Refining Generalized Entropy

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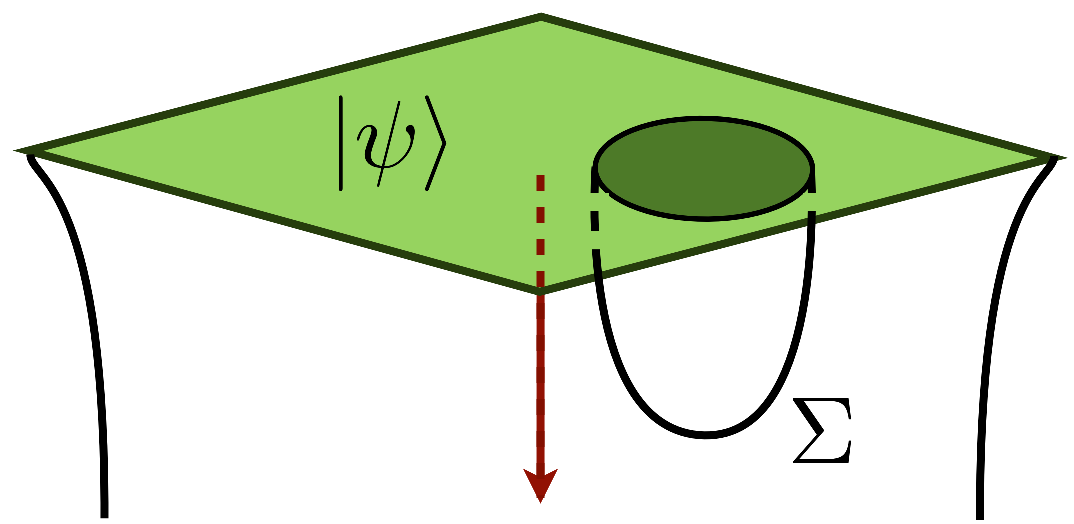
based on 1412.4093, with William R. Kelly (UCSB)

Ryu-Takayanagi

$$I = \frac{1}{16\pi G}\int \sqrt{-g}d^D x\, R$$



Extremal surface

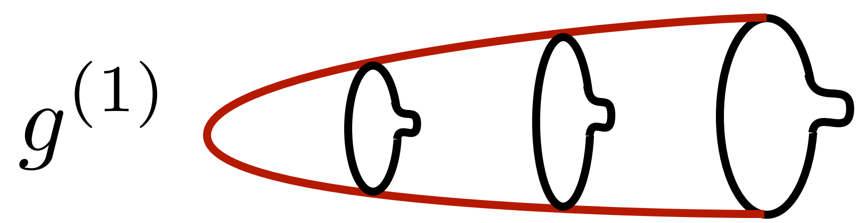
$$S = \frac{(\mathrm{Area}(\Sigma))_{D-2}}{4G}$$

Comments

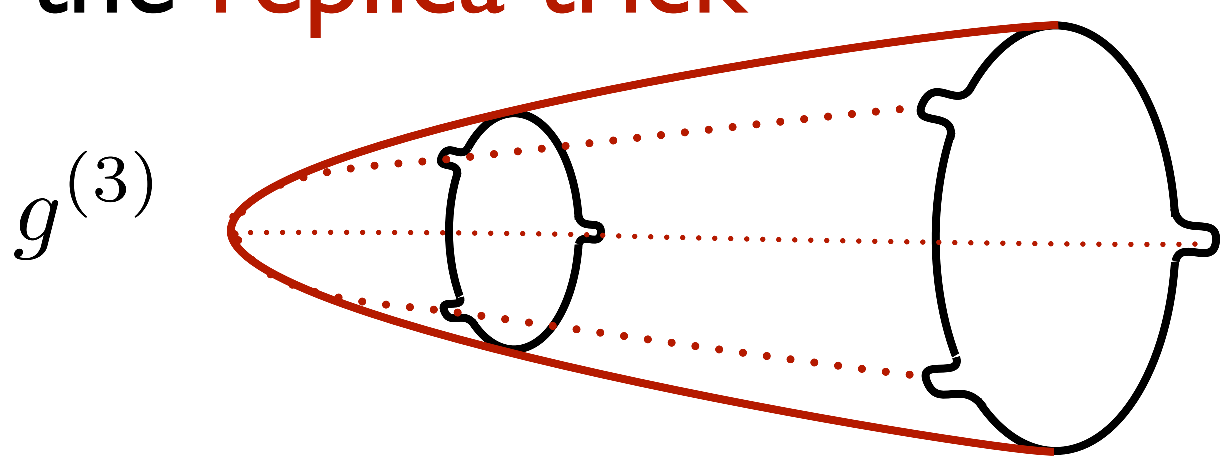
- Generalises Bekenstein-Hawking entropy
- Cumbersomely understood properties of entanglement become transparent, $\delta S = 0$
- A tool to potentially reconstruct space holographically
- We have a first-principles derivation of it: Lewkowycz-Maldacena

Outline of derivation

- Euclidean dual of thermal-like part function



- And the replica trick

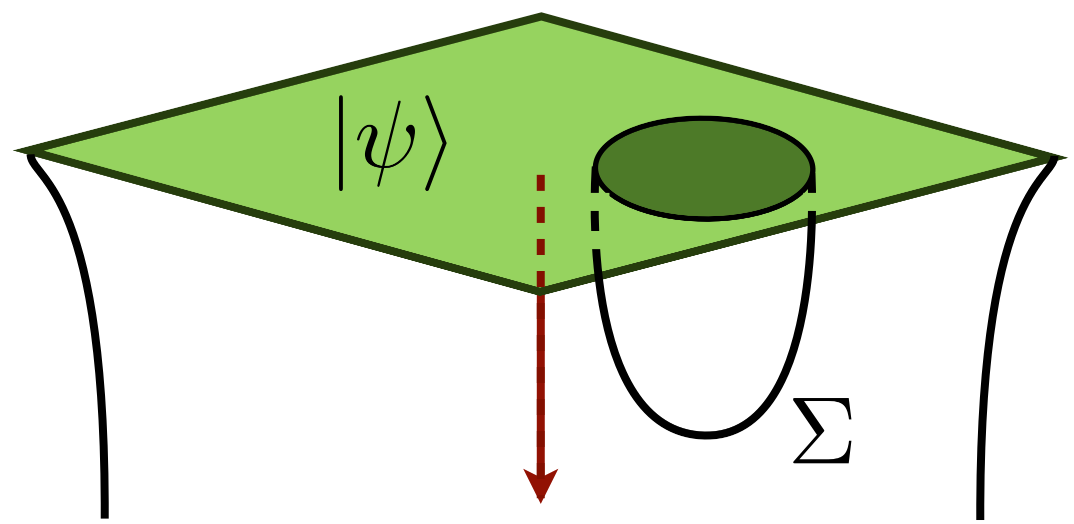


- Analytically continuing metrics $R_{\mu\nu}[g^{(n)}] = \text{finite}$
- Expanding around $n \sim 1$

More general proposal

Hung, Myers, Smolkin;
de Boer, Kulaxizi, Parnachev

$$I = \frac{1}{16\pi G}\int \sqrt{-g}d^D x\ (R + \alpha' \mathrm{Riem}^2 + \ldots)$$



Extremises entropy?

$$S \sim \int_\Sigma \sqrt{\gamma}d^{D-2}\sigma\ \left(\frac{\partial \mathcal{L}}{\partial \mathrm{Riem}} + \frac{\partial^2 \mathcal{L}}{\partial \mathrm{Riem}^2}K^2\right)$$

