eta)$

$$\Phi(\rho,\eta) = -\frac{1}{2}\int_0^\infty d\eta_1 \lambda(\eta_1) G(\rho,\eta;\eta_1)\,, \qquad \lambda(\eta) = z(\rho = 0, \eta)\,,$$

$$G(\rho,\eta;\eta_1) = \frac{1}{\sqrt{\rho^2 + (\eta-\eta_1)^2}} - \frac{1}{\sqrt{\rho^2 + (\eta+\eta_1)^2}}\,, \quad G(\rho,\eta,\eta_1)\big|_{\eta=0} = 0\,.$$

Maldacena–Núñez

The Toda potential

$$e^{\Psi} = 4\,\frac{N^2 - z^2}{(1 - r^2)^2}\,, \quad z \in [0, N]\,, \quad r \in [0, 1]\,.$$

Ward's transformation $(r, z, \Psi) \mapsto (\rho, \eta, \Phi)$

$$\rho = \frac{2r\sqrt{N^2 - z^2}}{1 - r^2} \geqslant 0\,, \quad \eta = \frac{1 + r^2}{1 - r^2}\,z \geqslant 0\,, \quad \Phi = z - N \tanh^{-1}\frac{z}{N} + \frac{1 + r^2}{1 - r^2}\,z \ln r\,.$$

Its charge density at $\rho = 0$

$$\lambda(\eta) = z(\rho = 0, \eta) = \begin{cases} \eta\,, & 0 \leqslant \eta \leqslant N\,, \\ N\,, & \eta \geqslant N\,. \end{cases}$$

$AdS_7 \times S^4$

The Toda frame

$$e^{\Psi} = \coth^2 \zeta \,, \qquad r = \sinh^2 \zeta \sin^2 \vartheta \,, \qquad z = \cosh^2 \zeta \cos^2 \vartheta \,.$$

Ward's transformation $(r, z, \Psi) \mapsto (\rho, \eta, \Phi)$

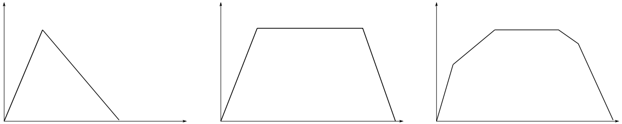
$$\rho = \frac{1}{2} \sinh 2\zeta \sin \vartheta \,, \quad \eta = \frac{1}{2} \cosh 2\zeta \cos \vartheta \,, \quad \Phi = \frac{1}{2} \left(\cos \vartheta \, (1 + \cosh 2\zeta \ln r) + \ln \tan \frac{\vartheta}{2} \right) \,.$$

Its charge density at $\rho = 0$

$$\lambda(\eta) = z(\rho = 0, \eta) = \begin{cases} 2\eta \,, & 0 \leqslant \eta \leqslant \frac{1}{2} \,, \\ \eta + \frac{1}{2} \,, & \eta \geqslant \frac{1}{2} \,. \end{cases}$$

ELECTROSTATICS AND BEYOND

Charge distributions with singularities – irregular punctures



Pathologies: ρ*-small – smearing and* ρ*-large – conical singularities:* $\mathbb{R}^4/\mathbb{Z}_k$.

Goal: Find Toda potentials which are not separable and depend on x, y, z.

Idea: Toda equation appears in Euclidean 4d metrics, i.e. RSD, Kählher and R=0, WSD and Einstein.

ELECTROSTATICS AND BEYOND

Genuine solutions known for four-dimensional instantons with only $SU(2)$ isometry.

Riemann self-dual by Atiyah–Hitchin (1985).
Kähler and $R = 0$ (WASD) by Pedersen–Poon (1990).
Weyl self-dual and Einstein by Tod (1994) & Hitchin (1995).

A toolkit for regular solutions in 11d SUGRA with only $SO(2,4) \times SO(3) \times U(1)_R$ isometry:

1. Singular "Atiyah–Hitchin" metrics – K. Sfetsos, P.M. Petropoulos and K.S. (2013).
2. Pedersen–Poon metrics K. Sfetsos, P.M. Petropoulos and K.S. (2014).

