Interactions in Vasiliev's Theory, Locality & Holography

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Based on:
arXiv:1508.04764 w. E. Skvortsov
arXiv:1508.04139 w. N. Boulanger, P. Kessel, E. Skvortsov
arXiv:1505.05887 w. G. Lucena Gómez, P. Kessel, E. Skvortsov

HS-holography: Status & Motivations

Singlet Sector of O(N)/U(n) Vector Model $\Longleftrightarrow$ Bosonic Vasiliev's theory

Klebanov-Polyakov, Sezgin-Sundell (2002), Leigh-Petkou (2003)

Many highly non-trivial checks so far:

- Some 3-pt functions from Vasiliev's equations (Giombi & Yin 2010):

$$\widetilde{D}C = \omega \star C - C \star \pi(\omega) + \dots$$

- 1-loop tests (Giombi, Klebanov, Tseytlin et al.): free energy, Casimir energy etc.

...but also some puzzles:

- Couplings beyond HS structure constants were subtle (Giombi & Yin 2011, Colombo & Sundell 2012):

$$\widetilde{D}C = \omega \star C - C \star \pi(\omega) + \boxed{\mathcal{V}(\Omega, C, C)}$$

- 1-loop tests so far rely on particle content (no need of fully non-linear theory)

Goals & Plan

The Goal:

Extract from Vasiliev's equations the first non-linear corrections to the Fronsdal equations.

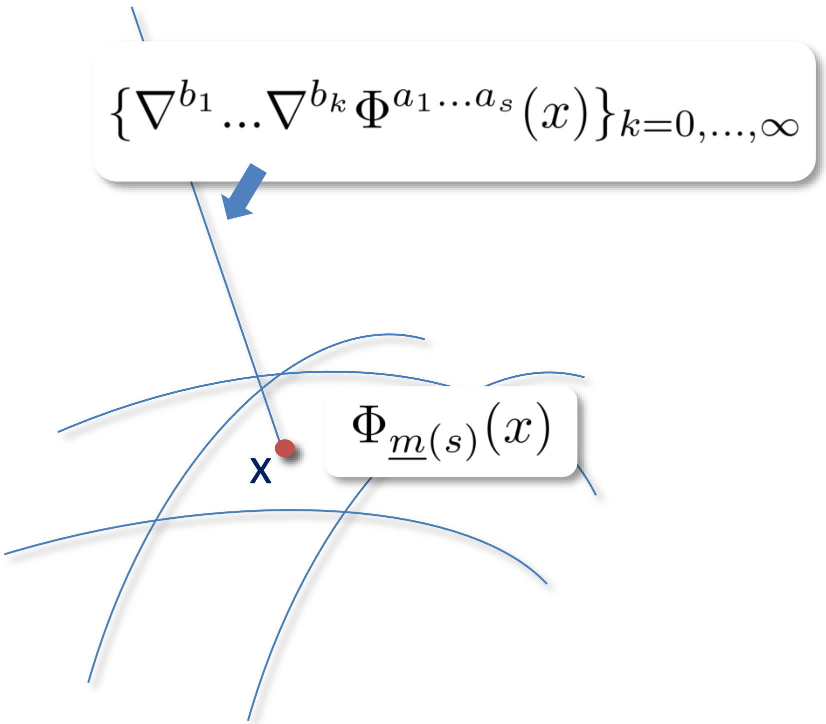
We want to extract the numbers in front of Metsaev's couplings(!)

Plan of the talk:

- 4d Vasiliev's Lorentz covariant perturbation theory
- Dictionary between unfolded formalism and Fronsdal's equations
- Improve our understanding of non-locality in Vasiliev's theory

$$\Box\Phi_{\underline{m}(s)}+\ldots=\sum_{k,l}a_{k,l}\Lambda^{-l}\nabla_{\underline{m}(s-k)\underline{n}(l)}\Phi\,\nabla_{\underline{m}(k)}{}^{\underline{n}(l)}\Phi+\ldots$$

Unfolding and Jet space



Jet Bundle: each field and all of its derivatives as independent coordinates

It can be decomposed as:

- Gauge invariant components
- Gauge dependent components
- Components set to zero on the on-shell surface

- Gauge dependent components encoded into 1-forms: $\delta\omega = d\Lambda + \ldots$
- Gauge invariant components encoded into 0-forms

$e^a \, , \omega^{ab}$ (frame-like)

$W_{\mu\nu\rho\sigma} \, , \nabla_\rho W_{\mu\nu\rho\sigma} \, , \ldots$

(Weyl and derivatives)