Dong; JC '13
see also Miao, Guo '14

What this talk is about

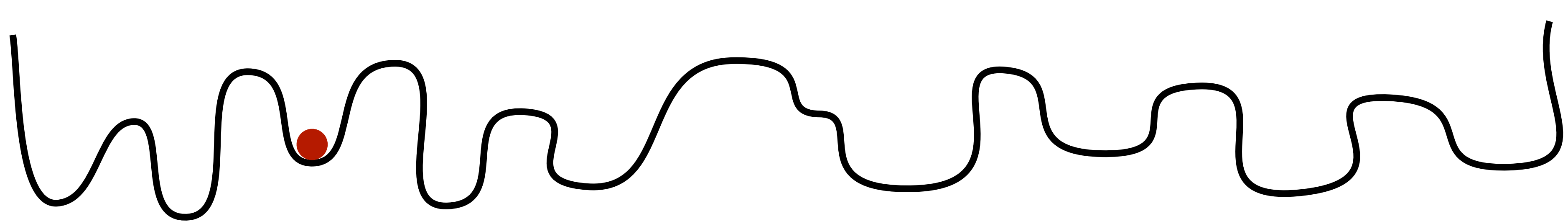
- Reviewing the replica trick in entanglement entropy calculations (and holographic dual)
- Discussing the structure of $g^{(n)}$ (analytic continuation of metrics) around $n \sim 1$

Conclusions

- Σ is determined by $\delta S = 0$ in GR and away (explicitly in one case)
- Ryu-Takayanagi follows independently from the assumption of replica symmetry
- Disclaimer: We do not conclude that replica symmetry is broken (but it may be)

Why?

- Higher derivatives: (a) Appear in the low energy limit of string theory; (b) Give a more general understanding
- Replica symmetry breaking: (a) Strengthens the derivation; (b) Hints to new possibilities
- In CMT, RSB explains glassy physics. Glasses are frustrated, with landscapes of vacua...



Lewkowycz-Maldacena

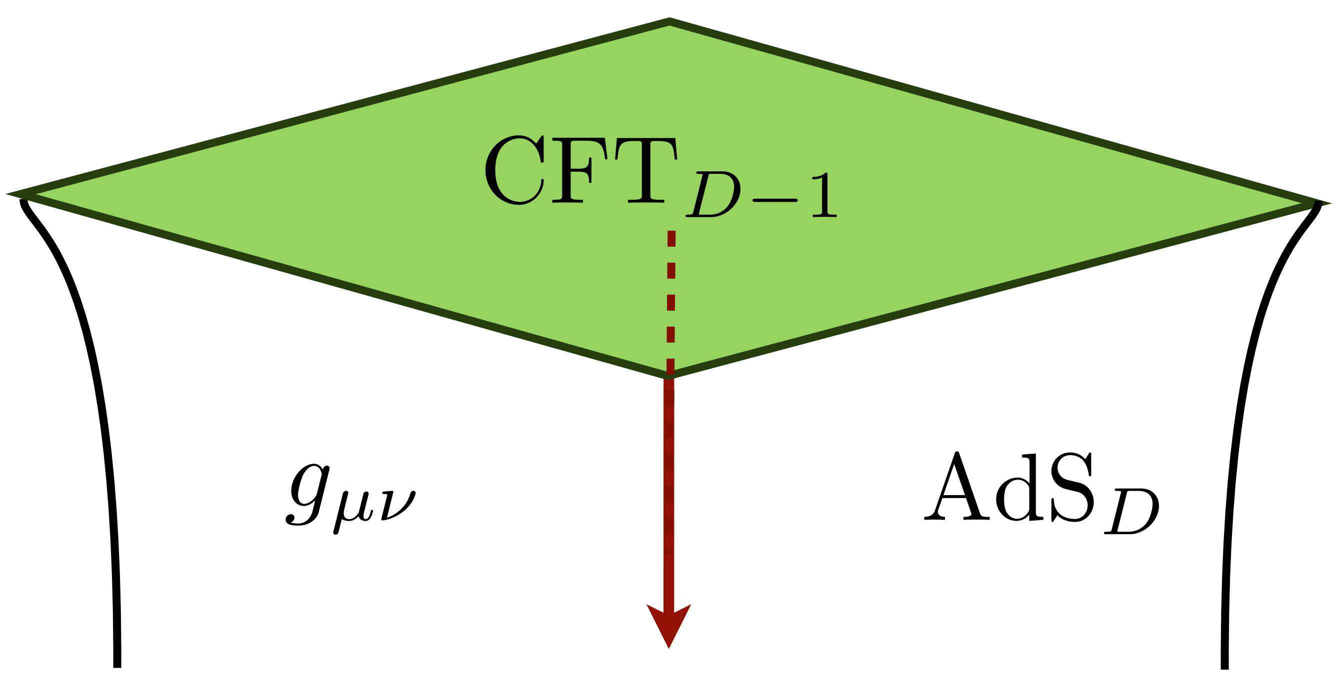
Ingredients of Generalized Entropy

- Replica trick

$$\mathrm{tr}(\rho \log \rho) = \lim_{n \to 1} \frac{\log(\mathrm{tr}\rho^n)}{n-1}$$

Expansion around $n \sim 1$

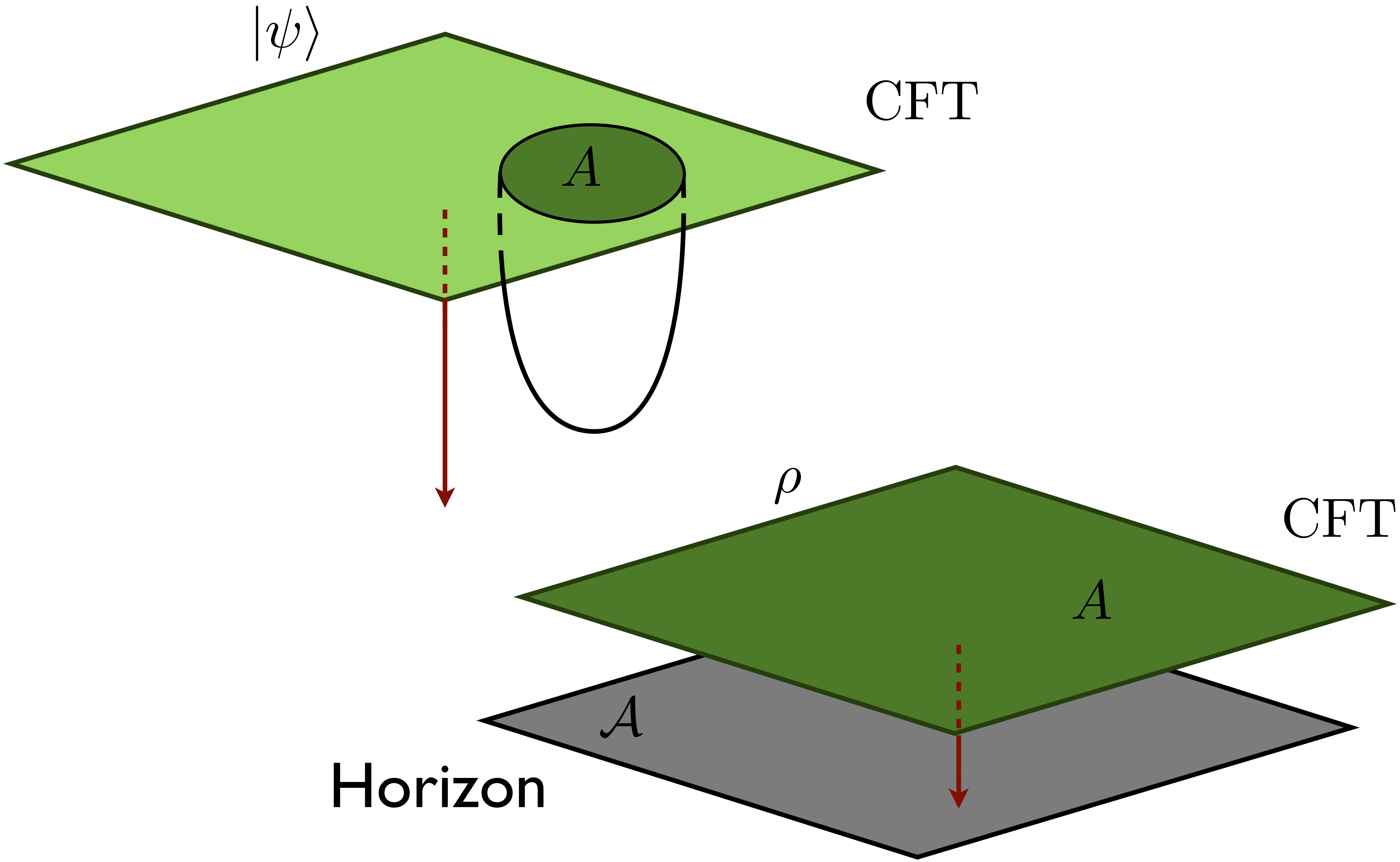
- Saddle point approximation to holography