Plan of the talk

LEBRUN METRICS

Kähler metric with symmetry and $R = 0$ – WASD (canonical orientation)

LeBrun (1991)

$$\mathrm{d}\ell^2_{\mathrm{LeBrun}} = V(\mathrm{d}\varphi + A)^2 + V^{-1}(\mathrm{d}z^2 + \mathrm{e}^{\Psi}(\mathrm{d}x^2 + \mathrm{d}y^2)),$$

where the $R = 0$ and Kähler conditions translate to:

$$\begin{cases} \left(\partial_x^2 + \partial_y^2\right)\Psi + \partial_z^2\left(\mathrm{e}^{\Psi}\right) = \mathrm{e}^{\Psi}\nabla^2\Psi = 0\,, \\ \left(\partial_x^2 + \partial_y^2\right)V^{-1} + \partial_z^2\left(V^{-1}\,\mathrm{e}^{\Psi}\right) = 0\,, \\ A = \partial_x V^{-1}\mathrm{d}y \wedge \mathrm{d}z + \partial_y V^{-1}\mathrm{d}z \wedge \mathrm{d}x + \partial_z\left(V^{-1}\,\mathrm{e}^{\Psi}\right)\mathrm{d}x \wedge \mathrm{d}y\,, \end{cases}$$

with Kähler form

$$J = (\mathrm{d}\varphi + A) \wedge \mathrm{d}z - V^{-1}\,\mathrm{e}^{\Psi}\mathrm{d}x \wedge \mathrm{d}y\,, \quad \mathrm{d}J = 0\,.$$

Solutions usually involve electrostatics, i.e. two commuting isometries.

Going beyond using solutions with no commuting isometries.

FOLIATIONS WITH $SU(2)$ ISOMETRY

A four dimensional metric:

$$d\ell^2 = \Omega_1\Omega_2\Omega_3\, dT^2 + \frac{\Omega_2\Omega_3}{\Omega_1}\sigma_1^2 + \frac{\Omega_1\Omega_3}{\Omega_2}\sigma_2^2 + \frac{\Omega_1\Omega_2}{\Omega_3}\sigma_3^2\,,$$

$$\sigma_1 + i\sigma_2 = -e^{i\psi}\left(i\, d\vartheta + \sin\vartheta\, d\varphi\right)\,, \quad \sigma_3 = d\psi + \cos\vartheta\, d\varphi\,, \quad d\sigma_i = \frac{1}{2}\,\varepsilon_{ijk}\sigma_j \wedge \sigma_k\,,$$

$$\psi \in [-2\pi, 2\pi]\,, \qquad \vartheta \in [0,\pi]\,, \qquad \varphi \in [0, 2\pi]\,.$$

The metric is invariant under the action of the Killing (right-invariant) fields

$$\begin{cases} \xi_1 = -\cos\varphi\cot\vartheta\,\partial_\varphi - \sin\varphi\,\partial_\vartheta + \frac{\cos\varphi}{\sin\vartheta}\,\partial_\psi\,, \\ \xi_2 = \sin\varphi\cot\vartheta\,\partial_\varphi - \cos\varphi\,\partial_\vartheta - \frac{\sin\varphi}{\sin\vartheta}\,\partial_\psi\,, \\ \xi_3 = \partial_\varphi \end{cases}$$

where $[\xi_i, \xi_j] = -\varepsilon_{ijk}\xi_k$ and $\nabla_i\xi_j + \nabla_j\xi_i = 0$.

PEDERSEN–POON METRICS

General diagonal Kähler and $R=0$ (Weyl anti-self-dual) metric with $SU(2)$ isometry.

Pedersen–Poon (1990)

$$\mathrm{d}\ell^2 = \Omega_1\Omega_2\Omega_3\,\mathrm{d}T^2 + \frac{\Omega_2\Omega_3}{\Omega_1}\,\sigma_1^2 + \frac{\Omega_1\Omega_3}{\Omega_2}\,\sigma_2^2 + \frac{\Omega_1\Omega_2}{\Omega_3}\,\sigma_3^2\,,$$

$$\Omega_1' = \Omega_2\Omega_3 - \alpha\,\Omega_1\,,\quad \Omega_2' = \Omega_1\Omega_3 - \alpha\,\Omega_2\,,\quad \Omega_3' = \Omega_1\Omega_2\,,\quad f' = \frac{\mathrm{d}f}{\mathrm{d}T}\,,$$

where α is a constant. For $\alpha=0$ – RSD Belinski–Gibbons–Page–Pope metric (1979).