Ingredients

- 4d isomorphism: $so(3,2) \sim sp(4,\mathbb{R})$

The HS algebra is conveniently formulated in the sp(4) *oscillator* language (Vasiliev 1988)

$$[\hat{Y}^A, \hat{Y}^B] = 2iC^{AB} \qquad Y_A = (y_\alpha, \bar{y}_{\dot{\alpha}})$$

$$sp(4): \quad T^{AB} = -\frac{i}{4}\{\hat{Y}^A, \hat{Y}^B\}$$

- Master fields:

$$\boxed{f \in hs : f(\hat{Y}) \sim f_{\alpha(n_1)\dot{\alpha}(n_2)} y^{\alpha(n_1)} \bar{y}^{\alpha(n_2)}\,, \qquad f(-\hat{Y}) = f(\hat{Y})}$$

- Operator product realised by the Moyal star-product

1-forms & 0-forms

Gauge dependent and gauge invariant components in the jet bundle form modules under the isometry group/HS algebra

- Gauge dependent component are represented by 1-forms (Vasiliev's 1980):

$$\omega^{(s)}(y,\bar{y}|x)=\sum_{k=0}^{s-1}\frac{1}{(s-1+k)!(s-1-k)!}\,\omega_{\alpha(s-1+k)\dot{\alpha}(s-1-k)}(x)y^{\alpha(s-1+k)}\bar{y}^{\dot{\alpha}(s-1-k)}$$

$$\omega_{\alpha(s-1+k)\dot{\alpha}(s-1-k)}\sim\nabla^k\Phi_{\alpha(s)}$$

- The gauge invariant components are instead HS Weyl tensor (anti-)selfdual components + the scalar (Weyl module)

$$C^{(s)}(y,\bar{y}|x)=\sum_{k=0}^{\infty}\frac{1}{(s+k)!k!}\,C_{\alpha(s+k)\dot{\alpha}(k)}(x)y^{\alpha(s+k)}\bar{y}^{\dot{\alpha}(k)}\qquad C_{\alpha(s+k)\dot{\alpha}(k)}\sim\nabla^k C_{\alpha(s)}$$

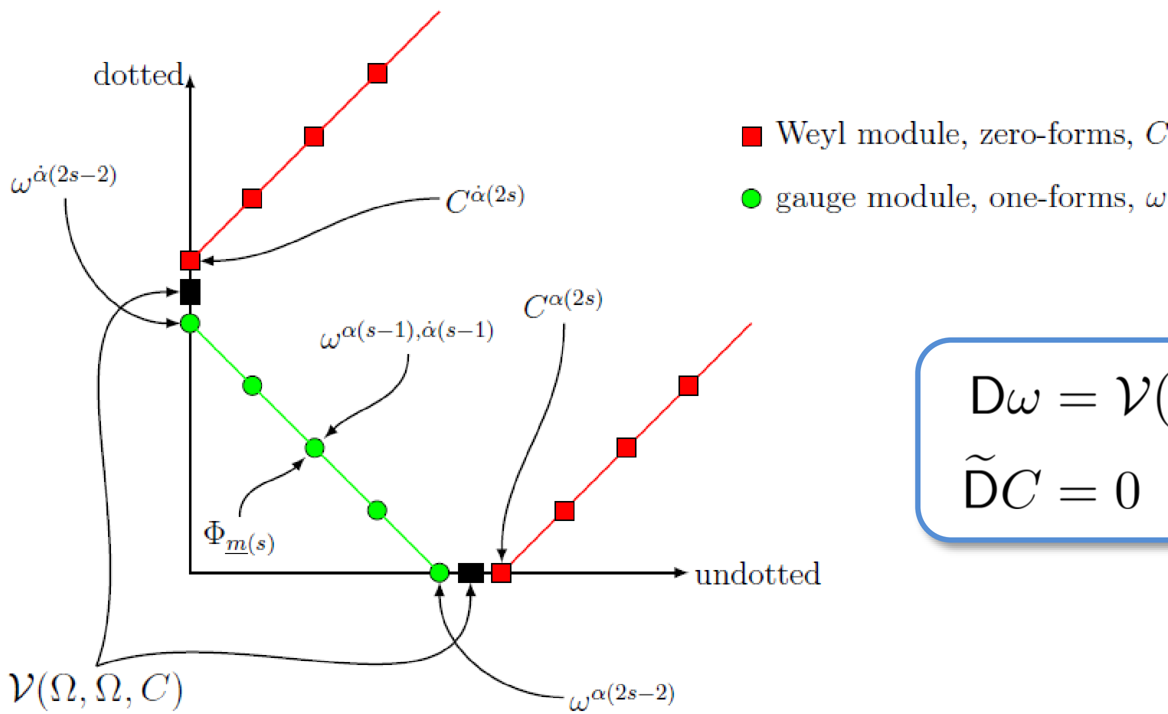
Backgrounds

- Vacuum solution = flat sp(4) Connection of HS algebra (C=0)

$$d\Omega(Y|x) = \Omega(Y|x) \star \Omega(Y|x) \longrightarrow \Omega = \frac{1}{2}\varpi^{\alpha\alpha}L_{\alpha\alpha} + h^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}} + \frac{1}{2}\varpi^{\dot{\alpha}\dot{\alpha}}\bar{L}_{\dot{\alpha}\dot{\alpha}}$$