$$Z_{\mathrm{CFT}_{D-1}} \approx e^{-I_E[g_{\mu\nu}]}$$

Setup

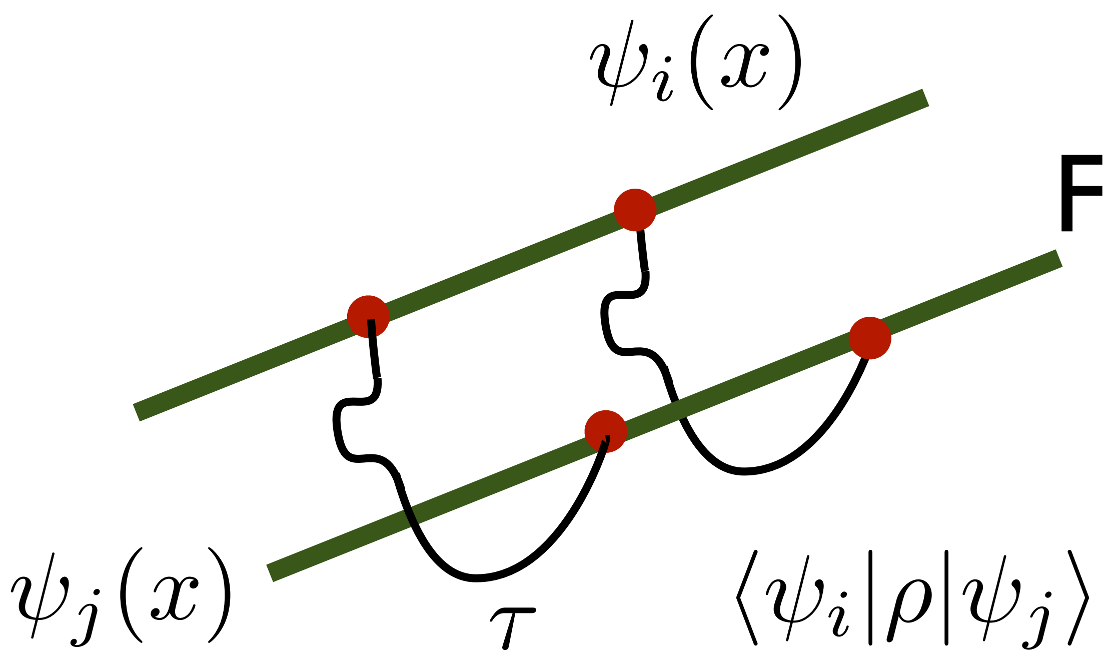
Casini, Huerta, Myers



States considered

- Generated by Euclidean path integrals

$$\rho = \mathcal{P}\, e^{-\int H(\tau) d\tau} \qquad \rho_T = \sum_i |i\rangle e^{-E_i \beta} \langle i|$$



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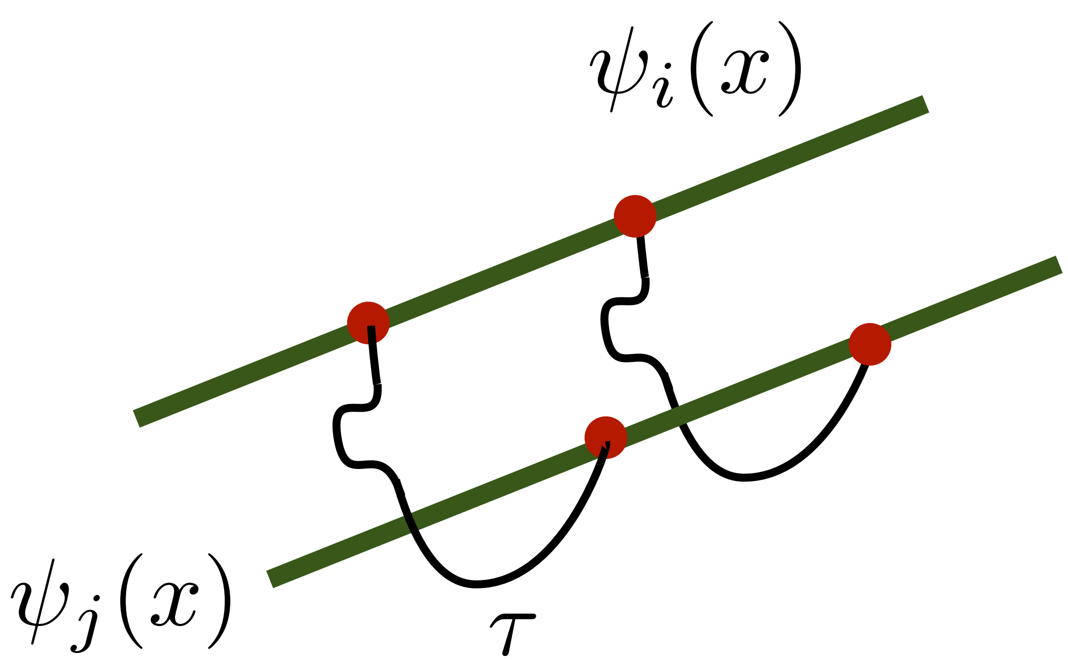