Their LeBrun frame

Tod (1995)

$$z = n_3\,\Omega_3\,,\quad x = e^{\alpha T}\,n_2\,\Omega_2\,,\quad y = e^{\alpha T}\,n_1\,\Omega_1\,,\quad e^{\Psi} = e^{-2\alpha T}\,,\quad n_i = (s_\vartheta\,c_\psi, s_\vartheta\,s_\psi, c_\vartheta)\,,$$

$$V = \frac{\Omega_2\Omega_3}{\Omega_1}\,n_1^2 + \ldots\,,\quad A_i\,\mathrm{d}x^i = V^{-1}\left(\left(\frac{\Omega_1\Omega_3}{\Omega_2} - \frac{\Omega_2\Omega_3}{\Omega_1}\right) s_\vartheta\,s_\psi\,c_\psi\,\mathrm{d}\vartheta + \frac{\Omega_1\Omega_2}{\Omega_3}\,c_\vartheta\,\mathrm{d}\psi\right).$$

APPLICATION IN 11D SUGRA

The axisymmetric solution

$$\Omega_1 = \Omega_2 = \frac{2\alpha\, w}{w^2 - 1}\,, \qquad \Omega_3 = \frac{2\alpha\, w^2}{w^2 - 1}\,, \qquad w = \alpha^{-1}\, e^{-\alpha\, T}\,.$$

The line charge density is regular if:

$$\exists\, t_1 : \qquad \Omega_1(t_1) = 0\,, \qquad \Omega_3(t_1) = 2\alpha\,, \qquad \boxed{\alpha \in \mathbb{N}^*}\,.$$

The generic non-$U(1)$ solution

$$\Omega_1 = \alpha w \cosh G\,, \quad \Omega_2 = \alpha w \sinh G\,, \quad \Omega_3 = -\alpha w \frac{\mathrm{d}G}{\mathrm{d}w}\,,$$

$$\text{where}: \quad \frac{\mathrm{d}}{\mathrm{d}w}\left(w\,\frac{\mathrm{d}G}{\mathrm{d}w}\right) = \frac{w}{2}\sinh(2G)\,.$$

1. It is regular at $z = 0$, i.e. $\vartheta = \pi/2$ or $\Omega_3(w_*) = 0$.
2. There is a regular puncture at $w = 0$ & $\vartheta = 0$ if: $\boxed{N_5 = \frac{\alpha\zeta}{2}\,, \quad 0 \leqslant \zeta < 1}$

Regularity demands in agreement: On the non-$U(1)$ solution and on the line charge density.

PLAN OF THE TALK

1. GRAVITY DUALS OF $\mathcal{N} = 2$ SCFTS
2. THE CONTINUAL TODA EQUATION
3. CONSTRUCTION OF A NON-$U(1)$ SOLUTION
4. EUCLIDEAN HOLOGRAPHY
5. DISCUSSION & OUTLOOK

PRZANOWSKI–TOD METRICS

W(A)SD and Einstein metrics with a symmetry – Przanowski (1991) – Tod (1997).

Conformal rescaling of LeBrun metrics:

$$\mathrm{d}\ell^2_{\mathrm{PT}} = \frac{1}{z^2}\,\mathrm{d}\ell^2_{\mathrm{LeBrun}} = \frac{1}{z^2}\left(V(\mathrm{d}\varphi + A)^2 + V^{-1}(\mathrm{d}z^2 + \mathrm{e}^{\Psi}(\mathrm{d}x^2 + \mathrm{d}y^2))\right)\,,$$

where the upliftability condition reads:

$$2k^2 V = 2 - z\partial_z\Psi\,,$$

with

$$R_{\mu\nu} = -3k^2 g_{\mu\nu}\,, \qquad \widehat{W}_2 = \star_4 \widehat{W}_2\,, \qquad \widehat{W}_2 = \mathcal{R}_2 + k^2 e\wedge e\,.$$

Additional $U(1)$ isometry – electrostatic potential – Calderbank–Pedersen (2001)

$$F = F(\rho,\eta) = \sqrt{\rho}\,\partial_\rho\Phi : \qquad \rho^2\left(F_{\rho\rho} + F_{\eta\eta}\right) = \frac{3}{4}F\,.$$

For example $\sqrt{\rho}F = \sqrt{\rho^2+\eta^2} - k^2$ yields H_4.

