Torsional Newton-Cartan geometry in Lifshitz holography and non-relativistic FTs

Aspects of fluid/gravity correspondence

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based on work with:

Jelle Hartong and Elias Kiritsis

1409.1519 [1] & 1409.1522 [2] & 1502.00228 [3] & to appear

and

Morten Holm Christensen, Jelle Hartong, Blaise Rollier

1311.4794 (PRD) & 1311.6471 (JHEP)

Introduction

- holography beyond original AdS-setup

- apply to study of strongly coupled CM systems
non-relativistic scaling -> Schroedinger, Lifshitz, hyperscaling violating geometries

- how general is the holographic paradigm ?
(nature of quantum gravity, black hole physics)

- appearance of novel geometric structures on the boundary (this talk: TNC)

- exotic theories of gravity can be viewed as Schwinger source functionals
of non-rel QFTs (``metric'' couples to stress tensor)
symmetries of FT -> symmetries of the coupled grav. theory and
constrain form of source functionals

This talk: direct implementation of this in class of examples characterized
by Lifshitz scaling symmetry and extended Schroedinger sym.
\+ holographic realization within context of bulk Lifshitz spacetime

Lifshitz symmetries

Many systems in nature exhibit critical points with non-relativistic scale invariance

Includes in particular scale invariance with dynamical exponent z>1

$$t \to \lambda^z t\,, \qquad \vec{x} \to \lambda \vec{x}$$

Such systems typically have Lifshitz symmetries:

$$\begin{aligned} D_z &: \quad \vec{x} \to \lambda \vec{x} \qquad t \to \lambda^z t\,, \\ H &: \quad t \to t + a\,, \\ P_i &: \quad x^i \to x^i + a^i\,, \\ J_{ij} &: \quad x^i \to R^i{}_j x^j\,. \end{aligned}$$

Lifshitz algebra (non-zero commutators, not involving rotations)

$$[D_z, H] = -zH\,, \qquad [D_z, P_i] = -P_i\,.$$

Schroedinger symmetries

example of symmetry group that also displays non-relativistic scaling and contains Lifshitz is Schroedinger group

additional symmetries:

Galilean boosts $G_i \ (x^i \to x^i + v^i t)$

particle number symmetry N

Schroedinger algebra

$$
\begin{aligned}
[D_z, H] &= -zH\,, & [D_z, P_i] &= -P_i\,, & [D_z, N] &= (z-2)N \\
[D_z, G_i] &= (z-1)G_i\,, & [H, G_i] &= P_i\,, & [P_i, G_j] &= N\delta_{ij}
\end{aligned}
$$

for z=2: additional special conformal generator K

Lifshitz spacetimes

Aim: construct holographic techniques for (strongly coupled) systems with NR symmetries

Lifshitz holography $$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dr^2 + d\vec{x}^2\right)$$

[Kachru,Liu,Mulligan]
[Taylor]

Taylor/Danielson,Thorlacius/Ross,Saremi/Ross/
Baggio,de Boer,Holsheimer/Mann,McNees/
Griffin,Horava,Melby-Thompson/
Korovin,Skenderis,Taylor/
Cheng,Hartnoll,Keeler/Baggio/Holsheimer/
Christensen,Hartong,NO,Rollier
Chemissany, Papadimitriou
Hartong,Kiritsis,NO

Some intuitions/expectations

- holography for bulk spacetimes with non-relativistic scaling
-> some type of non-relativistic geometry on the boundary
(Newton-Cartan or a generalization thereof)

besides energy momentum tensor, non-rel. theory also has a mass current T^μ
-> natural that there is an extra source coupling to it M_μ

- Newton-potential should enter the story

- expect Lifshitz sym (at the least, will see that there can be more)

Mini-intro to Newton-Cartan geometry

GR is a diff invariant theory whose tangent space is Poincare invariant

Newtonian gravity is diff invariant theory whose tangent space is the Bargmann algebra (non-rel limit of Poincare)

Andringa,Bergshoeff,Panda,de Roo

$$[J_{ab}, P_c] = \delta_{ac} P_b - \delta_{bc} P_a \, ,$$
$$[J_{ab}, J_{cd}] = \delta_{ac} J_{bd} - \delta_{ad} J_{bc} - \delta_{bc} J_{ad} + \delta_{bd} J_{ac} \, .$$

centrally extended Galilean algebra

$$[H, G_a] = P_a \qquad [P_a, G_b] = \delta_{ab} N \, ,$$
$$[J_{ab}, G_c] = \delta_{ac} G_b - \delta_{bc} G_a \, ,$$

here: only interested in geometrical framework; not in EOMs
boundary geometry in holographic setup is non-dynamical

From Poincare to GR

GR is a diff invariant theory whose tangent space is Poincare invariant

- make Poincare local (i.e. gauge the translations and rotations)

$$
\begin{aligned}
A_\mu &= P_a e^a_\mu + \frac{1}{2} J_{ab} \omega_\mu{}^{ab} \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu \, , A_\nu] = P_a R_{\mu\nu}{}^a(P) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J) \\
\delta A_\mu &= \partial_\mu \Lambda + [A_\mu \, , \Lambda] \, , \qquad \Lambda = \xi^\mu A_\mu + \frac{1}{2} J_{ab} \lambda^{ab}
\end{aligned}
$$

GR (Lorentzian geometry) follows from curvature constraint

$$
R_{\mu\nu}{}^a(P) = 0 \begin{cases} \omega_\mu{}^{ab} = \text{spin connection: expr. in terms of } e^a_\mu \\ \delta A_\mu = \mathcal{L}_\xi A_\mu + \frac{1}{2} J_{ab} \partial_\mu \lambda^{ab} + \frac{1}{2} [A_\mu \, , J_{ab}] \lambda^{ab} \\ R_{\mu\nu}{}^{ab}(J) = \text{Riemann curvature 2-form} \\ \nabla_\mu \ \text{ defined via vielbein postulate} \end{cases}
$$

From Bargmann to NC

Andringa,Bergshoeff,Panda,de Roo

Newtonian gravity is a diff invariant theory whose tangent space is Bargmann (make Bargmann local)