Vacuum AdS$_4$ solution

- Linear eoms can be written in terms of covariant derivatives on the jet bundle and read as *covariant (non-)constancy* conditions (BRST-like)



$$\mathsf{D} = \nabla - h^{\alpha\dot{\alpha}}[P_{\alpha\dot{\alpha}}, \bullet]_\star$$

$$\widetilde{\mathsf{D}} = \nabla + ih^{\alpha\dot{\alpha}}\{P_{\alpha\dot{\alpha}}, \bullet\}_\star$$

$$\mathsf{D}\omega = \mathcal{V}(\Omega,\Omega,C) \qquad \delta\omega = \mathsf{D}\xi$$

$$\widetilde{\mathsf{D}}C = 0 \qquad \delta C = 0$$

Vasiliev '88

Unfolding and Interactions

- From free theories to interacting theories Vasiliev's idea is:

$$\begin{cases} d\boldsymbol{\omega} = F^{\omega}(\boldsymbol{\omega}, \mathbf{C}) \\ d\mathbf{C} = F^{C}(\boldsymbol{\omega}, \mathbf{C}) \end{cases}$$

$$\boxed{d^2 = 0}$$

$$F^{\omega}(\boldsymbol{\omega}, \mathbf{C}) = \mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}) + \mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}, \mathbf{C}) + \mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}, \mathbf{C}, \mathbf{C}) + \ldots$$

$$F^{C}(\boldsymbol{\omega}, \mathbf{C}) = \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}) + \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}, \mathbf{C}) + \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}, \mathbf{C}, \mathbf{C}) + \ldots$$

C-expansion

- First coupling governed by HS algebra:

$$\mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}) = \boldsymbol{\omega} \star \boldsymbol{\omega} \qquad\qquad \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}) = \boldsymbol{\omega} \star \mathbf{C} - \mathbf{C} \star \pi(\boldsymbol{\omega})$$

- The system is endowed with fully non-linear gauge symmetries
- "Analogous" to String Field Theory

$$d \leftrightarrow Q$$
$$\star \leftrightarrow \text{Witten product}$$

Goal: Expand Vasiliev's equations and study the above couplings at second order

Vasiliev's Equations

The couplings are resummed in Vasiliev's equations with the help of an additional Z oscillator and new auxiliary fields S:

$$W = W_{\underline{m}}(Y,Z|x)\,dx^{\underline{m}} \qquad B = B(Y,Z|x) \qquad S_A = S_A(Y,Z|x)$$

Vasiliev 1990

(well known) subtleties:

- The Lorentz generators acquire field-dependent corrections in the Schwinger-Fock gauge (open algebra):

$$\omega^{\alpha\alpha} L^{\text{full}}_{\alpha\alpha} = \omega^{\alpha\alpha}\left(L^{\text{yz}}_{\alpha\alpha} - \frac{i}{4}\{S_\alpha, S_\alpha\}_\star\right)$$

Vasiliev 1999

Lorentz Covariant Perturbation Theory

Closed subsector of differential equations along the Z directions:

$$S_A \to Z_A + A_A \qquad W \to \widehat{\Omega}(\mathcal{S}) + W$$

fluctuations

Perturb around the vacuum

One obtains a set of recursive solutions for the B, W and A Z-dependence:

$$\begin{cases} B = C(Y) + \partial_Z^{-1}(\text{quadratic}) \\ A = \partial_Z^{-1}(\text{quadratic}) + \boxed{\partial_Z \epsilon} \\ W = \omega(Y) - \partial_Z^{-1}(\text{quadratic}) \end{cases}$$

$\Rightarrow$ Gauge Ambiguity!

Any choice gives a consistent unfolded system in Y variables

Need of gauge choice: simplest one is the Schwinger-Fock gauge

$$Z^A A_A = 0$$

$$\partial_A \epsilon = 0 \to \epsilon(Y, Z) \sim \epsilon(Y)$$

...but non-linear deformations in C are possible (!)

Physical Eqs. to the 2nd Order

$$\left\{\begin{aligned} \mathsf{D}\omega^{(2)} - \mathcal{V}(\Omega,\Omega,C^{(2)}) &= \omega\star\omega + \mathcal{V}(\Omega,\omega,C) + \mathcal{V}(\Omega,\Omega,C,C) \\ \widetilde{\mathsf{D}}C^{(2)} &= \omega\star C - C\star\pi(\omega) + \mathcal{V}(\Omega,C,C) \end{aligned}\right.$$