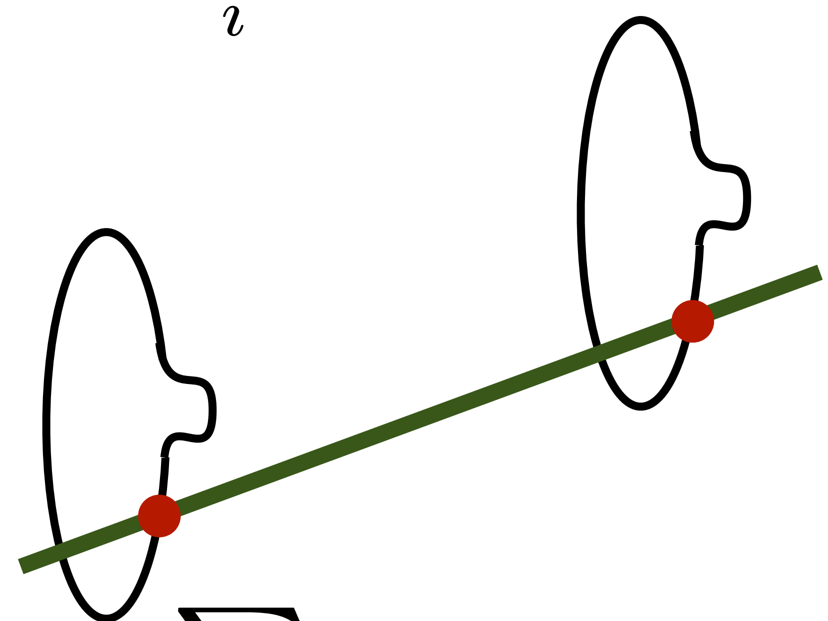

$$\sum_i \langle \psi_i | \rho | \psi_i \rangle = \mathrm{tr} \rho$$

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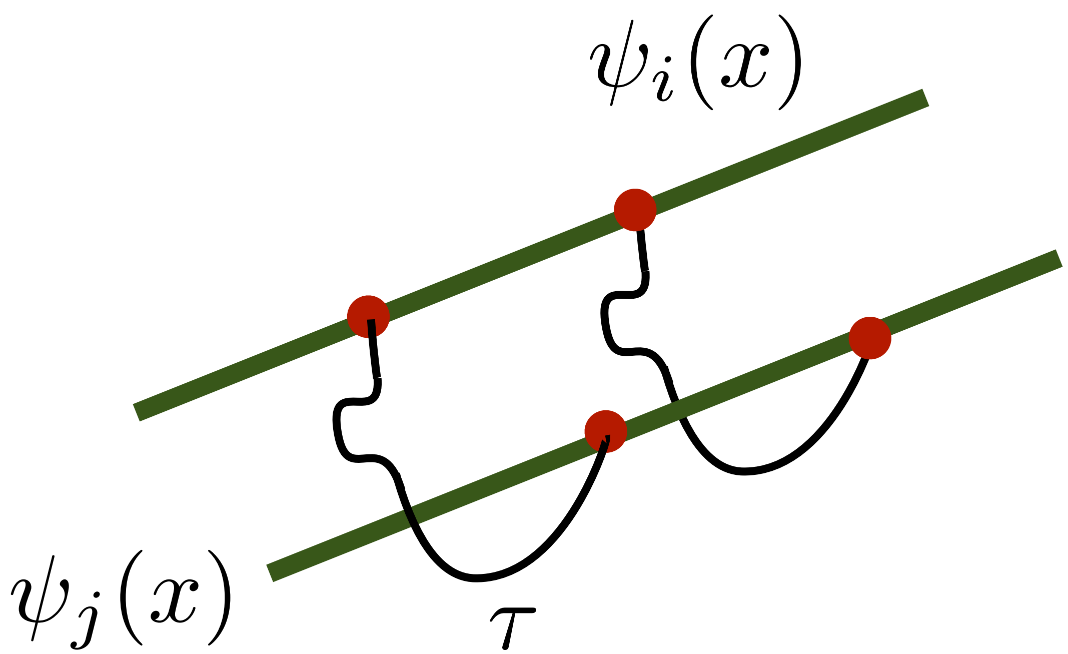

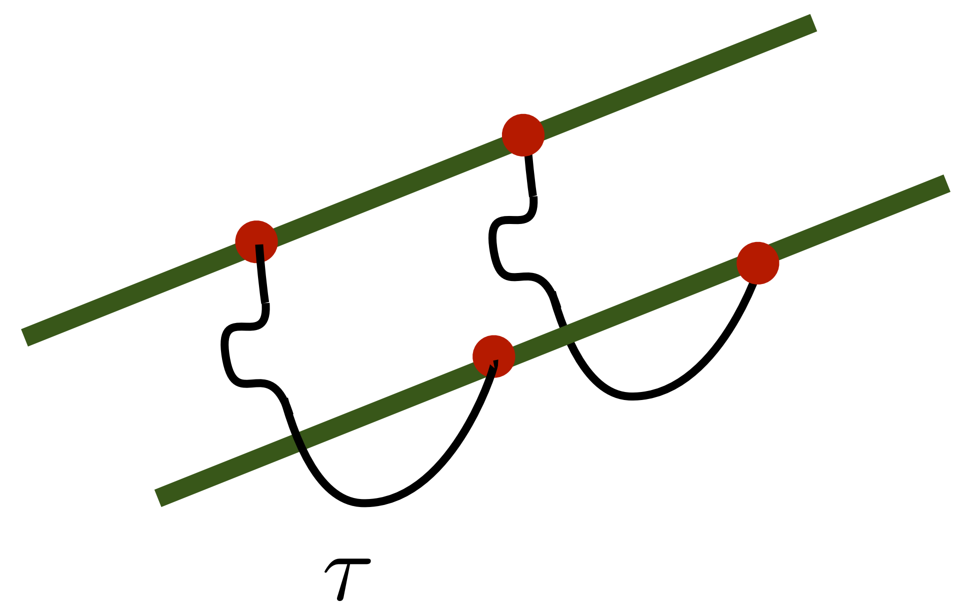

$$\mathrm{tr}\rho^2 = \sum_i \sum_j \langle \psi_i|\rho|\psi_j\rangle \langle \psi_j|\rho|\psi_i\rangle$$