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$R_{\mu\nu}(H)$
space translations	P_a	$e_\mu{}^a$	$\zeta^a(x^\nu)$	$R_{\mu\nu}{}^a(P)$
boosts	G_a	$\omega_\mu{}^a$	$\lambda^a(x^\nu)$	$R_{\mu\nu}{}^a(G)$
spatial rotations	J_{ab}	$\omega_\mu{}^{ab}$	$\lambda^{ab}(x^\nu)$	$R_{\mu\nu}{}^{ab}(J)$
central charge transf.	N	m_μ	$\sigma(x^\nu)$	$R_{\mu\nu}(N)$

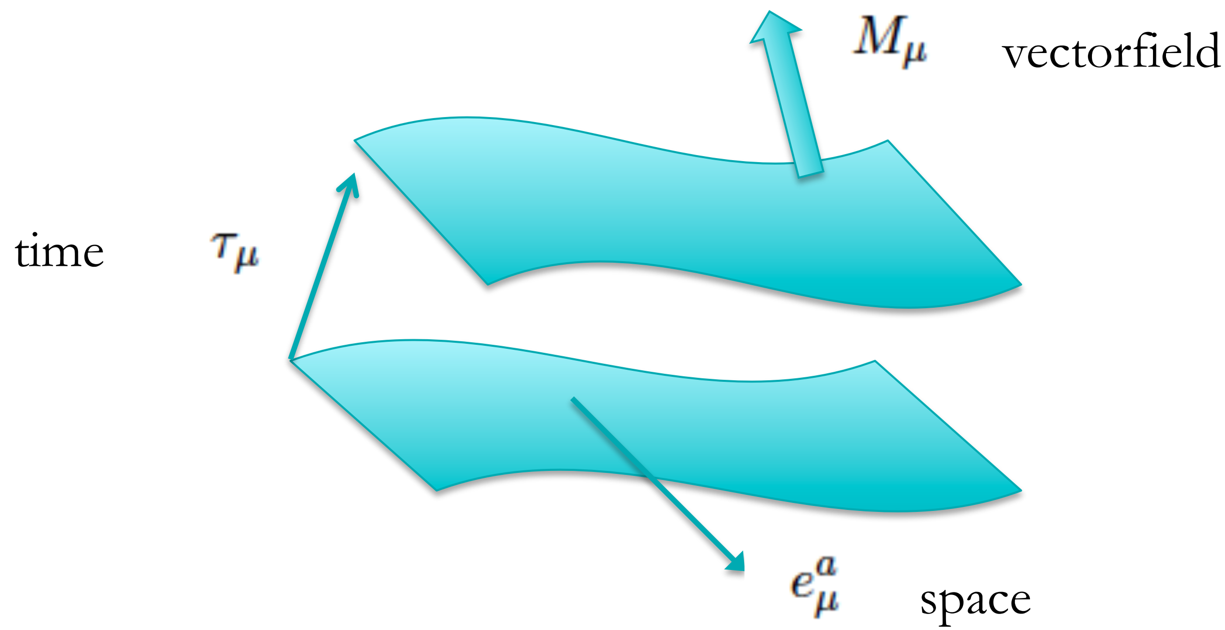
curvature constraints $R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = R_{\mu\nu}(N) = 0.$

leaves as independent fields: $\tau_\mu,\ e_\mu^a,\ m_\mu$

transforming as

$$\begin{aligned}\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu \\ \delta e_\mu^a &= \mathcal{L}_\xi e_\mu^a + \lambda^a \tau_\mu + \lambda^a{}_b e_\mu^b \\ \delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e_\mu^a\end{aligned}$$

Newton-Cartan geometry



- Newton-Cartan (NC): $\tau_\mu = \partial_\mu t$ notion of absolute time
- twistless torsional NC: $\tau_\mu = \mathrm{HSO}$ preferred foliation in equal time slices
- torsional NC (TNC): no conditions

(more on Christoffel connection and geodesics later)

Overview of recent background

- bdry geometry for Lifshitz spacetimes is torsional Newton-Cartan geometry
  (novel extension of NC)

- first observed for specific z=2 example (in 4D): [Christensen,Hartong,NO,Rollier]
  * Scherk-Schwarz dim. reduction (null on bdry) from 5D AlAdS solution

- generalized to large class of arbitrary z (in EPD model) [Hartong,Kiritsis,NO]1

  sources: use vielbein formalism + appropriate lin. combo
  coupling of geometry to bdry -> vevs (stress tensor, mass current)
  & WIs in TNC covariant form

- TNC geometry arises by gauging the Schroedinger algebra [Bergshoeff,Hartong,Rosseel]

- coupling of TNC to non-rel FTs also considered directly [Jensen/Jensen,Karch]

- recent activity using NC/TNC in CM
  (strongly-correlated electron system, FQH)
  [Son][Gromov,Abanov][Geracie,Son,Wu,Wu]
  [Brauner,Endlich,Monin,Penco][Geracie,Son]
  [Wu,Wu],[Geracie,Golkar,Roberts]

Symmetries: from Lifshitz to Schroedinger

- holography for Lifshitz spacetimes: Schroedinger symmetry acting on sources
  -> strongly suggest that bdry theory can have Schroedinger invariance

[Hartong,Kiritsis,NO]1,2,3

Main overall points of this talk:

- appearance of global symmetries in non-relativistic field theories exhibits a new mechanism:
  * interplay between conserved currents and space-time isometries is different compared to relativistic case

- supported by considering the Lifshitz vacuum:
  = holographic dual of flat NC spacetime

-> Lifshitz holography dual to field theories on TNC spacetime

Plan & preview

1. holography for Lifshitz spacetimes and TNC geometry

   time-like vielbein τ_μ, space-like vielbeins e^a_μ and a vector field M_μ,

2. scale invariant field theories on TNC backgrounds
   * the vector field can make a global U(1) into local sym.

3. flat NC spacetime
   * comes with function M (in $M_\mu = \partial_\mu M$)
   * local symmetries can generate non-trivial orbit of equivalent M

4. scale-invariant field theories on flat NC
   * novel mechanism: M can be eaten up by physical fields generating extra global symmetries (e.g. Galilean boost) beyond Lif (-> Sch.)

