Entanglement Entropy and Duality in AdS_4

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- AdS/CFT correspondence could suggest that these underlying degrees of freedom are the conformal field theory degrees of freedom.
- More recently, it has been suggested that the relation between gravity and thermodynamics should not be attributed to thermal statistics, but rather to quantum statistics related to quantum entanglement physics.
- Enforcement of entanglement thermodynamics first law at linear order is equivalent to Einstein's equations (for Dirichlet boundary conditions).

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- We study the duality in the bulk from the point of view of entanglement physics, as it may provide new light to its understanding
- Such study can act as a consistency benchmark for the RT conjecture

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Entanglement between systems A and A^C is encoded to the spectrum of the reduced density matrix.
We define the entanglement entropy as the von Neumann entropy on the reduced density matrix ρ_A,

$$S := -\mathrm{Tr} \rho_A \ln \rho_A.$$

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If we assume a variation in the pure state of the overall system, the variation of the entanglement entropy is given by

$$\delta S_A = \delta \langle H_A \rangle .$$

This is the direct analog of the first law of thermodynamics for entanglement physics.

Ryu-Takayanagi conjecture connects the entanglement entropy of a region A defined by the entangling surface ∂A in the boundary field theory to the area of an extremal co-dimension two open surface in the bulk gravitational dual theory with boundary ∂A.

$$S_A = \frac{\mathrm{Area}\left(A^{\mathrm{extr}}\right)}{4G_N}.$$

If region A is defined as the polar cap

$$t = t_0, \; \theta \leq \theta_0,$$

then the extremal surface is given by

$$t = t_0, \; \theta(r) = \arccos\left(\cos\theta_0 \sqrt{1 + \frac{1}{r^2}}\right), \quad \rho \in [\cot\theta_0, \infty), \; \varphi \in [0, 2\pi),$$

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RT conjecture yields

$$A = 2\pi \lim_{r\to\infty}(r\sin\theta_0 - 1).$$

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Varying the metric one finds

$$\delta A = \left.\frac{\delta A\left(g, X\right)}{\delta g}\right|_{g=g_0, X=X_0} \delta g + \left.\frac{\partial A\left(g, b\right)}{\partial b}\right|_{g=g_0, b=b_0} \delta b := \delta A_g + \delta A_b.$$

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The term occuring from the variation of the surface equations is vanishing as a consequence of the extremality of the unperturbed surface.

The first term can be calculated as

$$\delta A_g = \frac{1}{2} \int d^2\sigma \sqrt{\gamma_0} (\gamma_0)^{ab} \delta\gamma_{ab},$$

where γ_{ab} is the induced metric on the unperturbed extremal surface.

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The second term can be calculated by varying the unperturbed result with respect to θ_0.

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In some cases, the modular Hamiltonian generates a geometric flow and then it can admit a local expression.
A known example is that of the region A being a disk of radius R in a Minkowski boundary.
In such cases, the modular Hamiltonian can be expressed in terms of the holographic energy-momentum tensor and a conformal Killing vector that leaves invariant the entangling surface ∂A and its causal development,

$$\delta E = \int_C d\Sigma^\mu T_{\mu\nu} \zeta^\nu ,$$

where C is any spacelike surface with boundary ∂A.

Our case can be connected with that of a disk in Minkowski through a coordinate transformation.

$$\zeta = \frac{2}{\sin\theta_0}\left[\left(\cos\left(t-t_0\right)\cos\theta-\cos\theta_0\right)\partial_t-\sin\left(t-t_0\right)\sin\theta\partial_\theta\right].$$

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Selecting C to coincide with A,

$$\delta E = \frac{\pi}{\sin\theta_0}\int_0^{\theta_0} d\theta \sin\theta\left(\cos\theta - \cos\theta_0\right) T_{tt}.$$

We consider linear metric perturbations around AdS_4 background

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$f(r) = 1 - \frac{\Lambda}{3}r^2.$$

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There are two classes of perturbations

- Axial perturbations

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + 2e^{-i\omega t}\sin\theta \frac{dP_l(\cos\theta)}{d\theta}\left(h_0(r)\,dt + h_1(r)\,dr\right)d\varphi.$$

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- Polar perturbations

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + e^{-i\omega t}P_l(\cos\theta)\Big[h_0(r)\left(f(r)\,dt^2 + \frac{dr^2}{f(r)}\right) + 2h_1(r)\,dtdr + K(r)\,r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\Big].$$

For both classes Einstein's equations become equivalent to the same effective Schrödinger problem with respect to the tortoise coordinate,

$$-\frac{d^2\Psi(x)}{dx^2}+\frac{l(l+1)}{\sin^2 x}\Psi(x)=\omega^2\Psi(x).$$

All functions of r in the metric can be expressed in terms of the solutions of the effective Schrödinger problem.

An asymptotic expansion of Ψ yields

$$\Psi = I_0 + \frac{I_1}{r} + \frac{I_2}{r^2} + \frac{I_3}{r^3} + \ldots$$

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Only two parameters are independent.
$l_0 = 0$ corresponds to Dirichlet boundary conditions.
$l_0 = 1$ corresponds to Neumann boundary conditions.
If both l_0 and l_1 are non-vanishing the solution obeys more general mixed boundary conditions.

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- Axial perturbations

Dirichlet conditions for Ψ correspond to Dirichlet conditions for the metric

- Polar perturbations

Dirichlet conditions for Ψ correspond to Neumann conditions for the metric

One can calculate the holographic energy-momentum and Cotton tensors for both kinds of perturbations to discover that

$$T_{ab}^{\mathrm{polar}} = C_{ab}^{\mathrm{axial}},$$
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if $\Psi_{\mathrm{polar}} = -\frac{2i}{\omega}\Psi_{\mathrm{axial}}$.