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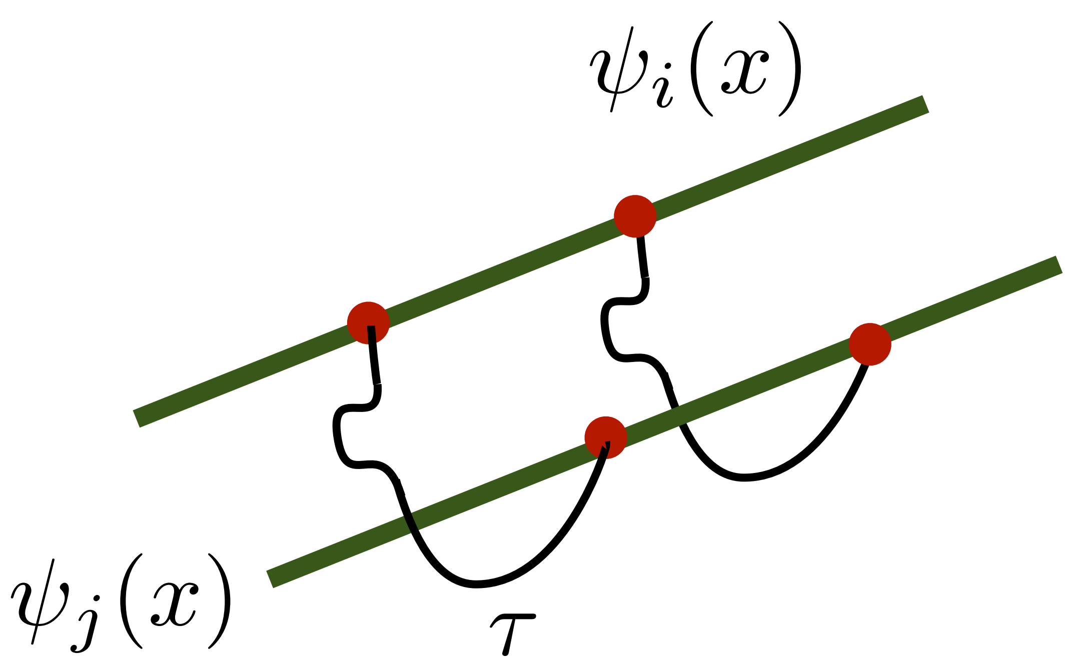
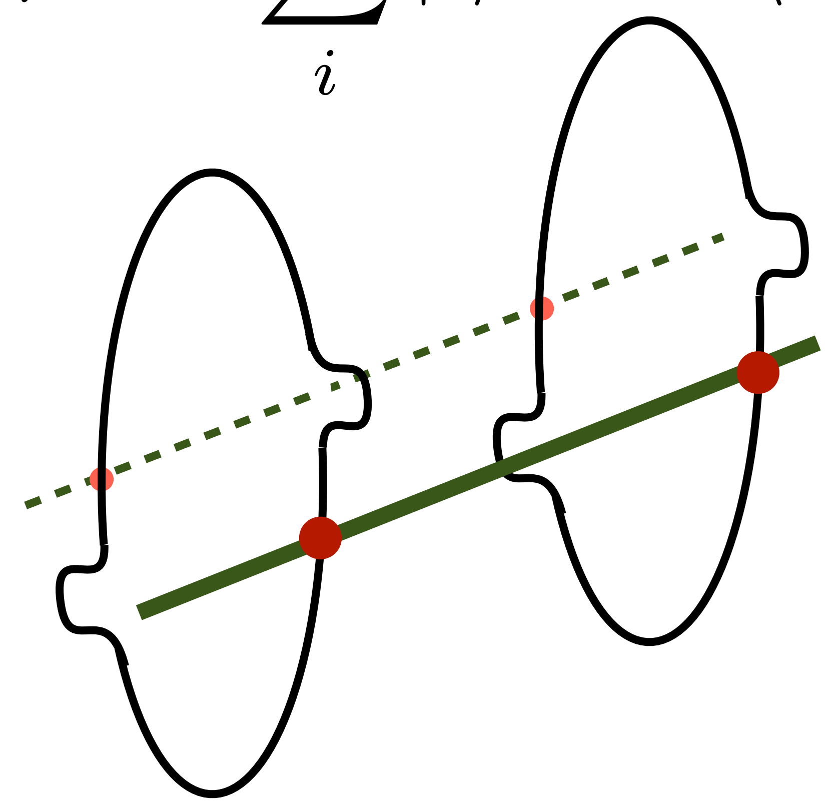
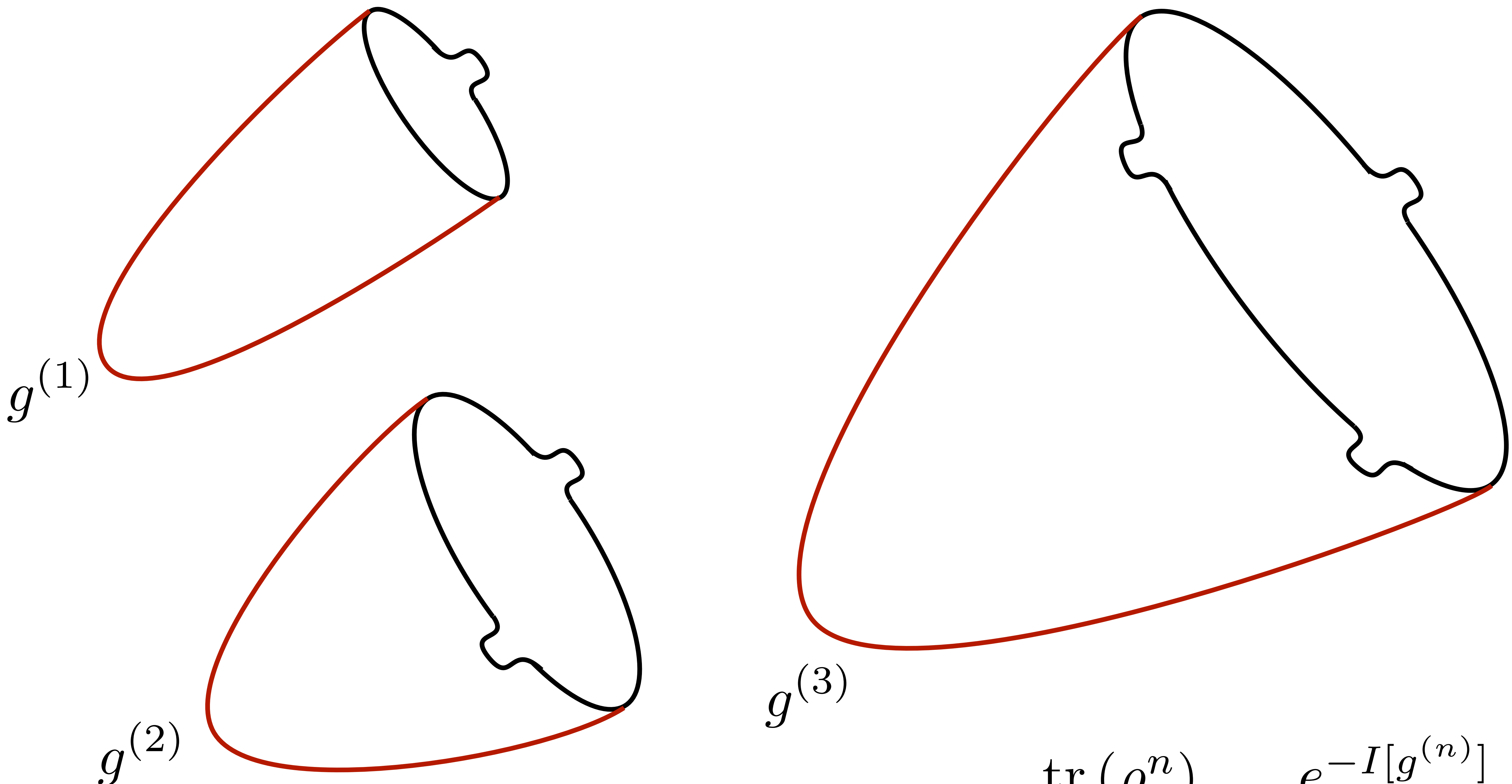




$$\mathrm{tr}\rho^2 = \sum_i \sum_j \langle \psi_i|\rho|\psi_j\rangle\langle\psi_j|\rho|\psi_i\rangle$$

Gravity dual

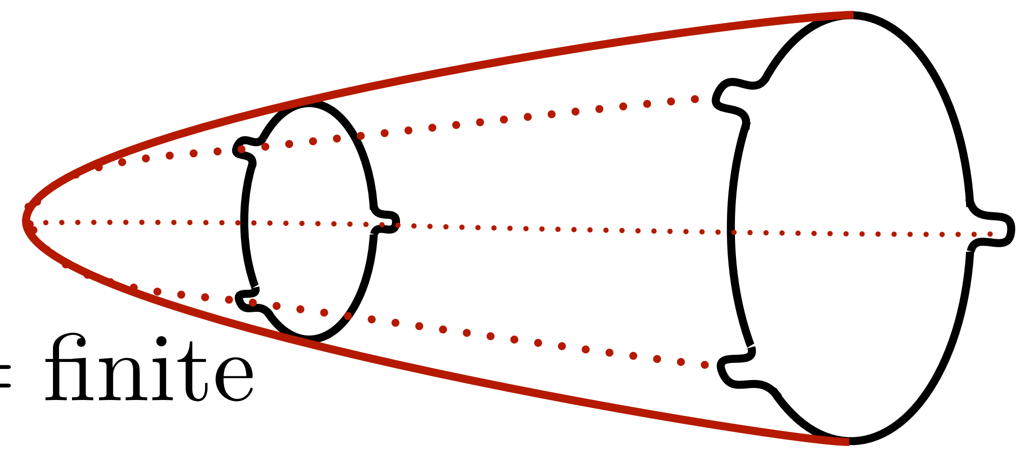


$$\frac{\mathrm{tr}\,(\rho^n)}{(\mathrm{tr}\rho)^n} \approx \frac{e^{-I[g^{(n)}]}}{e^{-nI[g^{(1)}]}}$$