5 . Lifsthiz vacuum
   * exhibits source M transforming under local Sch (Lif realized by Killing)
   * scalar probes on Lif bgr that are Sch invariant by similar mechanism as in FT
   * conserved (or improved) current: global U(1)

EPD model and AlLif spacetimes

- bulk theory

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}W(\Phi)B^2 - V(\Phi)\right)$$

- admits Lifshitz solutions with z>1

For AlLif BCs useful to write:

$$ds^2 = \frac{dr^2}{R(\Phi)r^2} - E^0E^0 + \delta_{ab}E^aE^b\,, \qquad B_M = A_M - \partial_M\Xi$$

[Ross],[Christensen,Hartong,NO,Rollier]
[Hartong,Kiritsis,NO]1

then AlLif BCs

$$E^0_\mu \simeq r^{-z}\tau_\mu\,, \qquad E^a_\mu \simeq r^{-1}e^a_\mu\,, \qquad \Phi \simeq r^\Delta\phi\,,$$

$$B_\mu - \alpha(\Phi)E^0_\mu \simeq -r^{z-2}M_\mu\,.$$

Stueckelberg decompostion: $M_\mu = \tilde{m}_\mu - \partial_\mu\chi\,.$

Transformation of sources

use local bulk symmetries:
local Lorentz, gauge transformations and diffs preserving metric gauge

these symmetries induce an action on sources: $\tau_\mu \quad e^a_\mu \quad M_\mu$

$=$ action of Bargmann algebra plus local dilatations $=$ Schroedinger

$$\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu + z\Lambda_D \tau_\mu \,, \\
\delta e^a_\mu &= \mathcal{L}_\xi e^a_\mu + \lambda^a \tau_\mu + \lambda^a{}_b e^b_\mu + \Lambda_D e^a_\mu \,, \\
\delta M_\mu &= \mathcal{L}_\xi M_\mu + e^a_\mu \lambda_a + (2-z)\Lambda_D M_\mu \,,
\end{aligned}$$

there is thus a Schroedinger Lie algebra valued connection given by

$$\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a \omega_\mu{}^a + \frac{1}{2} J_{ab} \omega_\mu{}^{ab} + N m_\mu + D b_\mu$$

with appropriate curvature constrains that reproduces trafos of the sources

Torsional Newton-Cartan (TNC) geometry

the bdry geometry is novel extension of NC geometry

- inverse vielbeins (v^μ, e^μ_a)

$$v^\mu \tau_\mu = -1\,, \qquad v^\mu e^a_\mu = 0\,, \qquad e^\mu_a \tau_\mu = 0\,, \qquad e^\mu_a e^b_\mu = \delta^b_a$$

can build Galilean boost-invariants

$$h_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$$

$$\hat{v}^\mu = v^\mu - h^{\mu\nu} M_\nu\,,$$
$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu M_\nu - \tau_\nu M_\mu\,,$$
$$\tilde{\Phi} = -v^\mu M_\mu + \tfrac{1}{2} h^{\mu\nu} M_\mu M_\nu\,,$$

affine connection of TNC

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}\right)$$

with torsion $\Gamma^\rho_{[\mu\nu]} = -\tfrac{1}{2}\hat{v}^\rho(\partial_\mu \tau_\nu - \partial_\nu \tau_\mu)$

$$\nabla_\mu \tau_\nu = 0\,, \qquad \nabla_\mu h^{\nu\rho} = 0\,,$$

Coupling FTs to TNC

[Hartong,Kiritsis,NO]

- action functional $$S = S[\hat{v}^\mu, h^{\mu\nu}, \tilde{\Phi}]\,.$$

EM tensor:	$T^\mu{}_\nu$
mass current	T^μ

energy current (density +flux) — momentum flux

$$\delta_{\rm bg} S = \int d^{d+1}x e \left[-\tau_\nu T^\nu{}_\mu \delta\hat{v}^\mu - \left(\hat{e}^a_\nu \hat{v}^\mu T^\nu{}_\mu\right) \hat{e}_{\sigma a} \tau_\rho \delta h^{\rho\sigma} + \frac{1}{2}\left(\hat{e}^b_\nu e^{\mu a} T^\nu{}_\mu\right) \hat{e}_{\rho b} \hat{e}_{\sigma a} \delta h^{\rho\sigma} + \tau_\mu T^\mu \delta\tilde{\Phi}\right]\,,$$

spatial stress — mass density

- off-shell WIs

$$e^{-1}\partial_\mu\left(e T^\mu\right) = \langle O_\chi\rangle\,,$$ particle number conservation (Stueckelberg U(1))

$$\hat{e}^a_\mu T^\mu - \tau_\nu e^{\mu a} T^\nu{}_\mu = 0\,.$$ mass current= momentum current (local boosts)

$$\hat{e}^{[a}_\nu e^{b]\mu} T^\nu{}_\mu = 0\,.$$ symmetric spatial stress (local rotations)

Diffeomorphism and scale Ward identities

- diffeos -> on-shell WI

$$0 = e^{-1}\partial_\nu \left(e T^\nu{}_\mu\right) + T^\rho{}_\nu \left(\hat{v}^\nu \partial_\mu \tau_\rho - e^\nu_a \partial_\mu \hat{e}^a_\rho\right) + \tau_\nu T^\nu \partial_\mu \tilde{\Phi}\,.$$

* conserved currents $\partial_\nu \left(e K^\mu T^\nu{}_\mu\right) = 0\,.$

for K a TNC Killing vector:

$$\mathcal{L}_\xi \hat{v}^\mu = 0\,, \qquad \mathcal{L}_\xi h^{\mu\nu} = 0\,, \qquad \mathcal{L}_\xi \tilde{\Phi} = 0\,,$$