HS algebra in terms of $*$ product while we concentrate on the quadratic terms in C:

generating Kernel

$$\boxed{\mathcal{V}(\Omega,\Omega,C,C) = \int d^4\xi\, d^4\eta\, \mathbf{J}(Y,\xi,\eta)\, C(\xi|x)C(\eta|x)}$$

$$\mathbf{J}(Y,\xi,\eta) = \underbrace{\int_0^1 dt \int_0^1 dq}_{\text{homotopy integrals}} \Big((h\wedge h)^{\alpha\alpha}(y+\ldots)_\alpha(y+\ldots)_\alpha Q_1\Big(iq^2t^2+\ldots\Big) + 3 \text{ terms} + h.c.\Big)$$

$$Q_1 = \exp i\left((qt(y+\eta)(y+\xi) + (\bar{y}-\bar{\eta})(\bar{y}+\bar{\xi}) + 2\theta\right)$$

- Factorially damped coefficients (no 1/□)
- Almost star-product! (e.g. String Field Theory)

$$C\star\pi(C)$$

$$\alpha^\mu_{-1} \leftrightarrow (y_\alpha, \bar{y}_{\dot\alpha})$$

Vasiliev vs. Fronsdal

Unfolded equations are over-complete set of equations written in terms of an over-complete set of dynamical variables

- The only dynamical variable is the Fronsdal field

$$\omega_0 = e_{\alpha(s-1)\dot\alpha(s-1)} y^{\alpha(s-1)} \bar{y}^{\dot\alpha(s-1)} = h^{\alpha\dot\alpha}\partial_\alpha\partial_{\dot\alpha}\phi + y^\alpha \bar{y}^{\dot\alpha} h_{\alpha\dot\alpha}\phi' + \text{Stueckelberg}$$

$$\omega_{\pm k} = \omega_{\alpha(s-1\pm k)\dot\alpha(s-1\mp k)}$$

$$\mathsf{D} = \nabla + Q_+ + Q_-$$

Grading with respect to $\dfrac{\#y - \#\bar{y}}{2} = k$

- The only dynamical equation are the Fronsdal equations, all the others are differential consequences thereof

Like in SUGRA we need to solve torsion to get the physical equations:

$$\boxed{\begin{aligned} \nabla\omega_0 + Q_+\omega_{-1} + Q_-\omega_{+1} &= \mathbf{J}_0 \\ \nabla\omega_{+1} + Q_+ e + Q_-\omega_{+2} &= \mathbf{J}_{+1} \end{aligned}}$$

Vasiliev vs. Fronsdal

- Torsion:

$$\nabla\omega_0 + Q_+\omega_{-1} + Q_-\omega_{+1} = \mathbf{J}_0 \quad \Rightarrow \quad \omega_{+1} = (Q_-)^{-1}\left(-\nabla\omega_0 + J_0\right)$$

- Plug back into ω_1 equation and look only at Fronsdal components:

$$\underbrace{\Big[-\nabla(Q_-)^{-1}\nabla e + Q_+ e\Big]_F}_{\text{Gives the Fronsdal tensor in metric-like language}} = \boxed{\left[(1-\nabla(Q_-)^{-1})\mathbf{J}\right]\Big|_{+1,F}}$$

Gives the Fronsdal tensor in metric-like language

Gives the non-linear source to Fronsdal

- Similar torsion eqs. arise for Weyl tensors – they acquire non-linear corrections

We get the explicit Fronsdal Currents

Examples: 0-0-2

The source to the linearised Einstein tensor reads:

$$\Box \Phi_{\underline{mm}} + \ldots = 2 \boxed{\cos 2\theta} \sum_l \Big(\tilde{a}_{l,1} \nabla_{\underline{mn(l)}} \Phi \nabla_{\underline{m}}{}^{\underline{n}(l)} \Phi$$

$$+ 2\tilde{a}_{l,0} \nabla_{\underline{mmn(l)}} \Phi \nabla^{\underline{n}(l)} \Phi + \tilde{c}_{l,0} g_{\underline{mm}} \nabla_{\underline{n}(l)} \Phi \nabla^{\underline{n}(l)} \Phi \Big)$$

$$a_{l,0} = \frac{1}{l!l!} \left(-\frac{3}{(2+l)^2} + \frac{7}{2(2+l)} - \frac{4}{3+l} + \frac{1}{2(4+l)} \right)$$

$$a_{l,1} = \frac{1}{l!l!} \left(\frac{1}{2(2+l)^2} - \frac{1}{4(2+l)} + \frac{1}{4(4+l)} \right)$$

$$c_{l,0} = \frac{1}{l!l!} \left(\frac{1}{12(1+l)^2} - \frac{3}{8(1+l)} + \frac{1}{2+l} - \frac{1}{8(3+l)} \right)$$

$$a_{l,0} l!l! \xrightarrow{l\to\infty} -\frac{2}{l^3}$$

$$a_{l,1} l!l! \xrightarrow{l\to\infty} \frac{1}{l^3}$$

$$c_{l,0} l!l! \xrightarrow{l\to\infty} \frac{1}{2l}$$

Some Comments

- θ enters 0-0-s via $\cos 2\theta$. The currents vanish for $\theta = \pi/4$ and the signs are opposite for A and B model. Puzzling in view of a slightly-broken HS symmetry (Maldacena-Zhiboedov)

- Fronsdal currents are pseudo-local expressions

$$\Box \Phi_{\underline{m}(s)} + \ldots = \sum_{k,l} \alpha_{k,l} \Lambda^{-l} \nabla_{\underline{m}(s-k)\underline{n}(l)} \Phi \, \nabla_{\underline{m}(k)}{}^{\underline{n}(l)} \Phi + \ldots$$

  while at cubic order all couplings have a number of derivatives bounded by the sum over spins. (Metsaev) – Not surprising: see string theory!