Recap

- Replica trick $S = \lim_{n \to 1} S_n = -\, \partial_n \log \frac{\mathrm{tr}\,(\rho^n)}{(\mathrm{tr}\rho)^n}\bigg|_{n=1}$

- Holography

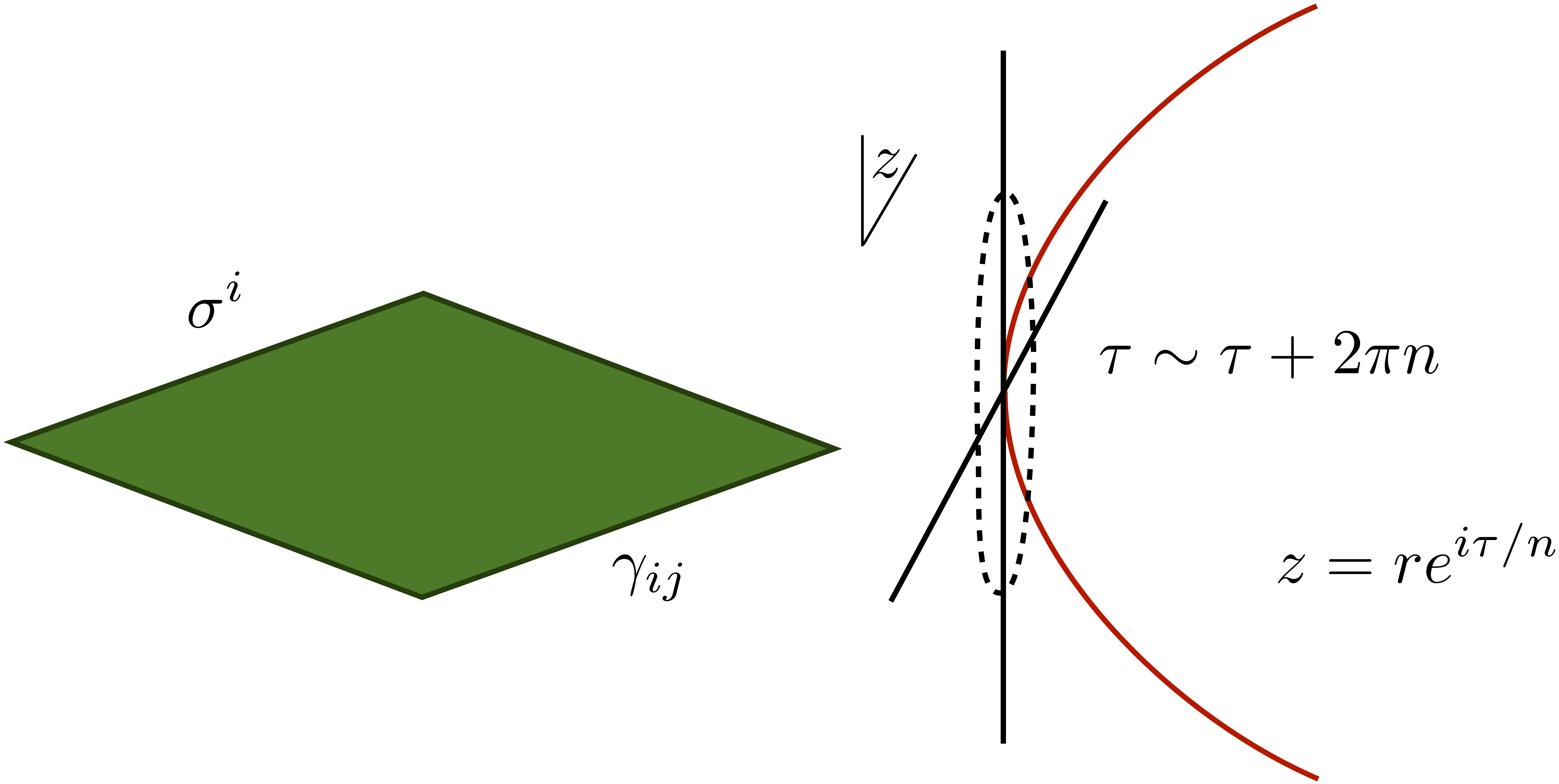


$$R_{\mu\nu}\left[g^{(n)}\right] = \text{finite}$$

- The calculation 'localises at the tip'