- if theory has scale invariance:
  can use TNC analogue of dilatation connection

$$-z\tau_\nu \hat{v}^\mu T^\nu{}_\mu + \hat{e}^a_\nu e^{\mu a} T^\nu{}_\mu + 2(z-1)\tau_\mu T^\mu \tilde{\Phi} = 0\,.$$

z-deformed trace WI

Schroedinger model & Lifshitz model

- simplest toy model for coupling non-rel. scale-inv theory to TNC (z=2)

$$S = \int d^{d+1}x e \left(-i\phi^\star \hat{v}^\mu \partial_\mu \phi + i\phi \hat{v}^\mu \partial_\mu \phi^\star - h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^\star - 2\tilde{\Phi}\phi\phi^\star - V_0 (\phi\phi^\star)^{\frac{d+2}{d}} \right)$$

-> gives Schr. equation

* can also consider deformations preserving local scale inv

- other possibility: do not couple to $\tilde{\Phi}$ -> e.g. z=2 Lifshitz model

$$S = \int d^{d+1}x e \left[\frac{1}{2} (\hat{v}^\mu \partial_\mu \phi)^2 - \frac{\lambda}{2} (h^{\mu\nu} \nabla_\mu \partial_\nu \phi)^2 \right]$$

- more generally: $S = S[\hat{v}^\mu, h^{\mu\nu}]$. --> $S = S[\hat{v}^\mu, g^{\mu\nu}]$.

$g^{\mu\nu} = -\hat{v}^\mu \hat{v}^\nu + h^{\mu\nu}$ situation considered in [Hoyos,Kim,Oz]

* special case: $S = S[g^{\mu\nu}]$.

Flat NC spacetime

to study FTs on flat NC: first need to define notion of flat NC

- use global inertial coordinates (t, x^i)

$$\tau_\mu = \delta_\mu^t \,, \qquad e_\mu^a = \delta_\mu^i \delta_i^a \,. \longrightarrow \begin{array}{l} h^{tt} = h^{ti} = 0 \,, \quad h^{ij} = \delta^{ij} \,, \\ v^\mu = -\delta_t^\mu \,, \\ h_{tt} = h_{ti} = 0 \,, \quad h_{ij} = \delta_{ij} \,. \end{array}$$

$$\Gamma^\rho_{\mu\nu} = 0 \to M_\mu = \partial_\mu M \,.$$

choice of vector field is motivated by looking at geodesics

- flat space should include M=const.
  * will see that we can allow for more general choices:
    equiv. to M=const by local syms of the theory
      -> defines the notion of orbit of M

intermezzo: geodesics on NC spacetime

- worldline action of non-rel particle of mass m on NC background

$$S = \int d\lambda L = \frac{m}{2} \int d\lambda \frac{\bar{h}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho}$$

[Kuchar],
[Bergshoeff et al]

- gives the geodesic equation with NC connection $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$,

* reduces to Newton's law $\frac{d^2 x^i}{dt^2} + \delta^{ij} \partial_j \Phi = 0$,

provided we take

$$M_t = \partial_t M + \Phi \,, \qquad M_i = \partial_i M \,,$$

- EM and mass current from the action

$$T^\mu{}_\nu = -P_\nu \dot{x}^\mu \qquad\qquad P_\mu =: p_\mu - m M_\mu$$

$$T^\mu = -m \dot{x}^\mu$$

residual coordinate trafos of flat NC

- trafos of the TNC geometry that leave flat NC invariant (up to local rescaling) ?

  (analogue of Poincare (conformal) for Minkowski)

* finite versions:

$$
\begin{array}{lll}
M'(x) = M(x) + C & & \\
t' = t + a & & M'(x') = M(x) \\
x'^i = x^i + a^i & & M'(x') = M(x) \\
x'^i = R^i{}_j x^j & & M'(x') = M(x) \\
t' = \lambda^z t & x'^i = \lambda x^i & M'(x') = \lambda^{2-z} M(x) \\
x'^i = x^i + v^i t & t' = t & M'(x') = M(x) - \frac{1}{2} v^i v^i t + v^i x^i
\end{array}
$$

plus special conformal transformation for z=2

$$
t' = \frac{t}{1 - ct}\,, \qquad x'^i = \frac{x^i}{1 - ct}\,, \qquad M'(x') = M(x) + \frac{c}{2}\frac{x^i x^i}{1 - ct}\,.
$$

Scale invariant FTs on flat NC

- role of M is non-trivial: consider the toy FT models

  - (deformed) Schroedinger model:

$$S=\int d^{d+1}x\left(-\varphi^2\left[\partial_t(\theta+M)+\frac{1}{2}\partial_i(\theta+M)\partial^i(\theta+M)+a\partial_i\partial^i(\theta+M)\right]\right.$$
$$\left.-\frac{1}{2}\partial_i\varphi\partial^i\varphi-V_0\varphi^{\frac{2(d+2)}{d}}\left(1+b\theta^2\right)\right),$$

  - Lifshitz model:

$$S=\int d^{d+1}x\left[\frac{1}{2}\left(\partial_t\phi+\partial^i M\partial_i\phi\right)^2-\frac{\lambda}{2}\left(\partial_i\partial^i\phi\right)^2\right].$$

can we remove M by local transformations (field redefinitions) ?
and get M=const. : depends on the model in question

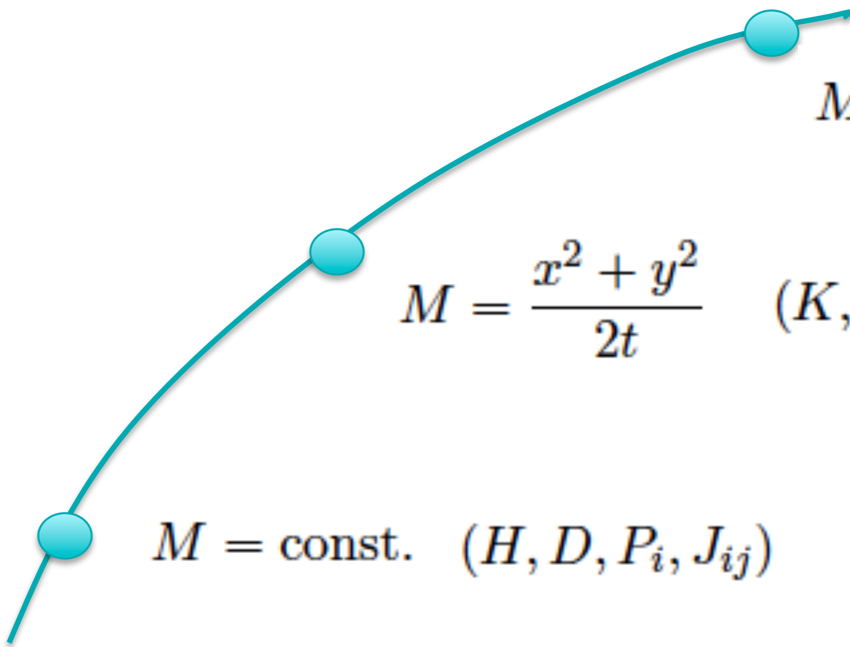
b=0: $\tilde{\theta}=\theta+M$ → Sch-invariant for a=0
Lif + Galilean boost for a not zero