- HS-algebra couplings are local (!): comparing the source for spin-s and the source to s=0 we see that the equations are not Lagrangian.

Locality & Field Redefinitions

- To bring the equations in their "Metsaev" frame needs a pseudo-local redefinition
- Pseudo-local redefinitions are not harmless... can remove the full current itself (!)

Canonical currents are exact in cohomology $\Rightarrow$

$$(\Phi^* \nabla^s \Phi) \sim D \left(\sum_l^\infty \Box^l \dots \right)$$

Projushkin-Vasiliev '99

Extending this we find:

$$\boxed{\mathbb{H}^2(D, CC) = \emptyset}$$

Kessel, Lucena-Gómez, Skvortsov & M.T. '15

AdS counterpart of the BRST-BV result in flat space by *Barnich & Henneaux 1993*

Without a proper definition of locality physical couplings become trivial...

Locality: the Flat Space Example

- The simplest example of pseudo-local tail is flat space:

$$\Box \Phi_{\underline{m}(s)} + \ldots = \sum_{k,l} a_{k,l} (\alpha')^l \partial_{\underline{m}(s-k)\underline{n}(l)} \Phi \, \partial_{\underline{m}(k)}{}^{\underline{n}(l)} \Phi + \ldots$$

$$\Downarrow$$

$$\delta^{(d)}(p_1 + p_2 + p_3)\,(p_2 \cdot p_3)^l (p_2 - p_3)_{\underline{m}(s)}\, \phi^{\underline{m}(s)}(p_1) \Phi^*(p_2) \Phi(p_3)$$

- Using momentum conservation and mass-shell:

$$(p_2 \cdot p_3)^l = \left[\frac{1}{2}(p_1^2 - p_2^2 - p_3^2)\right]^l = \left[\frac{1}{2}(m_1^2 - m_2^2 - m_3^2)\right]^l + \mathrm{Impr.}$$

- In the massless case each higher-derivative term *is* an improvement
- In the massive case this is not so, but one can classify all improvements(!) – *fix an orthogonal basis for currents*.

Current = Canonical + Improvements

Locality: String Field Theory

0-0-s vertex in open string theory is also pseudo-local!

$$\mathcal{V}_3 \sim |y_{12}y_{13}y_{23}| \exp\left[\alpha' \sum_{i\neq j} p_i \cdot p_j \ln|y_{ij}| - \frac{\sqrt{2\alpha'}}{y_{31}} p_3 \cdot \alpha_{+1}\right] \Phi^\star(p_2)\Phi(p_3)\Phi_{\underline{m}(s)}(p_1)\alpha_{-1}^{\underline{m}(s)}$$

One can split the latter in improvements and canonical currents:

s=1:

$$J^{\underline{m}} \sim \exp\left(-\alpha'\Box \ln\left|\frac{y_{23}}{y_{31}y_{12}}\right|\right) \Phi^\star \overleftrightarrow{\partial}^{\underline{m}}\Phi = \Phi^\star \overleftrightarrow{\partial}^{\underline{m}}\Phi - \alpha'\left(\ln\left|\frac{y_{23}}{y_{31}y_{12}}\right|\right)\underbrace{\Box\left(\Phi^\star \overleftrightarrow{\partial}^{\underline{m}}\Phi\right)} + \dots$$

- Canonical current comes out with a uniquely fixed coefficient compatible with Virasoro
- All infinite tail contributes! Otherwise y-dependence would not cancel out...

$$J^{\underline{m}(s)} \sim \frac{1}{s!}\Phi^\star \overleftrightarrow{\partial}^{\underline{m}(s)}\Phi + \mathrm{Impr.}$$

M.T. 2010, Sagnotti & M.T. 2010

Need to project the coupling into orthogonal basis

Locality: a General Criterion

In AdS we have a similar decomposition in canonical and improvement structures:

$$\sum_q \alpha^l_{s,q}\,(g^{\underline{nn}})^l \nabla_{\underline{m}(s-q)\underline{n}(l)}\Phi^*\nabla_{\underline{m}(q)\underline{n}(l)}\Phi + \mathcal{O}(\Lambda g_{\underline{mm}}) \approx C^{(s)}_{l+1}(\alpha) J^{\text{can.}}_{\underline{m}(s)} + J^{\text{impr.}}_{\underline{m}(s)}$$