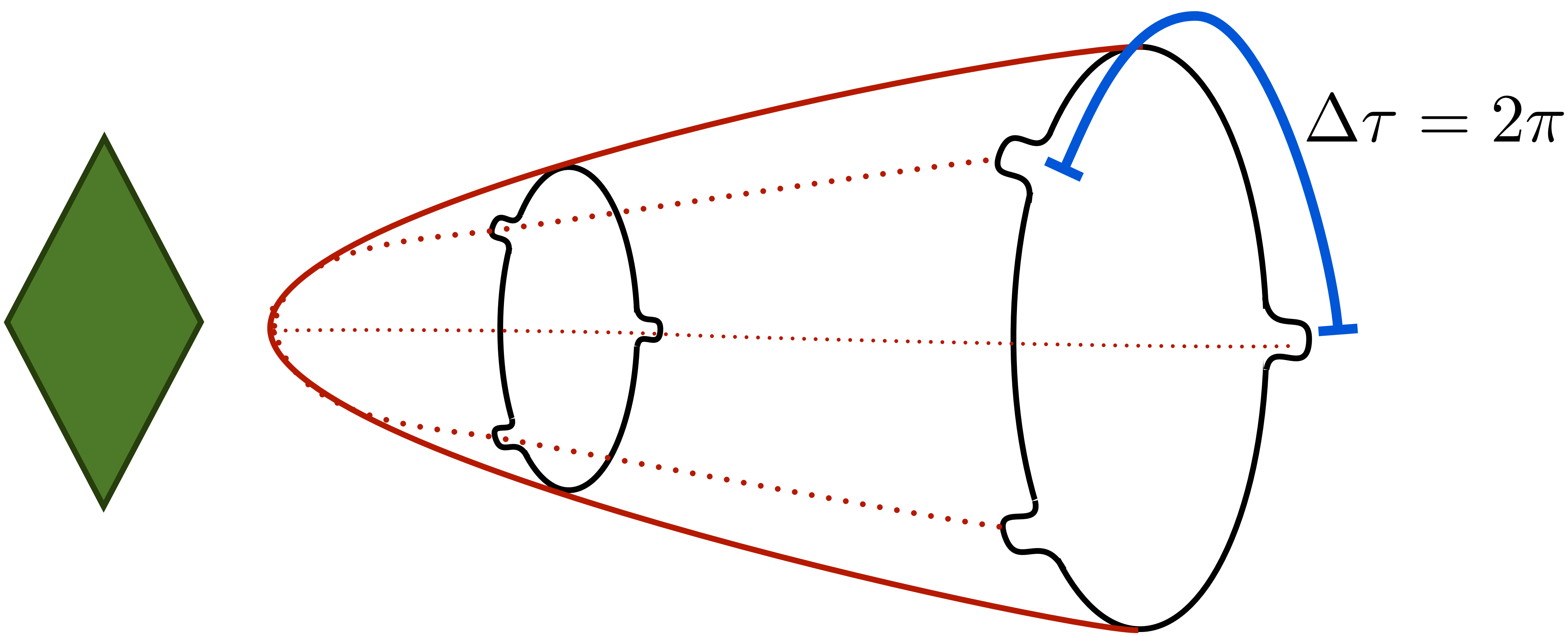
$$\mathbb{Z}_n$$

Zooming into the tip



$$g^{(n)} = dzd\bar{z} + \gamma_{ij}d\sigma^i d\sigma^j + \ldots$$

$\mathbb{Z}_n$ symmetry



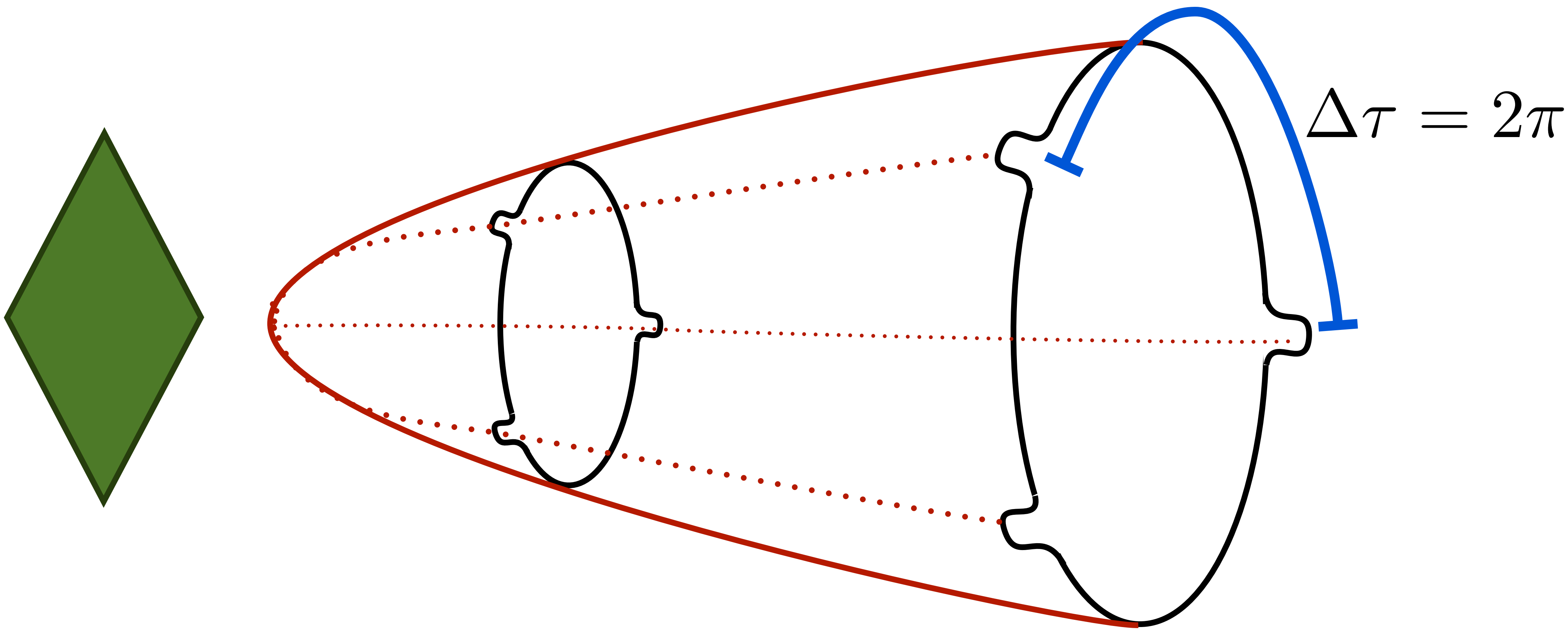
$$\tau \to \tau + 2\pi s$$

$$z = re^{i\tau/n} \to z\, e^{i2\pi s/n}$$

z^n

$z\bar{z}$

$\mathbb{Z}_n$ symmetry



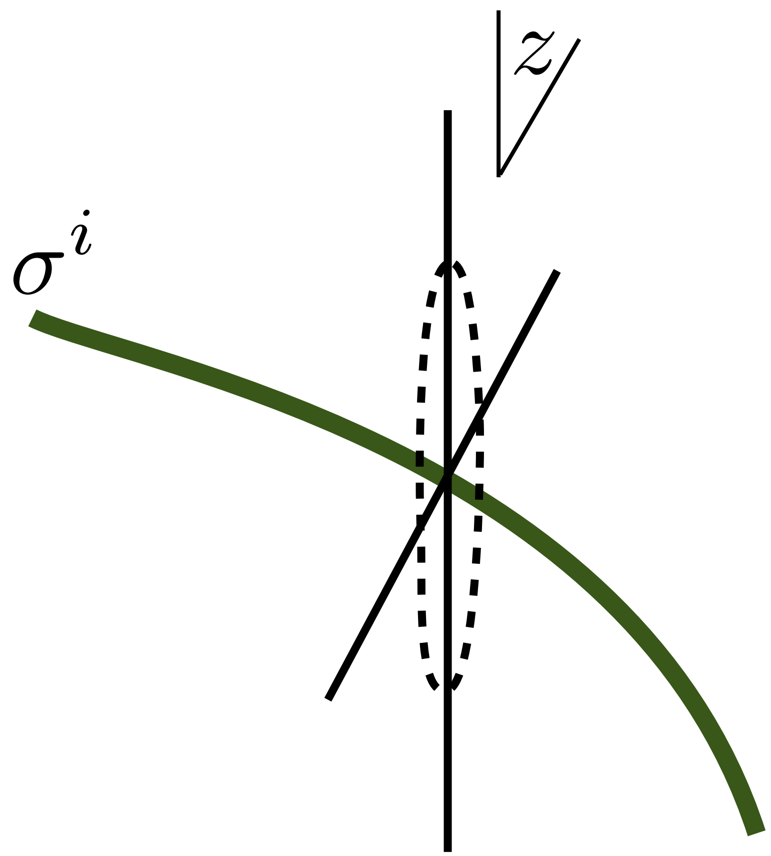
$$g^{(n)} = dz d\bar{z} + \gamma_{ij} d\sigma^i d\sigma^j + O(z^n) + O(z\bar{z})$$

$n \sim 1$: $O(z^n) \to O(z)$, $O(z\bar{z}) \to O(z^2)$

z^n

$z\bar{z}$

Interlude: Geometry of surfaces



$$g = dz d\bar{z} + \gamma_{ij} d\sigma^i d\sigma^j + O(z) + O(z^2)$$

$O(z)$: $1/\ell$

Extrinsic curvature	$K_{ijz} z \, d\sigma^i d\sigma^j$
Torsion of curve	$A_{iz\bar{z}}(\bar{z}dz - z d\bar{z}) d\sigma^i$
Gauge Christoffels	$\Gamma^z_{zz} z \, d\bar{z} \, dz$

$O(z^2)$: $1/\ell^2$ → $R_{\mu\nu\rho\sigma}$

Properties of $\mathbb{Z}_n$ metrics

$$g^{(n)} = dzd\bar{z} + \left(\gamma_{ij} + 2K_{ijz}z^n\right) d\sigma^i d\sigma^j + \dots$$

- Explicitly regular at finite n ✓
- At finite $n > 1$, $z = 0$ has no extrinsic curvature. At $n = 1$, K_{ijz} ✓
- Think about it in a double expansion

$$z \sim 0 \qquad n \sim 1$$

Properties of $\mathbb{Z}_n$ metrics

$$g^{(n)} = dz d\bar{z} + \left(\gamma_{ij} + 2K_{ijz} z^n\right) d\sigma^i d\sigma^j + \ldots$$

$$\mathrm{Riem} \sim \partial_z^2 g^{(n)}$$

$$R_{zz} = -(n-1) \frac{\gamma^{ij} K_{ijz}}{z} z^{n-1} + \ldots$$

Singular stress tensor for $n \sim 1$ unless $\boxed{\gamma^{ij} K_{ijz} = 0}$

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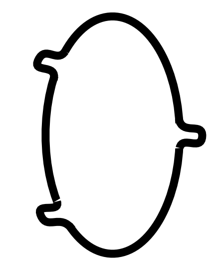
Caveats

- These metrics are not single-valued for non-integer n (just as z^n is not).
- While non-single valuedness comes as a branch cut, in the Riemann tensor we see issues at just one point ($z = 0$).
- This is related to the meaning of $\mathbb{Z}_n$
- But we can calculate...