(consequence of local U(1) symmetry)

b not zero
& Lifsthiz model → only Lif invariance

Orbits of M

- the M functions related to M=const by residual trafos define orbit
  * maximal orbit underlies Sch symmetry (as in undeformed Sch model)



$$M = -\frac{1}{2}V^i V^i t + V^i x^i$$

$$M = \frac{x^2+y^2}{2t} \qquad (K, D, G_i, J_{ij})$$

$$M = \text{const.} \qquad (H, D, P_i, J_{ij})$$

- for each choice of M: CKVs form a Lifshitz subalgebra
  - residual trafos of flat NC with $\delta M = 0$

-> will be useful later when we look at Lif vacuum (in holography)

Lifshitz vacuum (back to holography)

[Kiritsis,Hartong,NO]3

- sources in Lif holography transform under Sch

- can show that sources for Lif vacuum transform under Sch group (via bulk PBH trafos)
  * Killing symmetries = Lif subalgebra of Sch

- in suitable bulk coords this is dual to:
  flat NC with CKVs spanning Lif and Sch realized locally on $M_\mu = \partial_\mu M$.

have seen: FTs on flat NC realize Sch with mechanism in which M is ``eaten'' up
(generators outside Lif are realized as projective transformations)

-> projective realizations of spacetime syms cannot be predicted by looking at Killing vectors

- can construct z=2 probe actions on Lifshitz bulk geometries that are invariant under Sch (in same manner as in FT setting)

One Lif metric for all M

[Kiritsis,Hartong,NO]3

- Lif metric in Poincare coords $ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^{2z}} + \frac{1}{r^2} dx^i dx^i \qquad B = \frac{dt}{r^z}.$

  (corresponds to M=const)

Lif metric for any M in flat NC-orbit

$$ds^2 = \left(\frac{dr}{r} - \frac{1}{d} \partial_i \partial^i M dt \right)^2 - \frac{dt^2}{r^{2z}} + \frac{1}{r^2} \left(dx^i - \partial^i M dt \right)^2 .$$

* not generally in radial gauge: but can do coord trafo to radial gauge that does not modify the sources

- trafo that close to bdry is bdry dependent rescaling and bdry diffeo
+ order (r^2) trivial bulk diffeo (bringing back to radial gauge)

Symmeries of Lif vacuum

- what is bulk realization of residual syms of flat NC ?

-> bulk diffeos that preserve the form of the Lif(M)

- trafos for given M in M=const orbit: Lifshitz
- delta M trafos lie in Sch algebra

generators of PBH transformations that preserve the boundary conditions span the Schroedinger algebra

-> can give rise to global Schroedinger invariance

* possible to have conserved particle number associated to local shifts in M (generated by Galilean and special conformal)

Schroedinger invariant probe actions

natural probe action for (z=2, d=2) Lifshitz spacetime

use covariant characterization of Lif

$$ds^2 = (-B_M B_N + \gamma_{MN})\, dx^M dx^N$$

$$B^2 = -1 \qquad \gamma_{MN} \text{ is orthogonal to } B^M$$

$$S = \int d^4x \sqrt{-g}\,(\gamma^{MN}\partial_M\phi^\star\partial_N\phi + iq\phi^\star B^M\partial_M\phi - iq\phi B^M\partial_M\phi^\star - (m^2-q^2)\phi^\star\phi)$$

omit $-B^M\partial_M\phi^\star B^N\partial_N\phi$ in $S = \int d^4x\sqrt{-g}\,(D_M\phi^\star D^M\phi - m^2\phi^\star\phi)$

equation of motion is

$$r^2\left(\partial_i\partial^i\phi + 2iqD_t\phi\right) + r^2\partial_r^2\phi - 3r\partial_r\phi - (m^2-q^2)\phi = 0\,.$$

$$D_t = \partial_t + \partial^i M\partial_i + \frac{1}{2}\partial^2 M r\partial_r$$

eat up M: $\phi = \exp[-iqM - \frac{i}{4}qr^2\partial^2 M]\tilde{\phi}$ + use all props of M

$$r^2\left(\partial_i\partial^i\tilde{\phi} + 2iq\partial_t\tilde{\phi}\right) + r^2\partial_r^2\tilde{\phi} - 3r\partial_r\tilde{\phi} - (m^2-q^2)\tilde{\phi} = 0$$

-> Sch in UV, flow to Lif in IR ?

Summary

→ defined sources for AlLif spacetimes and shown that they

- transform under local Schroedinger group
- describe torsional NC boundary geometry
- lead to Sch Ward identities for bdry stress tensor and mass current

→ have shown:

Lif vacuum dual to flat NC has local action of Sch group acting on remaining source (M): subgroup is Lif, generated by Killing vectors

boundary theory can have conserved current related to particle number

- both precisely in same manner as Sch syms arise in FTs on flat NC

\- to show that boundary theory is Sch invariant under global Sch syms
-> need to know type of matter fields living on space & coupling to geometry

one can indeed construct scalar probes on bulk Lif that are inv. under Sch

fluid/gravity ?