The above duality can be better understood in terms of the Weyl tensor. If one defines the dual Weyl tensor like

$$\tilde{C}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\kappa\lambda} C_{\kappa\lambda\rho\sigma} ,$$

then a dual metric can be defined as,

$$\tilde{C}_{\mu\nu\rho\sigma}\left(g\right) = C_{\mu\nu\rho\sigma}\left(\tilde{g}\right) .$$

At linear level, the classification of the perturbations to axial and polar modes resolves the highly non-trivial, non-local duality relations,

$$\tilde{g}^{\mathrm{polar}} = g^{\mathrm{axial}}, \qquad \tilde{\Psi}_{\mathrm{polar}} = -\frac{2i}{\omega}\Psi_{\mathrm{axial}},$$
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In both cases the relation between $\tilde{\Psi}$ and Ψ enforces that dual perturbations are corresponding to the same solution of the identical effective Schrödinger problems and thus, they are characterised by identical frequency and boundary conditions for $\Psi(x)$.

In four dimensions, the energy-momentum and Cotton tensors are given by appropriate elements of the Weyl tensor like

$$T_{ab} = - \lim_{r \to \infty} r^3 C_{arbr} ,$$
$$C_{ab} = \lim_{r \to \infty} r^3 \tilde{C}_{arbr} ,$$

giving the interpretation of the relation between the holographic energy-momentum and Cotton tensor as an electric-magnetic duality with respect to the radial ADM decomposition of the Weyl tensor.

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- Polar perturbations

$$\delta A_g = -2\pi e^{-i\omega t}\left[\lim_{r\to\infty}\left[rJ_1 + J_0\left(l\left(l+1\right)-\omega^2\right)\right]\sin\theta_0 P_l\left(\cos\theta_0\right)\right.$$
$$\left. + \cot\theta_0 \frac{l\left(l+1\right)}{2l+1} J_0\left(P_{l+1}\left(\cos\theta_0\right) - P_{l-1}\left(\cos\theta_0\right)\right)\right].$$

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- Axial perturbations

$$\theta_0(t) = \theta_0$$

- Polar perturbations

$$\theta_0(t) = \theta_0 + \left(J_1 + \frac{J_0 \left(l(l+1) - 2\omega^2 \right)}{2r} \right) e^{-i\omega t} \tan\theta_0 P_l(\cos\theta_0) .$$

Calculation of δA_b

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- Polar perturbations

$$\delta A_b = 2\pi \lim_{r\to\infty} \left[J_1 r + J_0 \left(\frac{l(l+1)}{2} - \omega^2 \right) \right] e^{-i\omega t} \sin\theta_0 P_l(\cos\theta_0) \, .$$

Calculation of δS

- Axial perturbations

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- Polar perturbations

$$\delta S = -2\pi \frac{l\left(l+1\right)}{2l+1} J_0 e^{-i\omega t_0} \cot\theta_0 \left(P_{l+1}\left(\cos\theta_0\right) - P_{l-1}\left(\cos\theta_0\right)\right) - \pi l\left(l+1\right) J_0 e^{-i\omega t_0} \sin\theta_0 P_l\left(\cos\theta_0\right).$$

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The result is finite for both kinds of perturbations

Calculation of δE

- Axial perturbations

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- Polar perturbations

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In all cases we find

$$\delta S = \delta E,$$

which is the first law of thermodynamics for entanglement. This is shown for general boundary conditions.

We define the dual entanglement entropy as

$$\tilde{S}_A = \frac{\mathrm{Area}\left(\tilde{A}^{\mathrm{extr}}\right)}{4G_N},$$

where $\tilde{A}^{\mathrm{extr}}$ is the extremal surface with respect to the dual metric $\tilde{g}$ ($\tilde{C}_{\mu\nu\rho\sigma}(g) = C_{\mu\nu\rho\sigma}(\tilde{g})$)

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Since the unperturbed AdS_4 space is self dual $\tilde{g}^{AdS} = g^{AdS}$, the unperturbed dual extremal surface is identical to the unperturbed extremal surface.
Thus, the variation of the dual entanglement entropy for axial and polar perturbations can be calculated in the same way as the variation of entanglement entropy.

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$$\delta \tilde{E}^{\mathrm{axial}} = \delta E^{\mathrm{polar}},$$
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Since it is true that $\delta S = \delta E$ for each kind of perturbations, it is also true that

$$\delta \tilde{S} = \delta \tilde{E}$$

for each kind of perturbations.

- We calculated the variations of the entanglement entropy and the expectation value of the modular Hamiltonian for linear metric perturbations in AdS_4 background and general boundary conditions.

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- We show that validity of the entanglement thermodynamics first law demands an isoperimetric time evolution for the entangling curve.
- It would be interesting to study this kind of isoperimetric deformations of the entangling curve in the presence of metric perturbations which do not preserve the entangling curve symmetry, the axial symmetry in our case.

- The electric-magnetic duality can be implemented introducing a dual entanglement entropy and a dual modular Hamiltonian, such that a dual Ruy-Takayanagi conjuncture and a dual entanglement thermodynamics first law hold.

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- This is a positive consistency check for the validity of RT conjecture.
- It would be interesting to check whether such constructions can be achieved in other backgrounds, for example in asymptotically AdS_4 black holes.

- As a final comment, it would be very interesting to better understand the connection between the dual entanglement entropy and modular Hamiltonian and the reduced density matrix of the boundary conformal field theory.

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- Notice that the electric magnetic duality interchanges boundary conditions for the metric, thus, the time evolution of the entangling curve and thus the region *A* is in general different for the initial and dual CFTs.

Thank you very much