Some facts one can prove:

- There exist a basis that at this order is only made of local elements (as flat)
- Only few elements cannot be removed by ***a local*** redefinition, "primary" couplings – #∂ < s1+s2+s3
- Applying this projection to Vasiliev's backreaction we get (similarly as in ST):

$$J^{\underline{m}(s)}_{\text{Vas.}} = \left(\sum_{l=1}^{\infty} C^{(s)}_l(\alpha)\right) J^{\underline{m}(s)}_{\text{can.}} + \text{Improvements}$$

$$\boxed{C_l \sim l^{2s}(l!)^2}$$

Grows very fast with *l*

Skvortsov & M.T. 2015

- No redefinition involved(!)

see also Gelfond-Vasiliev 2010

Resumming Vasiliev's Backreaction

For our result the resummation produces infinite series:

$$\frac{\cos(2\theta)}{12}\left(\sum_{n=1}^{\infty} l\right) J^{\text{can.}}_{s=2}$$

The series diverge as l^{2s-3}

$$\frac{\cos(2\theta)}{3\cdot 7!}\left(\sum_{l=1}^{\infty}\frac{l(l+1)(l+2)^2(3l+11)(5l(l+4)+3)}{(l+3)(l+4)}\right) J^{\text{can.}}_{s=4}$$

$$\frac{3\cos(2\theta)}{5\cdot 11!}\left(\sum_{l=1}^{\infty}\frac{(l+4)!}{(l-1)!}\frac{(l+3)(5l+28)(7l(l+6)(3l(l+6)+19)+20)}{(l+5)(l+6)}\right) J^{\text{can.}}_{s=6}$$

Skvortsov & M.T. 2015

Possible ways out:

$$\Phi = \sum \mathrm{Tr}(\omega_0 \star \ldots \star \omega_0) + \mathrm{Tr}(\omega_0 \star \ldots \star \omega_0 \star C \star \ldots \star C)$$

$$W(Y,Z) \quad \supset \quad \omega(Y) \quad \supset \quad \Phi(Y)$$

$$\tfrac{1}{2}\omega^{\alpha\alpha}\left(L^{\text{yz}}_{\alpha\alpha} - \frac{i}{4}\{\mathcal{S}_\alpha, \mathcal{S}_\alpha\}\right) \rightarrow \tfrac{1}{2}\left(\omega + hCC + \ldots\right)^{\alpha\alpha}\left(L^{\text{yz}}_{\alpha\alpha} - \frac{i}{4}\{\mathcal{S}_\alpha, \mathcal{S}_\alpha\}\right)$$

Summary & Outlook

- We have extracted all vertices from Vasiliev's equations to the lowest non-trivial order in the SF gauge.
- The result is pseudo-local for a large class of vertices (as in ST).
- Projecting the result in an orthogonal basis of "primary" canonical currents and Improvements the coefficient of the "primary" canonical component is divergent – overall coefficients undetermined...
- This is in agreement and sheds light on the puzzles observed by Giombi & Yin. The divergent contribution is the overall coefficient of the "primary" canonical coupling.
- *(New) Puzzle:* how to fix the gauge ambiguity in Vasiliev's equations in a way that the unfolded equations one extracts are local – HS theory should be an "ordinary" field theory (like ST) – see Charlotte's talk.

Locality: the Result

- Having defined our functional class criterion, what needs to be done is to compute the coefficients C_l

Idea:

$$\Box^l - \mathbf{j}(\mathrm{D}\Delta\omega^{(2)}) = \boxed{C_l}\,\Box^0$$

Solved for by requiring all Box-term to be absent

Generic l-contracted derivative term

Generic current produced by a local redefinition

- The explicit form of C_l can be computed explicitly and is given in terms of hypergeometric functions (1508.04764). Below I give its asymptotics in 4d

- One may call a backreaction local if (notice that this has nothing to do with redefinitions):

$$\Box\Phi_{\underline{m}(s)} + \ldots = \sum_{l,q} \alpha^l_{s,q}\,(g^{\underline{nn}})^l \nabla_{\underline{m}(s-q)\underline{n}(l)}\Phi^*\nabla_{\underline{m}(q)\underline{n}(l)}\Phi + \mathcal{O}(\Lambda g_{\underline{mm}})$$

$$C_l \sim l^{2s}(l!)^2 \quad \Longrightarrow \quad \alpha^l_{s,q} \prec \frac{1}{l^{2s+1}}\left(\frac{1}{l!}\right)^2$$

Vasiliev Theory and Witten Diagrams

$$\langle \ldots \rangle \sim \int_{AdS_3} \mathrm{Tr}\,[\omega \star J] = \int_{AdS_3} \mathrm{Tr}\left[D\xi^{\mathbf{B.H.}} \star J\right] = \lim_{z\to 0} \int_{\partial AdS_3} \mathrm{Tr}\left[\boxed{\xi^{\mathbf{B.H.}}} \star J\right]$$