TNC of growing interest in cond-mat (str-el, mes-hall) literature

developments in Lifshitz holography can drive development of tools to study dynamics and hydrodynamics of non-rel. systems

Lifsthiz hdyro: [Hoyos,Kim,Oz]
Galilean: [Jensen]

(in parallel to progress in the last many years in relativistic fluids and superfluids inspired from the fluid/gravity correspondence in AdS)

TNC right ingredients to start constructing effective TNC theories and their coupling to matter (e.g. QH-effect)

Next steps

- new perspective on existing results: ncomparison to linearized perturbations
  relation between $\tilde{\Phi}$ and ψ.
- EMD model (emergence of TNC, and role of U(1) ?)
- adding other exponents: (logarithmic running of scalar) alpha/zeta-deformation

  $$A_a = r^{-z-\zeta}\alpha_{(0)}\tau_{(0)a}$$

  [Kiritsis,Goutereaux][Gath,Hartong,Monteiro,NO]
  [Khveshchenko][Karch][Hartnoll,Karch]
- most general soln of Lif for bdry = NC + Newton potential
- 3D bulk (Virasoro-Schroedinger) & connection to Warped CFTs [Hofman,Rollier]
- applications to hydrodynamics:
  black branes with zero/non-zero particle number density ? Galilean perfect fluids
- Schroedinger holography
- HL gravity and Einstein-aether theories
- adding charge

The end

The Schroedinger model and deformations

- simplest toy model for coupling non-rel scale-inv theory to TNC (z=2)

$$S = \int d^{d+1}x e \left(-i\phi^\star \hat{v}^\mu \partial_\mu \phi + i\phi \hat{v}^\mu \partial_\mu \phi^\star - h^{\mu\nu}\partial_\mu \phi \partial_\nu \phi^\star - 2\tilde{\Phi}\phi\phi^\star - V_0 (\phi\phi^\star)^{\frac{d+2}{d}} \right)$$

* consider deformations preserving local scale inv $\qquad \phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$

- change the potential $\quad V_0 \varphi^{\frac{2(d+2)}{d}} (1 + b\theta^2)$

- adding the term: $\quad -a \int d^{d+1}x e \varphi^2 h^{\mu\nu} \tilde{\nabla}_\mu \partial_\nu \theta \quad - a \int d^{d+1}x e \varphi^2 e^\mu_a \mathcal{D}_\mu M^a$

- can show that a-deformed model has local symmetry

$\delta M_\mu = \partial_\mu \alpha \,, \qquad \delta\theta = -\alpha \,, \qquad$ giving on-shell WI $\quad \partial_\mu \left(e T^\mu \right) = 0 \,,$

* diffeos + local boosts (+ possibly local scale) induce trafos of type:

$\tilde{N} \; : \qquad \delta v^\mu = 0 \,, \qquad \delta h^{\mu\nu} = 0 \,, \qquad \delta M_\mu = \partial_\mu \tilde{\sigma} \,,$

-> posssibility of extra global symmetries (intimately connected to vector field)

residual coordinate trafos of flat NC

- trafos of the TNC geometry that leave flat NC invariant?

$$
\begin{aligned}
\Lambda_D &= -\lambda - \delta_{z,2}\alpha t\,,\\
\xi^t &= a + z\lambda t + \delta_{z,2}\alpha t^2\,,\\
\xi^i &= v^i t + a^i + \lambda^i{}_j x^j + \lambda x^i + \delta_{z,2}\alpha t x^i\\
\lambda^i &= -v^i - \delta_{z,2}\alpha x^i\,,\\
\delta M &= \xi^t\partial_t M + \xi^i\partial_i M - (2-z)\lambda M - C - v^i x^i - \frac{1}{2}\delta_{z,2}\alpha x^i x^i\,.
\end{aligned}
$$

* finite versions:

$$
\begin{array}{lll}
M'(x) = M(x) + C & & \\
t' = t + a & & M'(x') = M(x)\\
x'^i = x^i + a^i & & M'(x') = M(x)\\
x'^i = R^i{}_j x^j & & M'(x') = M(x)\\
t' = \lambda^z t & x'^i = \lambda x^i & M'(x') = \lambda^{2-z} M(x)\\
x'^i = x^i + v^i t & t' = t & M'(x') = M(x) - \frac{1}{2}v^i v^i t + v^i x^i
\end{array}
$$

plus special conformal transformation for z=2

$$
t' = \frac{t}{1-ct}\,, \qquad x'^i = \frac{x^i}{1-ct}\,, \qquad M'(x') = M(x) + \frac{c}{2}\frac{x^i x^i}{1-ct}\,.
$$

More on the b=0 model

- in terms of the physical field $\tilde{\theta} = \theta + M$

$$S = \int d^{d+1}x \left(-\varphi^2 \left[\partial_t \tilde{\theta} + \frac{1}{2} \partial_i \tilde{\theta} \partial^i \tilde{\theta} + a \partial^2 \tilde{\theta} \right] - \frac{1}{2} \partial_i \varphi \partial^i \varphi - V_0 \varphi^{\frac{2(d+2)}{d}} \right)$$

Lifshitz invariance
+ Galilean boost

$$t = t' \,, \qquad x^i = x'^i - v^i t' \,,$$

$$\tilde{\theta} = \tilde{\theta}' + \frac{1}{2} v^i v^i t' - v^i x'^i \,,$$

Orbits of M

- the M functions related to M=const by residual trafos are characterized by

$$\tilde{\Phi} = \partial_t M + \frac{1}{2}\partial_i M \partial^i M = 0 \,.$$

$$0 = \partial_i \partial_j \partial^j M \,,$$

$$0 = \partial_i \partial_j M - \frac{1}{d}\delta_{ij}\partial_k \partial^k M \,.$$

* maximal orbit underlies Sch symmetry (as in undeformed Sch model)

-> will be useful later when we look at Lif vacuum (in holography)