PRZANOWSKI–TOD AND HOLOGRAPHY

Stationary solutions admit (locally at least) two commuting Killing isometries:

$$\mathrm{d}\ell^2 = \frac{1}{z^2}\left(V(\mathrm{d}\varphi + A)^2 + V^{-1}(\mathrm{d}z^2 + \mathrm{e}^{\Psi}(\mathrm{d}r^2 + r^2\,\mathrm{d}\beta^2))\right)\,, \quad (V, A, \Psi) = \text{depend on } (r, z)\,.$$

Double pole at $z = 0$ – Potential conformal boundary – Fefferman–Graham (FG) expansion near $z = 0$.

The Gaussian normal coordinates (R, ζ) centred at $z = 0$:

$$z(R,\zeta) = \frac{1}{R}\left(1 + \frac{a(\zeta)}{R} + \frac{b(\zeta)}{R^2} + \frac{c(\zeta)}{R^3}\right) + \dots\,, \quad \rho(R,\zeta) = \zeta + \frac{e(\zeta)}{R} + \frac{f(\zeta)}{R^2} + + \frac{g(\zeta)}{R^3} + \dots\,.$$

Recasting the metric in the FG gauge:

$$\mathrm{d}s^2 = \left(\theta^R\right)^2 + \delta_{\mu\nu}\theta^\mu\theta^\nu\,, \quad \theta^R = N\,\frac{\mathrm{d}R}{kR}\,, \quad \theta^\mu = N^\mu \mathrm{d}R + \tilde{\theta}^\mu\,, \quad \mu = (\zeta, \varphi, \beta)\,,$$
$$N = 1\,, \qquad N^\zeta = 0\,, \qquad A = A_\varphi(z,\rho)\mathrm{d}\varphi\,,$$

yields $a, b, c, e \dots$, regularity conditions and validity constraints for the FG gauge at $z = 0$:

$$V = {}^1\!/_{k^2}\,, \quad z\partial_z\Psi = 0\,, \quad \dots\,, \quad \text{and} \quad \partial_z\left(V^2\mathrm{e}^{\Psi}\right) = 0\,, \quad \partial_z A_\varphi = 0\,, \quad \dots$$

PRZANOWSKI–TOD AND HOLOGRAPHY

The boundary is odd-dimensional and there are no anomalies:

$$g = \delta_{\mu\nu}\,\frac{\tilde{\theta}^\mu}{kR}\,\frac{\tilde{\theta}^\nu}{kR} = g_{(0)} + \cancelto{0}{\frac{1}{kR}\,g_{(1)}} + + \frac{1}{k^2R^2}\,g_{(2)} + \frac{1}{k^3R^3}\,g_{(3)} + \ldots\,, \qquad g_{(0)} = {}^1\!/_{k^2R^2}\,\mathrm{d}s^2\big|_{\partial\mathcal{M}}$$

Boundary data:

- Boundary metric:

$$g_{(0)} = (\mathrm{d}\chi + a_\beta\,\mathrm{d}\beta)^2 + \frac{e^\psi}{k^4}\left(\mathrm{d}r^2 + r^2\mathrm{d}\beta^2\right)\,, \quad a_\beta := A_\beta(r,z)\big|_{z=0}\,, \quad \psi := \Psi(r,z)\big|_{z=0}$$

- Stress–energy momentum tensor:

$$T_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{3k}{16\pi G_N}\,g_{(3)}\,, \quad g_{(3)} = \#k^3\left(-2(\mathrm{d}\chi + a_\beta\,\mathrm{d}\beta)^2 + \frac{e^\psi}{k^4}\left(\mathrm{d}r^2 + r^2\mathrm{d}\beta^2\right)\right)\,.$$

  It is traceless and divergenceless; provided $\# = \text{constant} = -{}^{16M}\!/_{k^2}$.

- Integrability condition: Leigh, Petkou 07; de Haro 08; Mansi et al 08; Miskovic–Olea 09

$$\boxed{\left(\pm C_{\mu\nu} - 8\pi\,G_N\,k^2\,T_{\mu\nu}\right)\mathrm{d}x^\mu\mathrm{d}x^\nu = 0}$$