Refinements

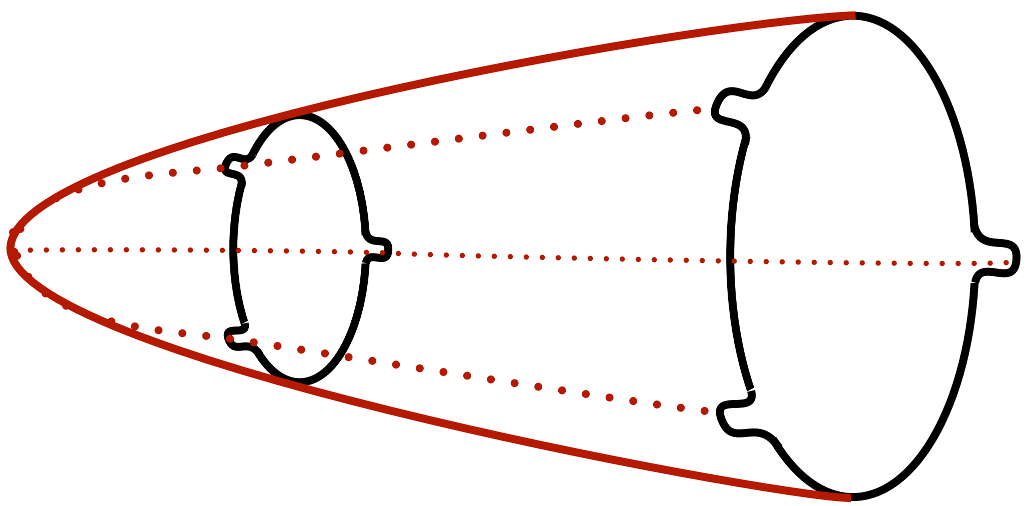
RSB: discussion

- By definition, the boundary is replica symmetric



- The bulk could be not, ~~$\mathbb{Z}_n$~~ , (SSB)
- The tip of the cigar can perhaps be defined as the fixed point of a residual symmetry
- However, recall we want to work to $O(n-1)$

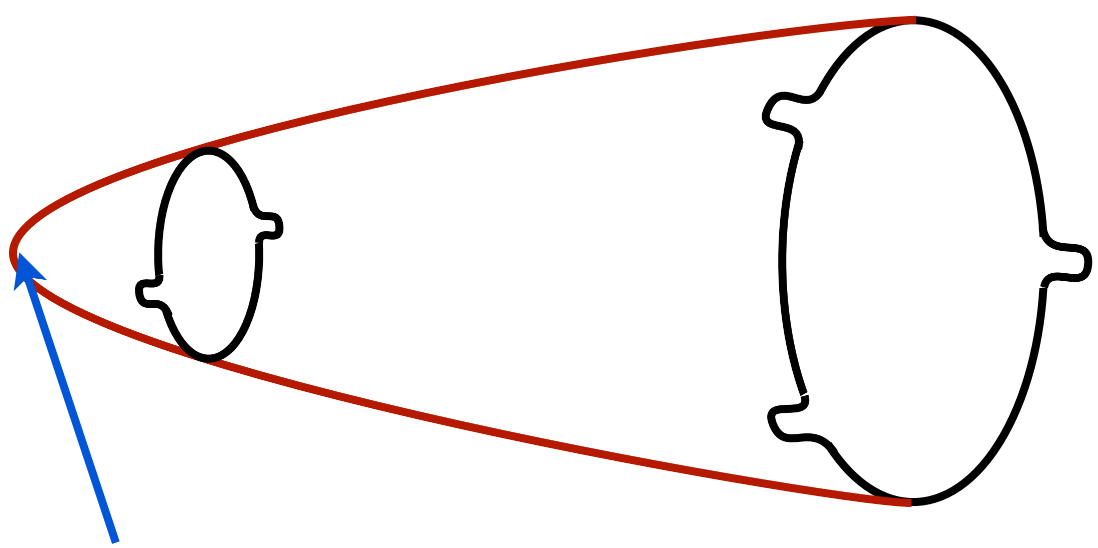
RSB terms



z^n

$z \to z\, e^{i2\pi \frac{s}{n}}$

$\boxed{\mathbb{Z}_n}$



Fixed point of residual symmetry

$z^{1+m(n-1)}$

$z \to z\, e^{i2\pi \frac{s}{1+m(n-1)}}$

$\boxed{\mathbb{Z}_{1+m(n-1)}}$

RSB terms

$$K_{ijz} z^n \to \sum_{m>0} K^{(m)}_{ijz}\, z\, z^{m(n-1)}$$

- we forbid contributions $\propto z\, d\sigma^i d\sigma^j$, that do not preserve any 'subsymmetry' ✓
- Explicitly regular at finite n ✓
- At finite $n > 1$, $z = 0$ has no extrinsic curvature. At $n = 1$, $K_{ijz} = \sum_{m>0} K^{(m)}_{ijz}$

Geometrically located, even when RSB

RSB terms

$$K_{ijz} z^n \to \sum_{m>0} K^{(m)}_{ijz}\, z\, z^{m(n-1)}$$

$$R_{zz} \sim \frac{n-1}{z} \sum_{m>0} m\, \gamma^{ij} K^{(m)}_{ijz}\, z^{m(n-1)}$$

$$\gamma^{ij} K^{(m)}_{ijz} = 0 \quad \longrightarrow \quad \gamma^{ij} \sum_{m>0} K^{(m)}_{ijz} = 0$$

RSB terms

$$K_{ijz} z^n \to \sum_{m>0} K^{(m)}_{ijz}\, z\, z^{m(n-1)}$$

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RT

$$\gamma^{ij} K^{(m)}_{ijz} = 0 \quad \longrightarrow \quad \gamma^{ij} \sum_{m>0} K^{(m)}_{ijz} = 0$$

Further refinements

- Essential for higher derivative gravity
- Crucial to allow for terms that are pure gauge at $n \sim 1$, e.g.

$$g^{(n)} = dz\, d\bar{z} + \sum_m \Gamma^{(m)}_{\bar{z}\bar{z}z}\, z\, (z\bar{z})^{m(n-1)} dz\, d\bar{z} + \ldots \checkmark$$

- All components $R_{\mu\nu} - \alpha' H_{\mu\nu} = \text{finite}$ $\boxed{\delta S = 0}$
- RSB allowed as before

Summary

- Analytically continue $g^{(n)}$ in n, expand around $n \sim 1$

- Allow all terms in $g^{(n)}$ consistent with hypotheses (RSB, pure gauge at $n = 1$)
- RT and generalisations follow: $\boxed{\delta S = 0}$