- families of M solutions:

$$M = C + \frac{(x^i - x_0^i)(x^i - x_0^i)}{2(t - t_0)}$$

$$M = C - \frac{1}{2}V^i V^i t + V^i x^i$$

TNC Killing vectors

[Kiritsis,Hartong,NO]2,3

- consider residual trafos with $\delta M = 0$

* correspond to conformal Killing vectors

$$
\begin{aligned}
\mathcal{L}_K \tau_\mu &= -z\Omega\tau_\mu \,, & \mathcal{L}_K \hat{v}^\mu &= z\Omega\hat{v}^\mu \,, & \mathcal{L}_K \bar{h}_{\mu\nu} &= -2\Omega\bar{h}_{\mu\nu} \\
\mathcal{L}_K h^{\mu\nu} &= 2\Omega h^{\mu\nu} \,, & \mathcal{L}_K \Phi_N &= 2(z-1)\Omega\Phi_N \,, & \mathcal{A}\Omega &= 0
\end{aligned}
$$

$$
\begin{aligned}
M &= \text{cst} & & H, D, P_i, J_{ij} \,, \\
M &= \frac{x^2+y^2}{2t} & & K, D, G_i, J_{ij} \,, \\
M &= -\frac{1}{2}V^iV^it + V^ix^i & & H, D, P_i, J_{ij} \,,
\end{aligned}
$$

- for each choice of M: CKVs form a Lifshitz subalgebra

$$
\begin{aligned}
H &= \partial_t \,, & P_i &= \partial_i \,, \\
G_i &= t\partial_i \,, & J_{ij} &= x_i\partial_j - x_j\partial_i \,, \\
D &= zt\partial_t + x^i\partial_i \,, & K &= t^z\partial_t + t^{z-1}x^i\partial_i \,,
\end{aligned}
$$

Local realization of Schr on M

CKVs can be used to generate maximal orbit: of Sch sym

$$
\begin{aligned}
H &= \partial_t\,, & P_i &= \partial_i\,,\\
G_i &= t\partial_i + x_i\tilde{N}\,, & J_{ij} &= x_i\partial_j - x_j\partial_i\,,\\
D &= zt\partial_t + x^i\partial_i\,,
\end{aligned}
$$

$\tilde{N}$ shifts of M

$$K = t^2\partial_t + tx^i\partial_i + \frac{1}{2}x^i x^i\tilde{N}\,.$$

Transforming to radial gauge (details)

$$M = x^i x^i / 2t \qquad ds^2 = \left(\frac{dr}{r} - \frac{dt}{t}\right)^2 - \frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dx^i - \frac{x^i}{t}dt\right)^2 \qquad B = \frac{dt'}{r'^2}$$

$$t' = -\frac{1}{T}, \qquad r' = -\frac{R}{T}, \qquad x'^i = -\frac{x^i}{T}.$$

$$ds^2 = -\frac{dT^2}{R^4} + \frac{dR^2}{R^2} + \frac{1}{R^2}dX^i dX^i, \qquad B = \frac{dT}{R^2}$$

$$T = -\frac{1}{t}\frac{1}{1 - \frac{1}{4}\frac{r^4}{t^2}},$$
$$R = -\frac{r}{t}\frac{1}{\left(1 - \frac{1}{4}\frac{r^4}{t^2}\right)^{1/2}},$$
$$X^i = -\frac{x^i}{t}.$$

$$ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^4} + \frac{1}{r^2}\delta_{ij}\left(1 - \frac{1}{4}\frac{r^4}{t^2}\right)\left(dx^i - \frac{x^i}{t}dt\right)\left(dx^j - \frac{x^j}{t}dt\right)$$

$$B = \frac{1 + \frac{1}{4}\frac{r^4}{t^2}}{1 - \frac{1}{4}\frac{r^4}{t^2}}\frac{dt}{r^2} - \frac{\frac{r^2}{t}}{1 - \frac{1}{4}\frac{r^4}{t^2}}\frac{dr}{r}.$$

Particle number current

local transformations of source M -> WI for $\partial_\mu T^\mu$

$$\delta S^{\rm ren}_{\text{on-shell}}[M] = -\int d^{d+1}x \partial_\mu T^\mu \delta M$$

can show

$$\begin{aligned}\partial_\mu T^\mu &= -\partial_t \lambda_1 - \partial_i(\lambda_1 \partial^i M) - \partial_i \partial_j \partial^j \lambda^i + \left(\partial_i \partial_j - \frac{1}{d}\delta_{ij}\partial_k \partial^k\right)\lambda^{ij} \\ &= -\partial_t \lambda_1 - \partial_i(\lambda_1 \partial^i M) + \left(\partial_i \partial_j \Lambda^{ij} - \frac{1}{d}\partial_i \partial^i \Lambda^k{}_k\right) \equiv \partial_\mu J^\mu ,\end{aligned}$$

-> local Sch inv of on-shell action with flat NC bcs can lead to conserved current

$$\partial_\mu \left(T^\mu - J^\mu\right) = 0 .$$

so possible to have conserved particle number associated to local shifts in M (generated by Galilean and special conformal)

Lessons from z=2 holographic Lifshitz model

[Christensen,Hartong,NO,Rollier]

considered first a specific z=2 example (in 4D) that can
be obtained by Scherk-Schwarz dim. reduction (null on bdry)
froma 5D AlAdS solution

[Donos,Gauntlet][Cassani,Faedo][Chemissany,Hartong]

counterterms and reduction: [Papadimitriou][Chemissany,Geisbuehler,Hartong,Rollier]

important lessons:

- use of vielbeins highly advised (see also [Ross])
- identification of sources requires appropriate lin. combo of timelike vielbein and bulk gauge field (-> crucial for boundary gauge field)
- bdr. geometry is torsional Newton-Cartan
- can compute unique gauge and tangent space inv. bdry stres tensor
- WIs take TNC covariant form
- conserved quantities from WIs and TNC (conformal) Killing vectors

EPD model and AlLif spacetimes

bulk theory

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}W(\Phi)B^2 - V(\Phi)\right)$$