Brown-Henneaux boundary behavior

We have computed the Witten diagram associated to: $\Rightarrow$ $\Box^l J^{\mathrm{can.}}$

$$\left(\frac{z}{x^+x^- + z^2}\right)^{\lambda+s+l} \sim \sum_{k=0}^{s+l} \# z^{2-\lambda-s-l+2k} (\partial_{x^+}\partial_{x^-})^k \delta^{(2)}(x)$$

$$\boxed{\langle \ldots \rangle \sim \left(\sum_l a_l^{(s)}\, C_l^{(s)}\right) \frac{1}{|x_{12}|} \left(\frac{x_{12}^+}{x_{31}^+ x_{23}^+}\right)^s}$$

Puzzle: the naive field theory computation fails $\overset{?}{\Rightarrow}$ $\int_{AdS} \sum_l \neq \sum_l \int_{AdS}$

Can we analytically continue the divergent sum?

Comments on the Result

- From the vertex just shown one can extract the gauge invariant part or the HS stress tensor:

$$\mathbf{J}^{s.t.} = \int_0^1 dt \int_0^1 dq \Big[H^{\alpha\alpha}(y+\xi)_\alpha (y+\eta)_\alpha Q_1 \Big(iq^2t^2 + (\bar{\xi}\bar{\eta}) \frac{qt(1-qt)}{2} \Big) + \\ - \frac{i}{2} H^{\dot{\alpha}\dot{\alpha}} \bar{\xi}_{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}} \Big(Q_1 - (1-t)P_1 \Big) + h.c. \Big]$$

$$\mathsf{D}\mathbf{J}^{s.t.} \approx 0$$

- The above decomposes into independently conserved subsectors corresponding to canonical currents and improvements (see also Gelfond-Vasiliev 2010)

$$\mathbf{J}^{can.} = \int_0^1 dt \int_0^1 dq \Big[\frac{1}{4} H^{\alpha\alpha} \partial_\alpha \partial_\alpha Q_1 \Big(i + (\bar{\xi}\bar{\eta}) \frac{(1-qt)}{2qt} \Big) \\ + \frac{i}{8} H^{\dot{\alpha}\dot{\alpha}} \partial_{\dot{\alpha}} \partial_{\dot{\alpha}} \Big(Q_1 - (1-t)P_1 \Big) + h.c. \Big]$$

- $C^{(2)}$ is not the linearized Weyl tensor but a non-linear deformation thereof(!)

Fronsdal vs. Weyl

The equations for C encode first of all the s=0,1 dynamics plus the equations for the Weyl tensors obtained by s-curl of the Fronsdal equations

- We need to perform a similar procedure as for solving torsion(!)

$$\widetilde{\mathrm{D}}C^{(2)}(y,\bar{y}|x) = \mathbf{P}(y,\bar{y}|x) = h_{\gamma\dot{\gamma}}P^{\gamma\dot{\gamma}} \qquad \mathbf{P} = \omega\star C - C\star\pi(\omega) + \mathcal{V}(\Omega,C,C)$$

- Upon solving for $h^{\alpha\dot{\alpha}}\partial_\alpha\partial_{\dot{\alpha}}C$ in terms of $\mathbf{P} - \nabla C$ we get:

$$(\Box - 2(2+\bar{N}))C^{(2)}(0,\bar{y}) = (I - \nabla(\widetilde{Q}_-)^{-1})\mathbf{P}\Big|_{W,\,y=0}$$

- For the scalar the above equation encodes all the dynamics but for s>0 this is not true (!) [generalized torsion to be solved for at higher levels]

$$\mathrm{D}\omega^{(2)}_{s-1} = \mathbf{R}_{s-1} + \mathcal{V}(\Omega,\Omega,C^{(2)})_s \qquad \omega^{(2)}_k = (Q_-)^{-1}(\mathbf{R}_{k-1} - \nabla\omega^{(2)}_{k-1} - Q_+\omega^{(2)}_{k-2})$$

$$\Downarrow$$

$$\boxed{C^{(2)} = (\nabla^s\Phi) - 2e^{i\theta}\mathrm{R}^{\partial\partial}_{eff}\Big|_{\bar{y}=0} + 2e^{-i\theta}\bar{\mathrm{R}}^{\partial\partial}_{eff}\Big|_{y=0}}$$

A convenient basis

The decomposition of the backreaction in the various currents and improvements is most easily obtained up to a choice of basis:

Prokushkin-Vasiliev 1999

$$\begin{cases} C(y) = \int d^2\xi e^{iy\xi}\hat{C}(\xi) \\ J(y) \sim \int d^2\xi d^2\eta K(\xi,\eta,y)C(\xi)C(\eta) \end{cases}$$