Quaternionic Taub–NUT

The metric reads:

$$\mathrm{d}s^2_{\text{bulk}} = \frac{\mathrm{d}\xi^2}{V(\xi)} + V(\xi)\left(\mathrm{d}\varphi + 4n\sin^2 \vartheta/2\,\mathrm{d}\beta\right)^2 + (\xi^2 - n^2)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\beta^2\right),$$

$$V(\xi) = \frac{\xi - n}{\xi + n}\left(1 + k^2(\xi - n)(\xi + 3n)\right).$$

In Przanowski–Tod coordinates:

$$z = \frac{1}{\xi - n}, \quad r = \cot\vartheta/2, \quad A = 4n\sin^2\vartheta/2\,\mathrm{d}\beta,$$

$$V = \frac{1 + 2nz}{k^2 + 4nk^2 z + z^2}, \qquad \mathrm{e}^{\Psi} = 4\,\frac{k^2 + 4nk^2 z + z^2}{(1 + r^2)^2}.$$

The boundary data are built from:

$$\mathrm{e}^{\psi} = \frac{4k^2}{(1 + r^2)^2}, \qquad a_\beta = \frac{4n}{r^2 + 1}, \qquad M = n(1 - 4k^2 n^2),$$

the latter ensures the anti-self-duality and the regularity; nut at $\xi = n$ – locally R^4 in polar coordinates.

8D/7D HOLOGRAPHY

Self-duality in 8 Euclidean dimensions – octonions $f_{ab}{}^{cd}$.

1. Riemann two-form: $\mathcal{R}_2 \in \mathbf{28}$ of $SO(8)$.
2. $\mathbf{28} = \mathbf{21} \oplus \mathbf{7}$ under $Spin(7) \subset SO(8)$.
3. Self-duality and anti-self-duality corresponds to $P_{\mathbf{21}}\,\mathcal{R}_2 = 0$ or $P_{\mathbf{7}}\,\mathcal{R}_2 = 0$, where: B. De Wit–H. Nicolai (1984)

$$P_{\mathbf{7}} = \frac{1}{4}\left(\delta_{ab}{}^{cd} - \frac{1}{2} f_{ab}{}^{cd}\right), \quad P_{\mathbf{21}} = \frac{3}{4}\left(\delta_{ab}{}^{cd} + \frac{1}{6} f_{ab}{}^{cd}\right),$$

$$P_{\mathbf{7}}^2 = P_{\mathbf{7}}\,, \quad P_{\mathbf{21}}^2 = P_{\mathbf{21}}\,, \quad P_{\mathbf{7}} + P_{\mathbf{21}} = \delta_{ab}{}^{cd}\,, \quad P_{\mathbf{7}}\,P_{\mathbf{21}} = P_{\mathbf{21}}\,P_{\mathbf{7}} = 0.$$

4. In 8-dim the Riemann tensor splits to $\mathbf{336} = \mathbf{300} \oplus \mathbf{35} \oplus \mathbf{1}$ as:

$$\mathcal{S}_{\mathbf{21}} = P_{\mathbf{21}}\,\mathcal{R}_2 = W^{\mathbf{168}}\phi_{\mathbf{21}} + s^{\mathbf{1}}\phi_{\mathbf{21}} + W^{\mathbf{105}}\chi_{\mathbf{7}}\,, \qquad \mathcal{A}_{\mathbf{7}} = P_{\mathbf{7}}\,\mathcal{R}_2 = W^{\mathbf{27}}\chi_{\mathbf{7}} + s^{\mathbf{1}}\chi_{\mathbf{7}} + S^{\mathbf{35}}\phi_{\mathbf{21}}\,.$$

5. "Quaternionic" spaces:

$$\begin{cases} S^{\mathbf{35}} = 0,\ s^{\mathbf{1}} \neq 0, & \text{Einstein} \\ W^{\mathbf{27}} = 0, & \text{Weyl self-dual} \end{cases}$$

6. Seven-dimensional boundary:

$$T_{\mu\nu} := \frac{\delta S^{\mathrm{CS}}_{7-\mathrm{dim}}}{\delta g^{\mu\nu}}\,,$$

integrability and generalized filling-in problem.

Plan of the talk

1. Gravity duals of $\mathcal{N}=2$ SCFTs
2. The continual Toda equation
3. Construction of a non-$U(1)$ solution
4. Euclidean holography
5. Discussion & Outlook

Discussion & Outlook

Use of the Toda potential for studying:

- 11d supergravity solutions, which are regular and have no smearing.
- Toda frame thought Kähler metrics with $SU(2)$ isometry and $R = 0$.
- Emergence of an extra $U(1)$ symmetry – in agreement with electrostatic description.
- Quaternionic 4d-spaces with a symmetry and Euclidean holography.
- Going beyond the integrability condition.