- admits Lifshitz solutions with z>1

For AlLif BCs useful to write:

$$ds^2 = \frac{dr^2}{R(\Phi)r^2} - E^0E^0 + \delta_{ab}E^aE^b\,, \qquad B_M = A_M - \partial_M\Xi$$

then AlLif BCs

[Ross],[Christensen,Hartong,NO,Rollier]
[Hartong,Kiritsis,NO]1

$$\begin{array}{rclrcl} E^0_\mu & \propto & r^{-z}\tau_\mu + \ldots & E^a_\mu & \propto & r^{-1}e^a_\mu + \ldots \\ A_\mu - \alpha(\Phi)E^0_\mu & \propto & r^{z-2}\tilde{m}_\mu + \ldots & A_r & = & (z-2)r^{z-3}\chi + \ldots \\ \Xi & = & r^{z-2}\chi + \ldots & \Phi & = & r^{\Delta}\phi + \ldots \end{array}$$

Transformation of sources

use local bulk symmetries:
local Lorentz, gauge transformations and diffs preserving metric gauge

these symmetries induce an action on sources: $\tau_\mu,\ e^a_\mu,\ \tilde{m}_\mu,\ \chi$

$=$ action of Bargmann algebra plus local dilatations $=$ Schroedinger

there is thus a Schroedinger Lie algebra valued connection given by

$$A_\mu = H\tau_\mu + P_a e^a_\mu + M m_\mu + \frac{1}{2} J_{ab}\omega_\mu{}^{ab} + G_a \omega_\mu{}^a + D b_\mu$$

with $\tilde{m}_\mu = m_\mu - (z-2)\chi b_\mu$

with appropriate curvature constrains that reproduces trafos of the sources

Torsional Newton-Cartan (TNC) geometry

the bdry geometry is novel extension of NC geometry

source	ϕ	τ_μ	e_μ^a	v^μ	e_a^μ	$\tilde{m}_0$	$\tilde{m}_a$	χ
scaling dimension	Δ	$-z$	-1	z	1	$2z-2$	$z-1$	$z-2$

includes inverse veilbeins (v^μ, e_a^μ)

$$v^\mu \tau_\mu = -1\,, \qquad v^\mu e_\mu^a = 0\,, \qquad e_a^\mu \tau_\mu = 0\,, \qquad e_a^\mu e_\mu^b = \delta_a^b$$

from (inverse) vielbeins and vector: $M_\mu = \tilde{m}_\mu - \partial_\mu \chi$

can build Galilean boost-invariants

$$\begin{aligned} h^{\mu\nu} &= \delta^{ab} e_a^\mu e_b^\nu\,, & \hat{v}^\mu &= v^\mu - h^{\mu\nu} M_\nu \\ \bar{h}_{\mu\nu} &= \delta_{ab} e_\mu^a e_\nu^b - \tau_\mu M_\nu - \tau_\nu M_\mu\,, & \Phi_N &= -v^\mu M_\mu + \tfrac{1}{2} h^{\mu\nu} M_\mu M_\nu \end{aligned}$$

affine connection of TNC

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu} \right)$$

with torsion $\Gamma^\rho_{[\mu\nu]} = -\tfrac{1}{2} \hat{v}^\rho (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu)$

Vevs, EM tensor and mass current

assuming holographic renormalizability
-> general form of variation of on-shell action

$$\delta S^{\mathrm{os}}_{\mathrm{ren}} = \int d^3x e \left[-S^0_\mu \delta v^\mu + S^a_\mu \delta e^\mu_a + T^0 \delta \tilde{m}_0 + T^a \delta \tilde{m}_a + \langle O_\chi \rangle \delta\chi + \langle \tilde{O}_\phi \rangle \delta\phi - \mathcal{A}_{(\prime)} \frac{\delta r}{r} \right]$$

local bulk symmetries induce transformation on vevs (cf. sources)
-> exhibit again Schroedinger symmetry

from vevs & sources:
- bdyr EM tensor
- - mass current

$$\begin{aligned} \mathcal{T}^\mu{}_\nu &= -\left(S^0_\nu + T^0 \partial_\nu \chi\right) v^\mu + \left(S^a_\nu + T^a \partial_\nu \chi\right) e^\mu_a \\ T^\mu &= -T^0 v^\mu + T^a e^\mu_a \end{aligned}$$

tangent space projections provide
energy density, energy flux, momentum density, stress, mass density, mass current

$\mathcal{T}_\mu{}^\nu \tau_\nu \hat{v}^\mu$	$\mathcal{T}_\mu{}^\nu \tau_\nu e^\mu_a$	$\mathcal{T}_\mu{}^\nu \hat{e}^a_\nu \hat{v}^\mu$	$\mathcal{T}_\mu{}^\nu \hat{e}^a_\nu e^\mu_b$	$T^\mu \tau_\mu$	$T^\mu \hat{e}^a_\mu$
$z+2$	3	$2z+1$	$z+2$	$4-z$	3

Covariant Ward identities

Ward identities: (ignore for simplicity dilaton scalar)

$$
\begin{aligned}
0 &= -\hat{e}^a_\mu T^\mu + \tau_\mu e^{\nu a} \mathcal{T}^\mu{}_\nu && \text{boosts} \\
0 &= \hat{e}^a_\mu e^{\nu b} \mathcal{T}^\mu{}_\nu - (a \leftrightarrow b) && \text{rotations} \\
\mathcal{A} &= -z \hat{v}^\nu \tau_\mu \mathcal{T}^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a \mathcal{T}^\mu{}_\nu + 2(z-1)\Phi_N \tau_\mu T^\mu && \text{dilatations} \\
\langle O_\chi \rangle &= e^{-1} \partial_\mu \left(e T^\mu \right) && \text{gauge trafos} \\
0 &= \nabla_\mu \mathcal{T}^\mu{}_\nu + 2\Gamma^\rho_{[\mu\rho]} \mathcal{T}^\mu{}_\nu - 2\Gamma^\mu_{[\nu\rho]} \mathcal{T}^\rho{}_\mu \\
&\quad - T^\mu \hat{e}^a_\mu \mathcal{D}_\nu M_a + \tau_\mu T^\mu \partial_\nu \Phi_N && \text{diffs}
\end{aligned}
$$

- uses Galilean boost invariant vielbeins and density e

- ∇_μ contains affine TNC connection

- $\mathcal{D}_\mu$ contains Bargmann boost and rotation connections