Three possible tensor contraction can be defined:

$$\xi^\alpha\eta_\alpha \quad y^\alpha\eta_\alpha \quad y^\alpha\xi_\alpha$$

D is diagonal if we consider monomials of the type:

$$[y^\alpha(\xi+\eta)_\alpha]^n[y^\alpha(\xi-\eta)_\alpha]^m f(\xi^\alpha\eta_\alpha)$$

$$D\left([y^\alpha(\xi+\eta)_\alpha]^n[y^\alpha(\xi-\eta)_\alpha]^m f(\xi^\alpha\eta_\alpha)\right) = [y^\alpha(\xi+\eta)_\alpha]^n[y^\alpha(\xi-\eta)_\alpha]^m(\mathcal{D}f)(\xi^\alpha\eta_\alpha)$$

Examples: 0-0-s

The source to the linearized Fronsdal tensor reads:

$$\Box\Phi_{\underline{m}(s)} + \ldots = 2\cos 2\theta \sum_{l,k} \left(\tilde{a}_{l,k}\, \nabla_{\underline{m}(s-k)\underline{n}(l)}\Phi \nabla_{\underline{m}(k)}{}^{\underline{n}(l)}\,\Phi + \tilde{c}_{l,k}\, g_{\underline{mm}} \nabla_{\underline{m}(s-2-k)\underline{n}(l)}\Phi\nabla^{\underline{m}(k)\underline{n}(l)}\Phi\right)$$

$$a_{l,k} = \frac{(-)^k s!s!}{l!l!k!k!(s-k)!(s-k)!}\frac{\left(2k^2-2ks+s^2\right)}{2sl^3} + O\Big(\frac{1}{l^4 l!l!}\Big)$$

$$c_{l,k} = \frac{(-)^k (s-2)!(s-2)!}{l!l!k!k!(s-k-2)!(s-k-2)!}\frac{(2s+3)}{l(s+2)(s+3)} + O\Big(\frac{1}{l^2 l!l!}\Big)$$

If we just look at the canonical current sector:

$$\Box\Phi_{\underline{m}(s)} + \ldots = 2\cos 2\theta \sum_{l,k} \tilde{a}_{l,k} \nabla_{\underline{m}(s-k)\underline{n}(l)}\Phi\nabla_{\underline{m}(k)}{}^{\underline{n}(l)}\,\Phi\,,$$

$$a_{l,k} = \frac{(-)^k s!s!}{l!l!k!k!(s-k)!(s-k)!}\frac{s(2l(s-1)+s(2s-1))}{8(s-1)(l+s)^2(l+s+1)^2}$$

Locality: Another Simple Example

- The simplest case beyond flat, is the cubic scalar coupling where we can write infinitely many possible terms at this order

$$a_l \int \Phi \nabla_{\underline{m}(l)} \Phi \nabla^{\underline{m}(l)} \Phi \sim a_l C_l \int \Phi^3$$

- Each of them is however proportional on-shell to the same lowest derivative coupling up to a proportionality factor C_l
- Notice that in AdS there is a non-vanishing mass also for massless fields and therefore higher-derivative couplings will not be improvement but will contain a non-vanishing component along "primary" canonical couplings

$$\Box \phi_{\underline{m}(s)} - \nabla_{\underline{m}} \nabla^{\underline{n}} \phi_{\underline{nm}(s-1)} + \frac{1}{2} \nabla_{\underline{m}} \nabla_{\underline{m}} \phi^{\underline{n}}{}_{\underline{nm}(s-2)} - m_\phi^2 \phi_{\underline{m}(s)} + 2\Lambda g_{\underline{mm}} \phi^{\underline{n}}{}_{\underline{nm}(s-2)} = 0$$

$$m_\phi^2 = -\Lambda[(d+s-3)(s-2)-s]$$

Locality: a General Criterion

- Once an orthogonal basis has been selected all improvement can be removed one-by-one, each time with a local redefinition
- Restricting the functional space of allowed redefinitions to such class one cannot change the overall coefficient of the "primary"-canonical current
- Still one is left with an infinite series which in a local theory should be convergent:

$$\sum_{l=1}^{\infty} C_l^{(s)}(\alpha) < \infty$$

- The above defines an inner product on the space of currents that can be tested against AdS/CFT correspondence by computing Witten diagrams (1508.04764):

$$\left\langle \sum_q \alpha^l_{s,q}\,(g^{\underline{nn}})^l \nabla_{\underline{m}(s-q)\underline{n}(l)}\Phi^* \nabla_{\underline{m}(q)\underline{n}(l)}\Phi + \mathcal{O}(\Lambda g_{\underline{mm}}) \,\middle|\, J^{(s)}_{\text{can.}} \right\rangle = C^{(s)}_{l+1}(\alpha)$$

- One gets the same result by first projecting in such basis or by directly computing Witten diagrams