Extensions to 8d/7d holography.

Appendix: $\mathcal{N} = 2$ SCFTs

A class of 4d $\mathcal{N} = 2$ SCFTs can be viewed:

- As generalised quivers (elementary fields and strongly coupled field theories as building blocks).

  Gaiotto (2009)

- By taking the IR limit of N M5 branes wrapping a two dimensional Riemann surface Σ_2.

  Maldacena–Nũnéz (2001)

  - The geometry contains AdS_5 and S^4 fibered over Σ_2.
  - The fibration involves a twist preserving four dimensional $\mathcal{N} = 2$ SUSY.
  - Σ_2 may be sphere or torus or higher genus surface...
  - Σ_2 has constant curvature and it is a quotient of hyperbolic space.
  - Additional non-compact branes may intersect Σ_2 at points–punctures; z_c.

Maldacena–Nũnéz geometry

- M5 theory ($N \gg 1$) on Σ_2 flowing from UV to IR.
- At UV the geometry $\simeq AdS_7 \times S^4 \supseteq \mathbb{R}^4 \times \Sigma_2$, S^4 wrapped on $\Sigma_2 \Longrightarrow$ 8 supercharges.

Appendix: Comments on electrostatics

Line charge density – related to the M5 sources

Gaiotto–Maldacena, Reid-Edwards–Stefanski (2011), Donos–Simon (2011) & Aharony–Berdichevsky–Berkooz (2012)

- Extra $U(1)$ isometry endows a smearing process with the typical validity limitations.
- Regularity of spacetime imposes constraints on $\lambda(\eta)$, arising from 4-flux quantisation on punctures.
  - $\lambda(\eta)$ is continuous and piecewise segment, i.e. $a_n\eta + q_n$, where $a_n \in \mathbb{Z}$.
  - Kinks occur at integer values of η.
  - $\lambda(0) = 0$ and $a_{n-1} - a_n = k_n \in \mathbb{Z}_+$
  - Metric singularity around that point $\quad \simeq \quad AdS_5 \times S^2 \times \mathbb{R}^4/\mathbb{Z}_{k_n}$

Region of validity: $\rho_{\rm sm} \ll \rho \ll \rho_{U(1)}$. An exception is the Maldacena–Núñez solution.

A class of 4d $\mathcal{N} = 2$ SCFTs can be viewed as generalised quiver gauge theories

Gaiotto (2009)

- $\exists\, SU(\lambda_n)$ gauge group $\forall \lambda_n = \lambda(\eta)_{\eta=n} : \lambda_n \leqslant \lambda_{n+1} \leqslant N \,\&\, k_n := 2\lambda_n - \lambda_{n-1} - \lambda_{n+1} \geqslant 0\,.$
- $\forall\, \mathrm{Kink}_{\eta=\mathrm{n}}$, $\exists k_n = a_{n-1} - a_n$ fundamental hypermultiplets charged under the $SU(\lambda_n)$.
- In total, this is a quiver with gauge group $\prod_n SU(n)$ described at strong coupling by supergravity.

APPENDIX: ATIYAH–HITCHIN AND 11D SUGRA

A more complicated solution – K. Sfetsos, P.M. Petropoulos and K.S. (2013)

1. Riemann self-duality on foliations of Bianchi-IX.
2. Lagrange or Darboux–Halphen 1$^{\text{st}}$-order differential systems.

$$\Omega_1' = \Omega_2\Omega_3 - \lambda\,\Omega_1(\Omega_2 + \Omega_3)\,, \qquad \lambda = 0, 1.$$

3. Darboux–Halphen possesses an $SL(2,\mathbb{R})$ covariance.
4. Translational or Rotational Killing vectors.
5. Darboux–Halphen – rotational Killing vectors – continual Toda equation – LeBrun $V^{-1} = \partial_z\Psi$.
6. Regularity of the 11d metric yields irregular 4d metrics – Kretschmann scalar $\sim (\partial_z\Psi)^{-6}$.
7. Halphen (Atiyah–Hitchin) is a non-singular solution of the Darboux–Halphen $SL(2,\mathbb{R})$ family.
8. We satisfy the b.c. for 11d regularity by transforming the Atiyah–Hitchin under this $SL(2,\mathbb{R})$.
9. There is no axisymmetric limit – electrostatics description is inapplicable.